# NEW GENERALIZED INEQUALITIES USING ARBITRARY OPERATOR MEANS AND THEIR DUAL

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Abstract. In this article, we present some operator inequalities via arbitrary operator means and unital positive linear maps. For instance, we show that if $A, B \in \mathbb{B}(\mathscr{H})$ are two positive invertible operators such that $0 < m \leq A, B \leq M$ and $\sigma$ is an arbitrary operator mean, then

$$\Phi^p(A\sigma B) \leq K^p(h)\Phi^p(B\sigma^{\perp}A),$$

where $\sigma^{\perp}$ is dual $\sigma$, $p \geq 0$ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the classical Kantorovich constant. We also generalize the above inequality for two arbitrary means $\sigma_1, \sigma_2$ which lie between $\sigma$ and $\sigma^{\perp}$.

## 1. Introduction and preliminaries

In this paper, $\mathbb{B}(\mathscr{H})$ denote the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H}, \langle \cdot, \cdot \rangle)$. $I$ stands for the identity operator. A self-adjoint operator $A \in \mathbb{B}(\mathscr{H})$ is said to be positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathscr{H}$, and in this case we write $A \geq 0$. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, the order relation $A \leq B$ means that $B - A \geq 0$. A linear map $\Phi$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital provided that it preserves the identity operator, that is, $\Phi(I) = I$.

The axiomatic theory for pairs of positive operators has been developed by Kubo and Ando [9].

If $A, B \in \mathbb{B}(\mathscr{H})$ be two positive invertible operators, then the $\nu-$weighted arithmetic mean, geometric mean and harmonic mean of $A$ and $B$ denoted by $A\nabla_\nu B$, $A\sharp_\nu B$ $A!_\nu B$, respectively, are defined follows as

$$A\nabla_\nu B = \nu A + (1-\nu)B, \qquad A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu}A^{\frac{1}{2}},$$

and

$$A!_\nu B = (\nu A^{-1} + (1-\nu)B^{-1})^{-1},$$

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respectively. When $\nu = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$ and $A!B$ for the arithmetic mean, geometric mean and harmonic mean, respectively. The $\nu-$weighted arithmetic-geometric (AM-GM) operator inequality, which is proved in [15] says that if $A, B \in \mathbb{B}(\mathscr{H})$ are two positive operators and $0 \leq \nu \leq 1$, then $A\sharp_\nu B \leq A\nabla_\nu B$. For a particular case, when $\nu = \frac{1}{2}$, we obtain the AM-GM operator inequality

$$A\sharp B \leq \frac{A+B}{2}. \tag{1.1}$$

For two positive operators $A, B \in \mathbb{B}(\mathscr{H})$, the Löwner–Heinz inequality states that, if $A \leq B$, then

$$A^p \leq B^p, \qquad (0 \leq p \leq 1). \tag{1.2}$$

In general (1.2) is not true for $p > 1$.
Lin [12, Theorem 2.1] showed a squaring of a reverse of (1.1), namely that the inequality

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^2 \Phi^2(A\sharp B) \tag{1.3}$$

as well as

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^2 (\Phi(A)\sharp\Phi(B))^2 \tag{1.4}$$

where $\Phi$ is a positive unital linear map.
The Löwner–Heinz inequality and two inequalities (1.3) and (1.4) follow that for $0 < p \leq 2$,

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^p \Phi^p(A\sharp B) \tag{1.5}$$

and

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^p (\Phi(A)\sharp\Phi(B))^p \tag{1.6}$$

In [6], the authors showed that inequalities (1.5) and (1.6) for $p \geq 2$ hold.
For more improvements and refinements on the above inequalities see [13, 14] and references therein.
Let $\sigma$ be an operator mean with the representing function $f$. The operator mean with the representing function $\frac{t}{f(t)}$ is called the dual of $\sigma$ and denoted by $\sigma^\perp$. For $A, B \in \mathbb{B}(\mathscr{H})$,

$$A\sigma^\perp B = (B^{-1}\sigma A^{-1})^{-1}.$$

It is trivial that for two invertible operators $A, B \in \mathbb{B}(\mathscr{H})$, $A\nabla^{\perp}B = A!B$ and $A!B \leq A\sharp B$.
Let $0 < m \leq A, B \leq M$, $\Phi$ be a positive unital linear map and $\sigma, \tau$ be two arbitrary means between the harmonic and arithmetic means. In [7], the authors obtained the following inequality:

$$\Phi^2(A\sigma B) \leq K^2(h)\Phi^2(A\tau B), \tag{1.7}$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.
The authors in [5] generalized inequality (1.7) for the higher powers as follows:

$$\Phi^p(A\sigma B) \leq K^p(h)\Phi^p(A\tau B), \tag{1.8}$$

where $p > 0$.
Motivated by the above discussion, in this paper we first obtain the following inequality:

$$\Phi^2(A\sigma B) \leq K^2(h)\Phi^2(B\sigma^{\perp}A) \tag{1.9}$$

where $0 < m \leq A, B \leq M$, $\sigma$ is an arbitrary mean and $\sigma^{\perp}$ is its dual and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant. Then, we generalize inequality (1.9) for two arbitrary means $\sigma_1$ and $\sigma_2$ between $\sigma$ and $\sigma^{\perp}$.

## 2. Main results

To obtain the main results we need to recall the following Lemmas.

**Lemma 2.1.** [3](*Choi's inequality*) *Let $A \in \mathbb{B}(\mathscr{H})$ be positive and $\Phi$ be a positive unital linear map. Then*

$$\Phi(A)^{-1} \leq \Phi\left(A^{-1}\right). \tag{2.1}$$

**Lemma 2.2.** [15] *Suppose that $0 < m \leq A \leq M$. Then*

$$A + MmA^{-1} \leq M + m.$$

**Lemma 2.3.** [4, 1, 2] *Let $A, B \in \mathbb{B}(\mathscr{H})$ be positive and $\lambda > 0$. Then*
(i) $||AB|| \leq \frac{1}{4}||A + B||^2$.
(ii) *If $\lambda > 1$, then* $||A^{\lambda} + B^{\lambda}|| \leq ||(A + B)^{\lambda}||$.
(iii) $A \leq \lambda B$ *if and only if* $||A^{\frac{1}{2}}B^{-\frac{1}{2}}|| \leq \lambda^{\frac{1}{2}}$.

**Lemma 2.4.** [8] *Let* $X \in \mathbb{B}(\mathscr{H})$. *Then* $\|X\| \le t$ *if and only if*

$$\begin{pmatrix} tI & X \\ X^* & tI \end{pmatrix} \ge 0.$$

**Theorem 2.5.** *Let* $0 < m \le A, B \le M$ *such that* $0 < m < M$ *and* $\sigma$ *be an arbitrary mean. Then*

$$\Phi^2(A\sigma B) \le K^2(h)\Phi^2(B\sigma^{\perp}A), \tag{2.2}$$

*where* $\sigma^{\perp}$ *is dual* $\sigma$ *and* $K(h) = \frac{(M+m)^2}{4Mm}$ *is the Kantorovich constant.*

*Proof.* It follows from $0 < m \le A, B \le M$ that $(M-A)(m-A)A^{-1} \le 0$ and $(M-B)(m-B)B^{-1} \le 0$. Therefore

$$A + MmA^{-1} \le M + m \quad \text{and} \quad B + MmB^{-1} \le M + m.$$

Now, the subadditivity and monotonicity properties of the operator mean to conclude that

$$\begin{aligned} A\sigma B + Mm(A^{-1}\sigma B^{-1}) &\le (A + MmA^{-1})\sigma(B + MmB^{-1}) \\ &\le (M+m)\sigma(M+m) \\ &= M + m. \end{aligned}$$

Using the linearity and positivity of $\Phi$ and the latter inequality, we get

$$\Phi(A\sigma B) + Mm\Phi(A^{-1}\sigma B^{-1}) \le M + m. \tag{2.3}$$

Applying two inequalities (2.1) and (2.3), respectively, we have

$$\begin{aligned} \Phi(A\sigma B) + Mm\Phi^{-1}(B\sigma^{\perp}A) &\le \Phi(A\sigma B) + Mm\Phi(B\sigma^{\perp}A)^{-1} \\ &\le \Phi(A\sigma B) + Mm\Phi(A^{-1}\sigma B^{-1}) \\ &\le M + m. \end{aligned}$$

By Lemma 2.3(i) and the latter inequality, we get

$$\begin{aligned} \left\|\Phi(A\sigma B)Mm\Phi^{-1}(B\sigma^{\perp}A)\right\| &\le \frac{1}{4}\left\|\Phi(A\sigma B) + Mm\Phi(B\sigma^{\perp}A)^{-1}\right\|^2 \\ &\le \Phi(A\sigma B) + Mm\Phi(A^{-1}\sigma B^{-1}) \\ &\le M + m. \end{aligned}$$

This proves the assertion as desired. □

*Remark* 2.6. In special case, when $\sigma = \nabla$, since $\sigma^{\perp} =!$ and $! \le \sharp$, inequality (2.2) becomes inequality (1.3).

**Corollary 2.7.** *Let $0 < m \leq A, B \leq M$ such that $0 < m < M$, $\sigma$ be an arbitrary mean and let $p \geq 0$. Then*

$$\Phi^p(A\sigma B) \leq K^p(h)\Phi^p(B\sigma^{\perp}A), \tag{2.4}$$

*where $\sigma^{\perp}$ is dual $\sigma$ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.*

*Proof.* If $0 \leq p \leq 2$, then $0 \leq \frac{p}{2} \leq 1$. Applying inequality (2.2) we obtain the desired result. If $p > 2$, then

$$\begin{aligned}
&\left\|\Phi^{\frac{p}{2}}(A\sigma B)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\| \\
&\leq \frac{1}{4}\left\|\Phi^{\frac{p}{2}}(A\sigma B) + M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\|^2 \text{ (by Lemma 2.3 (i))} \\
&\leq \frac{1}{4}\left\|\Phi(A\sigma B) + Mm\Phi^{-1}(B\sigma^{\perp}A)\right\|^p \text{ (by Lemma 2.3 (ii))} \\
&\leq \frac{1}{4}\left\|\Phi(A\sigma B) + Mm\Phi((B\sigma^{\perp}A))^{-1}\right\|^p \text{ (by (2.1))} \\
&= \frac{1}{4}\left\|\Phi(A\sigma B) + Mm\Phi(A^{-1}\sigma^{\perp}B^{-1})\right\|^p \\
&\leq \frac{1}{4}(M+m)^p \text{ (by inequality (2.3))}.
\end{aligned}$$

Therefore, by Lemma 2.3(iii) we have

$$\Phi^p(A\sigma B) \leq K^p(h)\Phi^p(B\sigma^{\perp}A).$$

□

*Remark* 2.8. Using the same reason as in Remark 2.6 says that inequality (2.4) is a generalization of inequality (1.5) which is presented in [6].

In the following theorem, we generalize inequality (1.7).

**Theorem 2.9.** *Let $0 < m \leq A, B \leq M$, $\sigma_1$ and $\sigma_2$ be two arbitrary means which lie between $\sigma$ and $\sigma^{\perp}$ and let $p \geq 0$. Then for every positive unital linear map $\Phi$,*

$$\Phi^p(A\sigma_2 B) \leq K^p(h)\Phi^p(B\sigma_1 A), \tag{2.5}$$

*where $\sigma^{\perp}$ is dual $\sigma$ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.*

*Proof.* To prove (2.5), let $\sigma_1 \geq \sigma^{\perp}$ and $\sigma_2 \leq \sigma$. Therefore,

$$\begin{aligned}\Phi(A\sigma_2 B) + Mm\Phi^{-1}(B\sigma_1 A) &\leq \Phi(A\sigma_2 B) + Mm\Phi(B\sigma_1 A)^{-1} \text{ (by (2.1))}\\ &\leq \Phi(A\sigma B) + Mm\Phi(B\sigma^{\perp} A)^{-1}\\ &= \Phi(A\sigma B) + Mm\Phi(A^{-1}\sigma B^{-1})\\ &\leq M + m \text{ (by (2.3))}.\end{aligned}$$

Using the same ideas as used in the proof of Theorem 2.5 and Corollary 2.7, one can obtain the desired result. □

To find a better bound than the obtained bound in inequality (2.4), we need to state the following Lemma.

**Lemma 2.10.** [12] *Let $0 < m \leq A, B \leq M$ and $\sigma$ be an arbitrary mean. Then for every positive unital linear map $\Phi$*

$$\|\Phi^2(A\sigma B) + M^2m^2\Phi^n((A\sigma B)^{-1})\| \leq M^2 + m^2.$$

**Theorem 2.11.** *Let $0 < m \leq A, B \leq M$, $\sigma$ be an arbitrary mean and $p \geq 4$. Then*

$$\Phi^p(A\sigma B) \leq \left(\frac{K(h)(M^2+m^2)}{2^{\frac{4}{p}}Mm}\right)^p \Phi^p(B\sigma^{\perp} A), \tag{2.6}$$

*where $\sigma^{\perp}$ is dual $\sigma$ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.*

*Proof.* By Theorem 2.5 we have

$$\Phi^{-2}(B\sigma^{\perp} A) \leq K^2(h)\Phi^{-2}(A\sigma B). \tag{2.7}$$

A simple computation shows that

$$\begin{aligned}&\left\|\Phi^{\frac{p}{2}}(A\sigma B)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(B\sigma^{\perp} A)\right\|\\ &\leq \frac{1}{4}\left\|K^{\frac{p}{4}}(h)\Phi^{\frac{p}{2}}(A\sigma B) + \left(\frac{M^2m^2}{K(h)}\right)^{\frac{p}{4}}\Phi^{-\frac{p}{2}}(B\sigma^{\perp} A)\right\|^2 \text{ ( by Lemmas 2.3(i) )}\\ &\leq \frac{1}{4}\left\|K\Phi^2(A\sigma B) + \frac{M^2m^2}{K(h)}\Phi^{-2}(B\sigma^{\perp} A)\right\|^{\frac{p}{2}} \text{ ( by Lemmas 2.3(ii) )}\\ &\leq \frac{1}{4}\left\|K(h)\Phi^2(A\sigma B) + M^2m^2K(h)\Phi^{-2}(A\sigma B)\right\|^{\frac{p}{2}} \text{ ( by (2.7) )}\\ &\leq \frac{1}{4}K^{\frac{p}{2}}(h)\left\|\Phi^2(A\sigma B) + M^2m^2\Phi^2(A\sigma B)^{-1}\right\|^{\frac{p}{2}} \text{ ( by (2.1))}\\ &\leq \frac{1}{4}\left(K(h)\left(M^2+m^2\right)\right)^{\frac{p}{2}} \text{ ( by Lemma 2.10)}.\end{aligned}$$

Therefore

$$\left\|\Phi^{\frac{p}{2}}(A\sigma B)\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\| \leq \frac{1}{4}\left(\frac{K(h)\,(M^2+m^2)}{Mm}\right)^{\frac{p}{2}}.$$

The latter relation is equivalent to

$$\Phi^p(A\sigma B) \leq \left(\frac{K(h)\,(M^2+m^2)}{2^{\frac{4}{p}}Mm}\right)^p \Phi^p(B\sigma^{\perp}A).$$

This proves the desired result. □

*Remark* 2.12. When $p \geq 4$, the derived result in Theorem 2.11 is tighter than inequality (2.4).

Moreover, we show that Theorem 2.11 holds for $0 \leq p \leq 4$.

**Corollary 2.13.** *Let $0 < m \leq A, B \leq M$, $\sigma$ be an arbitrary mean and let $0 \leq p \leq 4$. Then*

$$\Phi^p(A\sigma B) \leq \left(\frac{K(h)\,(M^2+m^2)}{2Mm}\right)^p \Phi^p(B\sigma^{\perp}A),$$

*where $\sigma^{\perp}$ is dual $\sigma$ and $K(h) = \frac{(M+m)^2}{4MM}$.*

*Proof.* By Theorem 2.5 we have

$$\Phi^4(A\sigma B) \leq \left(\frac{K(h)\,(M^2+m^2)}{2Mm}\right)^4 \Phi^4(B\sigma^{\perp}A).$$

If $0 \leq p \leq 4$, then $0 \leq \frac{p}{4} \leq 1$. With the aid of the latter inequality and inequality (1.2), we conclude the desired inequality. □

**Theorem 2.14.** *Let $0 < m \leq A, B \leq M$, $\sigma_1$ and $\sigma_2$ be two arbitrary means between $\sigma$ and $\sigma^{\perp}$, $1 < \alpha \leq 2$ and $p \geq 2\alpha$. Then for every positive unital linear map $\Phi$*

$$\Phi^p(A\sigma_2 B) \leq \frac{(K^{\frac{\alpha}{2}}(h)(M^{\alpha}+m^{\alpha}))^{\frac{2p}{\alpha}}}{16M^p m^p}\Phi^p(B\sigma_1 A) \tag{2.8}$$

*where $\sigma^{\perp}$ is dual $\sigma$ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.*

*Proof.* At once from inequality (2.5) follows that for $1 < \alpha \leq 2$

$$\Phi^{-\alpha}(B\sigma_1 A) \leq K^{\alpha}(h)\Phi^{-\alpha}(A\sigma_2 B). \tag{2.9}$$

Using the fact that $0 < m \leq A, B \leq M$, it deduces that $0 < m \leq A\sigma_2 B \leq M$. Now, the linearity property $\Phi$ results that $0 < m \leq \Phi(A\sigma_2 B) \leq M$. Since $1 < \alpha \leq 2$, one can easily prove that

$$\Phi^{\alpha}(A\sigma_2 B) + M^{\alpha}m^{\alpha}\Phi^{-\alpha}(A\sigma_2 B) \leq M^{\alpha} + m^{\alpha}. \tag{2.10}$$

Therefore

$$
\begin{aligned}
&\left\|M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{\frac{p}{2}}(A\sigma_2 B)\Phi^{-\frac{p}{2}}(B\sigma_1 A)\right\|\\
&\leq \frac{1}{4}\left\|K^{-\frac{p}{4}}(h)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(B\sigma_1 A)+K^{\frac{p}{4}}(h)\Phi^{\frac{p}{2}}(A\sigma_2 B)\right\|^2 \text{ ( by Lemma 2.3(i) )}\\
&\leq \frac{1}{4}\left\|\left(K^{-\frac{\alpha}{2}}(h)M^{\alpha}m^{\alpha}\Phi^{-\alpha}(B\sigma_1 A)+K^{\frac{\alpha}{2}}(h)\Phi^{\alpha}(A\sigma_2 B)\right)^{\frac{p}{2\alpha}}\right\|^2 \text{ ( by Lemma 2.3(ii) )}\\
&= \frac{1}{4}\left\|K^{-\frac{\alpha}{2}}(h)M^{\alpha}m^{\alpha}\Phi^{-\alpha}(B\sigma_1 A)+K^{\frac{\alpha}{2}}(h)\Phi^{\alpha}(A\sigma_2 B)\right\|^{\frac{p}{\alpha}}\\
&\leq \frac{1}{4}\left\|K^{\frac{\alpha}{2}}(h)M^{\alpha}m^{\alpha}\Phi^{-\alpha}(A\sigma_2 B)+K^{\frac{\alpha}{2}}(h)\Phi^{\alpha}(A\sigma_2 B)\right\|^{\frac{p}{\alpha}} \text{ ( by (2.9))}\\
&\leq \frac{1}{4}K^{\frac{p}{2}}(h)(M^{\alpha}+m^{\alpha})^{\frac{p}{\alpha}} \text{( by (2.10))},
\end{aligned}
$$

that is

$$\left\|\Phi^{\frac{p}{2}}(A\sigma_2 B)\Phi^{-\frac{p}{2}}(B\sigma_1 A)\right\| \leq \frac{K^{\frac{p}{2}}(h)(M^{\alpha}+m^{\alpha})^{\frac{p}{\alpha}}}{4M^{\frac{p}{2}}m^{\frac{p}{2}}},$$

or equivalently

$$\Phi^{p}(A\sigma_2 B) \leq \frac{\left(K^{\frac{\alpha}{2}}(h)(M^{\alpha}+m^{\alpha})\right)^{\frac{2p}{\alpha}}}{16Mm}\Phi^{p}(B\sigma_1 A).$$

□

*Remark* 2.15. In special case, for $\alpha = 2$, inequality (2.8) becomes inequality (2.6).

*Remark* 2.16. By taking $\sigma = \nabla$ in inequality (2.8), we get inequality (1.8).

**Theorem 2.17.** *Let $0 < m \leq A, B \leq M$ such that $0 < m < M$ and $\sigma$ be an arbitrary mean. Then for every positive unital linear map $\Phi$ and two arbitrary means $\sigma_1$ and $\sigma_2$ which lie between $\sigma$ and $\sigma^{\perp}$ and $p \geq 0$, the following inequality holds*

$$\Phi^{p}(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A)+\Phi^{-p}(B\sigma_1 A)\Phi^{p}(A\sigma_2 B) \leq 2K^{p}(h)\Phi^{p}(B\sigma_1 A) \tag{2.11}$$

*where $\sigma^{\perp}$ is dual $\sigma$ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.*

*Proof.* It follows from (2.5) that

$$\|\Phi^{p}(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A)\| \leq K^{p}(h). \tag{2.12}$$

Applying Lemma 2.4 we have

$$\begin{pmatrix} K(h)^{p}I & \Phi^{-p}(B\sigma_1 A)\Phi^{p}(A\sigma_2 B) \\ \Phi^{p}(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A) & K(h)^{p}I \end{pmatrix} \geq 0$$

and

$$\begin{pmatrix} K(h)^p I & \Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A) \\ \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B) & K(h)^p I \end{pmatrix} \geq 0.$$

Summing up two above inequalities, we obtain the following inequality

$$\begin{pmatrix} 2K(h)^p I & \beta_1 \\ \beta_2 & 2K(h)^p I \end{pmatrix} \geq 0,$$

where

$$\beta_1 = \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B) + \Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A)$$

and

$$\beta_2 = \Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A) + \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B).$$

Again using Lemma 2.4 we get the desired result. □

*Remark* 2.18. Put $\sigma = \nabla$, inequality (2.11) reduces to some results in [2]

## 3. A refined inequality for arithmetic-geometric mean

Let $A, B \in \mathbb{B}(\mathscr{H})$ be two invertible positive operators, $0 \leq \nu \leq 1$ and $-1 \leq q \leq 1$. We use from the notation $A\sharp_{q,\nu}B$ to define the power mean

$$A\sharp_{q,\nu}B = A^{\frac{1}{2}}\left((1-\nu)I + \nu\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)^q\right)^{\frac{1}{q}}A^{\frac{1}{2}}.$$

For more information see [10]. The authors in [11] proved that if $0 < m \leq A, B \leq M$ such that $0 < m < M$ and $0 < \nu \leq \mu < 1$, $-1 \leq q \leq 1$. Then for every positive unital linear map $\Phi$ and $p \geq 0$, the following inequality holds

$$\Phi^p\left(A\nabla_\nu B + \frac{\nu}{\mu}Mm\left(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_{q,\mu}B^{-1}\right)\right)$$
$$\leq K^p(h)\Phi^p(A\sharp_{q,\nu}B), \tag{3.1}$$

where $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.
Using the following theorem, we obtain a generalization from inequality (3.1).

**Theorem 3.1.** *Suppose that $0 < m \leq A, B \leq M$ such that $0 < m < M$ and $0 < \nu \leq \mu < 1$, $-1 \leq q \leq 1$ and $1 < \alpha \leq 2$. Then for every positive unital linear map $\Phi$ and $p \geq 0$, the following inequality holds*

$$\Phi^p\left(A\nabla_\nu B + \frac{\nu}{\mu}Mm(A^{-1}\nabla_\nu B^{-1} - A^{-1}\sharp_{q,\mu}B^{-1})\right)$$
$$\leq \frac{\left(K^{\frac{\alpha}{4}}(h)(M^\alpha + m^\alpha)\right)^{\frac{2p}{\alpha}}}{16M^p m^p}\Phi^p(A\sharp_{q,\nu}B), \tag{3.2}$$

*where* $K(h)=\frac{(M+m)^2}{4Mm}$ *is the Kantorovich constant.*

*Proof.* For $1<\alpha\leq 2$, by inequality (3.1), we have

$$\Phi^{\alpha}\left(A\nabla_{\nu}B+\frac{\nu}{\mu}Mm\left(A^{-1}\nabla_{\mu}B^{-1}-A^{-1}\sharp_{q,\mu}B^{-1}\right)\right)\leqq K^{\alpha}(h)\Phi^{\alpha}(A\sharp_{q,\nu}B) \tag{3.3}$$

The last inequality deduces using a process similar to inequality (2.10). This shows that

$$\begin{aligned}&\left\|\Phi^{\frac{p}{2}}\left(A\nabla_{\nu}B+\frac{\nu}{\mu}Mm\left(A^{-1}\nabla_{\mu}B^{-1}-A^{-1}\sharp_{q,\mu}B^{-1}\right)\right)\Phi^{-\frac{p}{2}}(A\sharp_{q,\nu}B)\right\|\\&\leq\frac{K^{\frac{p}{2}}(h)(M^{\alpha}+m^{\alpha})^{\frac{p}{\alpha}}}{4M^{\frac{p}{2}}m^{\frac{p}{2}}}.\end{aligned}$$

Then

$$\begin{aligned}&\Phi^{p}\left(A\nabla_{\nu}B+\frac{\nu}{\mu}Mm\left(A^{-1}\nabla_{\mu}B^{-1}-A^{-1}\sharp_{q,\mu}B^{-1}\right)\right)\\&\leq\frac{\left(K^{\frac{\alpha}{4}}(h)(M^{\alpha}+m^{\alpha})\right)^{\frac{2p}{\alpha}}}{16M^{p}m^{p}}\Phi^{p}(A\sharp_{q,\nu}B).\end{aligned}$$

□

*Remark* 3.2. Taking $\alpha=2$, inequality (3.2) becomes inequality (3.1).

*Remark* 3.3. By putting $\alpha=2,\mu=\frac{1}{2}$ and taking $q\to 0$, inequality (3.2) collapse to the derived result in [2].

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