ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 51 (2021), No. 6, 1941–1951

SOME EXTENSIONS OF BEREZIN NUMBER INEQUALITIES ON OPERATORS

MOJTABA BAKHERAD, MONIRE HAJMOHAMADI,
RAHMATOLLAH LASHKARIPOUR AND SATYAJIT SAHOO

We establish some upper bounds for Berezin number inequalities including inequalities for 2×2 operator matrices and their off-diagonal parts. Among other inequalities, it is shown that if $T=\left[\begin{smallmatrix}0 & X\\ Y & 0\end{smallmatrix}\right]$, then

$$\mathbf{ber}^r(T)\le 2^{r-2}\big(\mathbf{ber}(f^{2r}(|X|)+g^{2r}(|Y^*|))+\mathbf{ber}(f^{2r}(|Y|)+g^{2r}(|X^*|))\big)-2^{r-2}\inf_{\|(k_{\lambda_1},k_{\lambda_2})\|=1}\eta(k_{\lambda_1},k_{\lambda_2}),$$

where X, Y are bounded linear operators on a Hilbert space $\mathscr{H}=\mathscr{H}(\Omega)$, $r\ge 1$, f, g are nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t)=t$ $(t\in[0,\infty))$ and

$$\eta(k_{\lambda_1},k_{\lambda_2})=\big(\langle(f^{2r}(|X|)+g^{2r}(|Y^*|))k_{\lambda_2},k_{\lambda_2}\rangle^{\frac{1}{2}}-\langle(f^{2r}(|Y|)+g^{2r}(|X^*|))k_{\lambda_1},k_{\lambda_1}\rangle^{\frac{1}{2}}\big)^2.$$

1. Introduction and preliminaries

A functional Hilbert space $\mathscr{H}=\mathscr{H}(\Omega)$ is a Hilbert space of complex valued functions on a (nonempty) set Ω, which has the property that point evaluations are continuous; i.e., for each $\lambda\in\Omega$ the map $f\mapsto f(\lambda)$ is a continuous linear functional on $\mathscr{H}$. The Riesz representation theorem ensures that for each $\lambda\in\Omega$ there is a unique element $k_\lambda\in\mathscr{H}$ such that $f(\lambda)=\langle f,k_\lambda\rangle$, for all $f\in\mathscr{H}$. The collection $\{k_\lambda:\lambda\in\Omega\}$ is called the reproducing kernel of $\mathscr{H}$. If $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathscr{H}$, then the reproducing kernel of $\mathscr{H}$ is given by $k_\lambda(z)=\sum_n\overline{e_n(\lambda)}e_n(z)$; see [10, Problem 37]. For $\lambda\in\Omega$, let $\hat{k}_\lambda=k_\lambda/\|k_\lambda\|$ be the normalized reproducing kernel of $\mathscr{H}$. For a bounded linear operator A on $\mathscr{H}$, the function $\widetilde{A}$ defined on Ω by $\widetilde{A}(\lambda)=\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle$ is the Berezin symbol of A, which was first introduced by Berezin [5; 6]. The Berezin set and the Berezin number of the operator A are defined by

$$\mathbf{Ber}(A):=\{\widetilde{A}(\lambda):\lambda\in\Omega\}\quad\text{and}\quad\mathbf{ber}(A):=\sup\{|\widetilde{A}(\lambda)|:\lambda\in\Omega\},$$

respectively; see [12]. In some recent papers, several Berezin number inequalities have been investigated by authors [3; 9; 8; 12; 13; 16; 17]. The Berezin number of operators A and B satisfies the properties $\mathbf{ber}(\alpha A)=|\alpha|\,\mathbf{ber}(A)$ $(\alpha\in\mathbb{C})$, and $\mathbf{ber}(A+B)\le\mathbf{ber}(A)+\mathbf{ber}(B)$ and $\mathbf{ber}(A)\le\|A\|$, where $\|\cdot\|$ is the operator norm. Let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with the corresponding norm $\|\cdot\|$. Throughout this paper, the operator matrix $T=\left[\begin{smallmatrix}S & X\\ Y & R\end{smallmatrix}\right]$ is a matrix, where $S\in\mathscr{B}(\mathscr{H}_1)$, $X\in\mathscr{B}(\mathscr{H}_2,\mathscr{H}_1)$, $Y\in\mathscr{B}(\mathscr{H}_1,\mathscr{H}_2)$ and $R\in\mathscr{B}(\mathscr{H}_2)$. The authors in [4]

Bakherad would like to thank the Tusi Mathematical Research Group (TMRG).
2020 AMS *Mathematics subject classification:* primary 47A30; secondary 30E20, 47A12, 47A60.
Keywords and phrases: Berezin number, Berezin symbol, diagonal operator matrices, off-diagonal part.
Received by the editors on January 1, 2021, and in revised form on February 24, 2021.

DOI: 10.1216/rmj.2021.51.1941

showed an upper bound for the off-diagonal part of the operator matrix, which is

$$(\mathbf{ber}(T))^r \le \tfrac{1}{4}\|h(f^2(|Y|))+h(g^2(|Y|))\|+\tfrac{1}{4}\|h(f^2(|X|))+h(g^2(|X|))\|, \tag{1-1}$$

where $T=\left[\begin{smallmatrix}0 & X\\ Y & 0\end{smallmatrix}\right]$, h is a convex function and f, g are nonnegative continuous functions on $[0,\infty)$ such that $f(t)g(t)=t$ $(t\in[0,\infty))$.

The classical Young inequality says that if $p, q>1$ such that $1/p+1/q=1$, then $ab\le a^p/p+b^q/q$ for positive real numbers a, b. In [1], the authors showed that a refinement of the scalar Young inequality is

$$\left(a^{\frac{1}{p}}b^{\frac{1}{q}}\right)^m + r_0^m\left(a^{\frac{m}{2}}-b^{\frac{m}{2}}\right)^2 \le \left(\frac{a}{p}+\frac{b}{q}\right)^m,$$

where $r_0=\min\{1/p, 1/q\}$ and $m=1,2,\dots$. In particular, if $p=q=2$, then

$$(a^{\frac{1}{2}}b^{\frac{1}{2}})^m+\left(\tfrac{1}{2}\right)^m(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2\le 2^{-m}(a+b)^m. \tag{1-2}$$

We obtain some upper bounds for Berezin number inequalities for the off-diagonal part of an operator matrix and refinements of them. Moreover, we obtain Berezin number inequalities for the diagonal operator matrix.

2. Main results

To prove our results, we need the following lemmas.

Lemma 2.1 [2]. *Let $S\in\mathscr{B}(\mathscr{H}_1)$, $X\in\mathscr{B}(\mathscr{H}_2,\mathscr{H}_1)$, $Y\in\mathscr{B}(\mathscr{H}_1,\mathscr{H}_2)$ and $R\in\mathscr{B}(\mathscr{H}_2)$. Then the following statements hold*:

(a) $\mathbf{ber}\left(\left[\begin{smallmatrix}S & 0\\ 0 & R\end{smallmatrix}\right]\right)\le\max\{\mathbf{ber}(S),\mathbf{ber}(R)\}$.

(b) $\mathbf{ber}\left(\left[\begin{smallmatrix}0 & X\\ Y & 0\end{smallmatrix}\right]\right)\le\frac{1}{2}(\|X\|+\|Y\|)$. *In particular,*

$$\mathbf{ber}\left(\begin{bmatrix}0 & X\\ X & 0\end{bmatrix}\right)\le\|X\|. \tag{2-1}$$

(c) $\mathbf{ber}(S)=\sup_{\theta\in\mathbb{R}}\mathbf{ber}(\mathrm{Re}(e^{i\theta}S))$.

The next lemma follows from the spectral theorem for positive operators and the Jensen's inequality; see, e.g., [14].

Lemma 2.2. *Let $T\in\mathscr{B}(\mathscr{H})$, $T\ge 0$ and $x\in\mathscr{H}$ such that $\|x\|=1$. Then*

(a) $\langle Tx,x\rangle^r\le\langle T^rx,x\rangle$ *for* $r\ge 1$,

(b) $\langle T^rx,x\rangle\le\langle Tx,x\rangle^r$ *for* $0<r\le 1$.

Lemma 2.3 [14, Theorem 1]. *Let $T\in\mathscr{B}(\mathscr{H})$ and $x,y\in\mathscr{H}$ be any vectors. If f, g are nonnegative continuous functions on* $[0,\infty)$ *which are satisfying the relation $f(t)g(t)=t$ $(t\in[0,\infty))$, then*

$$|\langle Tx,y\rangle|^2\le\langle f^2(|T|)x,x\rangle\langle g^2(|T^*|)y,y\rangle.$$

Now, we are in position to demonstrate the main results of this section by using some ideas from [15; 4].

Theorem 2.4. *Suppose that* $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$, $r \geq 1$ *and* f, g *are nonnegative continuous functions on* $[0, \infty)$ *satisfying the relation* $f(t)g(t) = t$ $(t \in [0, \infty))$. *Then*

$$(\mathbf{ber}(T))^r \leq \frac{2^r}{2} \mathbf{ber}^{\frac{1}{2}}(f^{2r}(|X|) + g^{2r}(|Y^*|)) \, \mathbf{ber}^{\frac{1}{2}}(f^{2r}(|Y|) + g^{2r}(|X^*|)),$$

and

$$(\mathbf{ber}(T))^r \leq \frac{2^r}{2} \mathbf{ber}^{\frac{1}{2}}(f^{2r}(|X|) + f^{2r}(|Y^*|)) \, \mathbf{ber}^{\frac{1}{2}}(g^{2r}(|Y|) + g^{2r}(|X^*|)).$$

Proof. For $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$, let $\hat{k}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel in $\mathscr{H}_1 \oplus \mathscr{H}_2$. Then

$$\begin{aligned}
|\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle|^r &= |\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle + \langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|^r \\
&\leq \big(|\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle| + |\langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|\big)^r \qquad \text{(by the triangular inequality)} \\
&\leq \frac{2^r}{2}\big(|\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle|^r + |\langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|^r\big) \qquad \text{(by the convexity of } f(t) = t^r) \\
&\leq \frac{2^r}{2}\big((\langle f^2(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} \langle g^2(|X^*|)k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}})^r \\
&\qquad + \big(\langle f^2(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}} \langle g^2(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}}\big)^r\big) \qquad \text{(by Lemma 2.3)} \\
&\leq \frac{2^r}{2}\big(\langle f^{2r}(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} \langle g^{2r}|X^*|k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}} \\
&\qquad + \langle f^{2r}(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}} \langle g^{2r}(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}}\big) \qquad \text{(by Lemma 2.2(a))} \\
&\leq \frac{2^r}{2}\big(\langle f^{2r}(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle + \langle g^{2r}(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle\big)^{\frac{1}{2}} \\
&\qquad \times \big(\langle f^{2r}(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle + \langle g^{2r}(|X^*|)k_{\lambda_1}, k_{\lambda_1}\rangle\big)^{\frac{1}{2}} \\
&\qquad \text{(by the Cauchy–Schwarz inequality)} \\
&= \frac{2^r}{2}\langle (f^{2r}(|X|) + g^{2r}(|Y^*|))k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} \langle (f^{2r}(|Y|) + g^{2r}(|X^*|))k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}} \\
&\leq \frac{2^r}{2} \mathbf{ber}^{\frac{1}{2}}(f^{2r}(|X|) + g^{2r}(|Y^*|)) \, \mathbf{ber}^{\frac{1}{2}}(f^{2r}(|Y|) + g^{2r}(|X^*|)).
\end{aligned}$$

Therefore,

$$\mathbf{ber}^r(T) \leq \frac{2^r}{2} \mathbf{ber}^{\frac{1}{2}}(f^{2r}(|X|) + g^{2r}(|Y^*|)) \, \mathbf{ber}^{\frac{1}{2}}(f^{2r}(|Y|) + g^{2r}(|X^*|)).$$

Hence, we get the first inequality. For the proof of the second inequality, we have

$$\begin{aligned}
\text{(2-2)} \quad |\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle|^r &= |\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle + \langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|^r \\
&\leq \big(|\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle| + |\langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|\big)^r \qquad \text{(by the triangular inequality)} \\
&\leq \frac{2^r}{2}\big(|\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle|^r + |\langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|^r\big) \qquad \text{(by the convexity of } f(t) = t^r) \\
&\leq \frac{2^r}{2}\Big(\big(\langle f^2(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} \langle g^2(|X^*|)k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}}\big)^r \\
&\qquad + \big(\langle g^2(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}} \langle f^2(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}}\big)^r\Big) \qquad \text{(by Lemma 2.3).}
\end{aligned}$$

With a similar argument to the proof of the first inequality we have the second inequality and this completes the proof of the theorem. □

Theorem 2.4 includes some special cases as follows.

Corollary 2.5. *Let* $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$, $0 \le p \le 1$ *and* $r \ge 1$. *Then*

$$\mathbf{ber}^r(T) \le 2^{r-2}\,\mathbf{ber}^{\frac{1}{2}}\big(|X|^{2rp} + |Y^*|^{2r(1-p)}\big)\,\mathbf{ber}^{\frac{1}{2}}\big(|Y|^{2rp} + |X^*|^{2r(1-p)}\big)$$

and

$$\mathbf{ber}^r(T) \le 2^{r-2}\,\mathbf{ber}^{\frac{1}{2}}\big(|X|^{2rp} + |Y^*|^{2rp}\big)\,\mathbf{ber}^{\frac{1}{2}}\big(|Y|^{2r(1-p)} + |X^*|^{2r(1-p)}\big).$$

Proof. The result follows immediately from Theorem 2.4 for $f(t) = t^p$ and $g(t) = t^{1-p}$ $(0 \le p \le 1)$. □

Remark 2.6. Taking $f(t) = g(t) = t^{\frac{1}{2}}$ $(t \in [0, \infty))$ and $r = 1$ in Theorem 2.4, we get,

$$\mathbf{ber}\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \le \tfrac{1}{2}\,\mathbf{ber}^{\frac{1}{2}}(|X| + |Y^*|)\,\mathbf{ber}^{\frac{1}{2}}(|Y| + |X^*|).$$

If we put $Y = X$, $r = 1$ and $f(t) = g(t) = t^{\frac{1}{2}}$ in Theorem 2.4, then we get a refinement of inequality (2-1) as follows.

Corollary 2.7. *Assume that* $X \in \mathscr{B}(\mathscr{H})$, $r \ge 1$ *and* f, g *are nonnegative continuous functions on* $[0, \infty)$ *satisfying the relation* $f(t)g(t) = t$ $(t \in [0, \infty))$. *Then*

$$\mathbf{ber}\left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\right) \le \tfrac{1}{2}\,\mathbf{ber}(|X| + |X^*|) \le \|X\|.$$

In the following corollary we obtain the following inequality.

Corollary 2.8. *Let* $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$. *Then*

$$\mathbf{ber}(T) \le \tfrac{1}{2}\,\mathbf{ber}^{\frac{1}{2}}(|X| + |Y^*|)\,\mathbf{ber}^{\frac{1}{2}}(|Y| + |X^*|) \le \tfrac{1}{2}\max\{\mathbf{ber}(|X| + |Y^*|), \mathbf{ber}(|Y| + |X^*|)\}.$$

Proof. In Theorem 2.4, we put $r = 1$, $f(t) = g(t) = t^{\frac{1}{2}}$ and applying arithmetic-geometric mean, get

$$\begin{aligned} \mathbf{ber}(T) &\le \tfrac{1}{2}\,\mathbf{ber}^{\frac{1}{2}}(|X| + |Y^*|)\,\mathbf{ber}^{\frac{1}{2}}(|Y| + |X^*|) \\ &\le \tfrac{1}{2}\left(\frac{\mathbf{ber}(|X| + |Y^*|) + \mathbf{ber}(|Y| + |X^*|)}{2}\right) \\ &\le \tfrac{1}{2}\max\{\mathbf{ber}(|X| + |Y^*|), \mathbf{ber}(|Y| + |X^*|)\}. \end{aligned}$$ □

Applying inequality (1-2), we obtain the following theorem.

Theorem 2.9. *Suppose that* $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ *and* f, g *are nonnegative continuous functions on* $[0, \infty)$ *satisfying the relation* $f(t)g(t) = t$ $(t \in [0, \infty))$. *Then for* $r \ge 1$

$$\mathbf{ber}^r(T) \le 2^{r-2}\big(\mathbf{ber}(f^{2r}(|X|) + g^{2r}(|Y^*|)) + \mathbf{ber}(f^{2r}(|Y|) + g^{2r}(|X^*|))\big) - 2^{r-2}\inf_{\|(k_{\lambda_1}, k_{\lambda_2})\|=1}\eta(k_{\lambda_1}, k_{\lambda_2}),$$

where

$$\eta(k_{\lambda_1}, k_{\lambda_2}) = \big(\langle (f^{2r}(|X|) + g^{2r}(|Y^*|))k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} - \langle (f^{2r}(|Y|) + g^{2r}(|X^*|))k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}}\big)^2.$$

Proof. For $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$, let $\hat{k}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel in $\mathscr{H}_1 \oplus \mathscr{H}_2$. Then

$$\begin{aligned}
|\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle|^r &= |\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle + \langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|^r \\
&\leq \big(|\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle| + |\langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|\big)^r \qquad \text{(by the triangular inequality)} \\
&\leq \frac{2^r}{2}\big(|\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle|^r + |\langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|^r\big) \qquad \text{(by the convexity of } f(t)=t^r) \\
&\leq \frac{2^r}{2}\big(\langle f^2(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{r}{2}}\langle g^2(|X^*|)k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{r}{2}} \\
&\qquad + \langle f^2(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{r}{2}}\langle g^2(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{r}{2}}\big) \qquad \text{(by Lemma 2.3)} \\
&\leq \frac{2^r}{2}\big(\langle f^{2r}(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}}\langle g^{2r}(|X^*|)k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}} \\
&\qquad + \langle f^{2r}(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}}\langle g^{2r}(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}}\big) \\
&\leq \frac{2^r}{2}\big(\langle f^{2r}(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle + \langle g^{2r}(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle\big)^{\frac{1}{2}} \\
&\qquad \times \big(f^{2r}(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle + \langle g^{2r}(|X^*|)k_{\lambda_1}, k_{\lambda_1}\rangle\big)^{\frac{1}{2}} \\
&= \frac{2^r}{2}\langle (f^{2r}(|X|) + g^{2r}(|Y^*|))k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}}\langle (f^{2r}(|Y|) + g^{2r}(|X^*|))k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}} \\
&\leq \frac{2^r}{4}\big(\langle (f^{2r}(|X|) + g^{2r}(|Y^*|))k_{\lambda_2}, k_{\lambda_2}\rangle + \langle (f^{2r}(|Y|) + g^{2r}(|X^*|))k_{\lambda_1}, k_{\lambda_1}\rangle\big) \\
&\quad - \frac{2^r}{4}\big(\langle (f^{2r}(|X|) + g^{2r}(|Y^*|))k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} - \langle (f^{2r}(|Y|) + g^{2r}(|X^*|))k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}}\big)^2 \\
&\qquad \text{(by inequality (1-2))} \\
&\leq \frac{2^r}{4}\big(\mathbf{ber}(f^{2r}(|X|) + g^{2r}(|Y^*|)) + \mathbf{ber}(f^{2r}(|Y|) + g^{2r}(|X^*|))\big) \\
&\quad - \frac{2^r}{4}\big(\langle (f^{2r}(|X|) + g^{2r}(|Y^*|))k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} - \langle (f^{2r}(|Y|) + g^{2r}(|X^*|))k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}}\big)^2.
\end{aligned}$$

Taking the supremum over all unit vectors $\hat{k}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix} \in \mathscr{H}_1 \oplus \mathscr{H}_2$, we get the desired inequality. □

If we put $X = Y$ in Theorem 2.9, then we get next result.

Corollary 2.10. *Let $X \in \mathscr{B}(\mathscr{H})$ and f, g be nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, \infty))$. Then for $r \geq 1$*

$$\mathbf{ber}^r\left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\right) \leq 2^{r-1}\|f^{2r}(|X|) + g^{2r}(|X^*|)\| - 2^{r-2}\inf_{\|(k_{\lambda_1},k_{\lambda_2})\|=1}\eta(k_{\lambda_1}, k_{\lambda_2}),$$

where

$$\eta(k_{\lambda_1}, k_{\lambda_2}) = \big(\langle (f^{2r}(|X|) + g^{2r}(|X^*|))k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} - \langle (f^{2r}(|X|) + g^{2r}(|X^*|))k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}}\big)^2.$$

3. Generalizations of the Berezin number of an operator

In this section, we present some Berezin number inequalities for the generalized Aluthge transform, $\widetilde{T}_t$, and then we present some inequalities, which generalized known inequalities.

Let $T = U|T|$ (U is a partial isometry with $\ker U = \text{rang}\,|T|^{\perp}$) be the polar decomposition of T. For an operator $T \in \mathscr{B}(\mathscr{H})$, the generalized Aluthge transform, denoted by, $\widetilde{T}_t$, is defined as

$$\widetilde{T}_t = |T|^t U |A|^{1-t}, \quad (0 \le t \le 1).$$

In the next theorem, we obtain an upper bound for the Berezin number of generalized Aluthge transform of the off-diagonal operator matrix $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$.

Theorem 3.1. *Suppose that* $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$. *Then*

$$\mathbf{ber}(\widetilde{T}_t) \le \tfrac{1}{2}\big(\||Y|^t|X^*|^{1-t}\| + \||X|^t|Y^*|^{1-t}\|\big). \tag{3-1}$$

Proof. Let $X = U|X|$ and $Y = V|Y|$ be the polar decompositions of the operators X and Y. Then

$$\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} = \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix}\begin{bmatrix} |Y| & 0 \\ 0 & |X| \end{bmatrix}$$

is the polar decomposition of T. The generalized Aluthge transform of T is

$$\widetilde{T}_t = |T|^t \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} |T|^{1-t} = \begin{bmatrix} |Y|^t & 0 \\ 0 & |X|^t \end{bmatrix}\begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix}\begin{bmatrix} |Y|^{1-t} & 0 \\ 0 & |X|^{1-t} \end{bmatrix} = \begin{bmatrix} 0 & |Y|^t U |X|^{1-t} \\ |X|^t V |Y|^{1-t} & 0 \end{bmatrix}.$$

So

$$\begin{aligned} \mathbf{ber}(\widetilde{T}_t) &= \mathbf{ber}\left(\begin{bmatrix} 0 & |Y|^t U |X|^{1-t} \\ |X|^t V |Y|^{1-t} & 0 \end{bmatrix}\right) \\ &\le \tfrac{1}{2}\big(\||Y|^t U |X|^{1-t}\| + \||X|^t V |Y|^{1-t}\|\big) \qquad \text{(by Lemma 2.1(b)).} \end{aligned}$$

Since, $|X^*|^2 = XX^* = U|X|^2U^*$, so $|X|^{1-t} = U^*|X^*|^{1-t}U$. Thus,

$$\||Y|^t U |X|^{1-t}\| = \||Y|^t U U^* |X^*|^{1-t} U\| = \||Y|^t |X^*|^{1-t}\|.$$

Similarly, $\||X|^t V |Y|^{1-t}\| = \||X|^t |Y^*|^{1-t}\|$. Therefore,

$$\mathbf{ber}(\widetilde{T}_t) \le \tfrac{1}{2}\big(\||Y|^t|X^*|^{1-t}\| + \||X|^t|Y^*|^{1-t}\|\big). \qquad \square$$

Theorem 3.2. *Assume that* $T \in \mathscr{B}(\mathscr{H})$. *Then*

$$\mathbf{ber}(T) \le \tfrac{1}{4}\||T|^{2t} + |T|^{2(1-t)}\| + \tfrac{1}{2}\,\mathbf{ber}(\widetilde{T}_t), \tag{3-2}$$

where $t \in [0, 1]$.

Proof. Let $\hat{k}_\lambda \in \mathscr{H}$. We have

$$\begin{aligned} \operatorname{Re}\langle e^{i\theta} T \hat{k}_\lambda, \hat{k}_\lambda\rangle &= \operatorname{Re}\langle e^{i\theta} U|T| \hat{k}_\lambda, \hat{k}_\lambda\rangle \\ &= \operatorname{Re}\langle e^{i\theta} U|T|^t|T|^{1-t} \hat{k}_\lambda, \hat{k}_\lambda\rangle \\ &= \operatorname{Re}\langle e^{i\theta} |T|^{1-t} \hat{k}_\lambda, |T|^t U^* \hat{k}_\lambda\rangle \\ &= \tfrac{1}{4}\|(e^{i\theta}|T|^{1-t} + |T|^t U^*)\hat{k}_\lambda\|^2 - \tfrac{1}{4}\|(e^{i\theta}|T|^{1-t} - |T|^t U^*)\hat{k}_\lambda\|^2 \\ &\qquad\qquad \text{(by the polarization identity)} \\ &\le \tfrac{1}{4}\|(e^{i\theta}|T|^{1-t} + |T|^t U^*)\hat{k}_\lambda\|^2 \end{aligned}$$

$$\begin{aligned}
&= \tfrac{1}{4}\langle (e^{i\theta}|T|^{1-t}+|T|^{t}U^*)\hat{k}_\lambda, (e^{i\theta}|T|^{1-t}+|T|^{t}U^*)\hat{k}_\lambda\rangle \\
&= \tfrac{1}{4}\langle (e^{i\theta}|T|^{1-t}+|T|^{t}U^*)(e^{-i\theta}|T|^{1-t}+U|T|^{t})\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&= \tfrac{1}{4}\langle |T|^{2t}+|T|^{2(1-t)}+e^{i\theta}\widetilde{T}_t+e^{-i\theta}(\widetilde{T}_t)^*\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&= \tfrac{1}{4}\langle |T|^{2t}+|T|^{2(1-t)}\hat{k}_\lambda, \hat{k}_\lambda\rangle + \tfrac{1}{4}\langle e^{i\theta}\widetilde{T}_t+e^{-i\theta}(\widetilde{T}_t)^*\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&= \tfrac{1}{4}\langle |T|^{2t}+|T|^{2(1-t)}\hat{k}_\lambda, \hat{k}_\lambda\rangle + \tfrac{1}{2}\langle \mathrm{Re}(e^{i\theta}\widetilde{T}_t)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\le \tfrac{1}{4}\||T|^{2t}+|T|^{2(1-t)}\| + \tfrac{1}{2}\,\mathbf{ber}(\mathrm{Re}(e^{i\theta}\widetilde{T}_t) \\
&\le \tfrac{1}{4}\||T|^{2t}+|T|^{2(1-t)}\| + \tfrac{1}{2}\,\mathbf{ber}(\widetilde{T}_t).
\end{aligned}$$

By taking the supremum over $\lambda \in \Omega$, we get the desired result. □

Remark 3.3. By putting $t = \frac{1}{2}$ in Theorem 3.2, we get

$$\mathbf{ber}(T) \le \tfrac{1}{2}\|T\| + \tfrac{1}{2}\,\mathbf{ber}(\widetilde{T}_t), \tag{3-3}$$

where $T \in \mathscr{B}(\mathscr{H})$.

From Theorem 3.1 we deduce the next result for the off-diagonal operator matrix $T = \left[\begin{smallmatrix} 0 & X \\ Y & 0 \end{smallmatrix}\right]$.

Corollary 3.4. *Let $T = \left[\begin{smallmatrix} 0 & X \\ Y & 0 \end{smallmatrix}\right] \in \mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$. Then*

$$\mathbf{ber}(T) \le \tfrac{1}{2}\max\big(\|X\|, \|Y\|\big) + \tfrac{1}{4}\big(\||Y|^t|X^*|^{1-t}\| + \||X|^t|Y^*|^{1-t}\|\big). \tag{3-4}$$

Proof. It is immediately deduced from Theorem 3.1 and Remark 3.3. □

In the following we present some Berezin number inequalities for the operator matrix $T = \left[\begin{smallmatrix} S & X \\ Y & R \end{smallmatrix}\right]$.

Theorem 3.5. *Suppose that $T = \left[\begin{smallmatrix} S & X \\ Y & R \end{smallmatrix}\right] \in \mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$. Then*

$$\mathbf{ber}(T) \le \tfrac{1}{2}\,\mathbf{ber}(S) + \mathbf{ber}(R) + \tfrac{1}{2}\sqrt{\alpha^2\,\mathbf{ber}^2(S) + \|X\|^2} + \tfrac{1}{2}\sqrt{(1-\alpha)^2\,\mathbf{ber}^2(S) + \|Y\|^2} \tag{3-5}$$

for $0 \le \alpha \le 1$.

Proof.

$$\begin{aligned}
\mathbf{ber}(\mathrm{Re}(e^{i\theta}T)) &= \mathbf{ber}\left(\frac{e^{i\theta}T + e^{-i\theta}T^*}{2}\right) \\
&= \frac{1}{2}\,\mathbf{ber}\begin{bmatrix} 2\mathrm{Re}(e^{i\theta}S) & e^{i\theta}X + e^{-i\theta}Y^* \\ e^{i\theta}Y + e^{-i\theta}X^* & 2\mathrm{Re}(e^{i\theta}R) \end{bmatrix} \\
&\le \frac{1}{2}\left(\mathbf{ber}\begin{bmatrix} 2\alpha\mathrm{Re}(e^{i\theta}S) & e^{i\theta}X \\ e^{-i\theta}X^* & 0 \end{bmatrix} + \mathbf{ber}\begin{bmatrix} 2(1-\alpha)\mathrm{Re}(e^{i\theta}S) & e^{-i\theta}Y^* \\ e^{i\theta}Y & 0 \end{bmatrix}\right. \\
&\qquad \left. + \,\mathbf{ber}\begin{bmatrix} 0 & 0 \\ 0 & 2\mathrm{Re}(e^{i\theta}R) \end{bmatrix}\right) \\
&\le \frac{1}{2}\left(\begin{bmatrix} 2\alpha\,\mathbf{ber}(\mathrm{Re}(e^{i\theta}S)) & \|e^{i\theta}X\| \\ \|e^{-i\theta}X^*\| & 0 \end{bmatrix} + \begin{bmatrix} 2(1-\alpha)\,\mathbf{ber}(\mathrm{Re}(e^{i\theta}S)) & \|e^{-i\theta}Y^*\| \\ \|e^{i\theta}Y\| & 0 \end{bmatrix}\right. \\
&\qquad \left. + \begin{bmatrix} 0 & 0 \\ 0 & 2\,\mathbf{ber}(\mathrm{Re}(e^{i\theta}R)) \end{bmatrix}\right) \quad \text{(by Theorem 2.1 of [2])}
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}\left(\begin{bmatrix} 2\alpha\,\mathbf{ber}(S) & \|X\| \\ \|X\| & 0 \end{bmatrix} + \begin{bmatrix} 2(1-\alpha)\,\mathbf{ber}(S) & \|Y\| \\ \|Y\| & 0 \end{bmatrix} + 2\,\mathbf{ber}(R)\right)\\
&= \frac{1}{2}\Big[\alpha\,\mathbf{ber}(S) + \sqrt{\alpha^2\,\mathbf{ber}^2(S) + \|X\|^2} + (1-\alpha)\,\mathbf{ber}(S)\\
&\qquad + \sqrt{(1-\alpha)^2\,\mathbf{ber}^2(S) + \|Y\|^2} + 2\,\mathbf{ber}(R)\Big]\\
&\le \frac{1}{2}\Big[\mathbf{ber}(S) + 2\,\mathbf{ber}(R) + \sqrt{\alpha^2\,\mathbf{ber}^2(S) + \|X\|^2} + \sqrt{(1-\alpha)^2\,\mathbf{ber}^2(S) + \|Y\|^2}\Big].
\end{aligned}$$

Taking supremum over $\theta \in \mathbb{R}$, we get

$$\mathbf{ber}(T) \le \tfrac{1}{2}\,\mathbf{ber}(S) + \mathbf{ber}(R) + \tfrac{1}{2}\sqrt{\alpha^2\,\mathbf{ber}^2(S) + \|X\|^2} + \tfrac{1}{2}\sqrt{(1-\alpha)^2\,\mathbf{ber}^2(S) + \|Y\|^2}. \qquad \square$$

Using similar argument as used in previous theorem, we have the following result.

Theorem 3.6. *Assume that* $T = \left[\begin{smallmatrix} S & X \\ Y & R \end{smallmatrix}\right] \in \mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$. *Then*

(3-6) $$\mathbf{ber}(T) \le \tfrac{1}{2}\,\mathbf{ber}(R) + \mathbf{ber}(S) + \tfrac{1}{2}\sqrt{\alpha^2\,\mathbf{ber}^2(R) + \|Y\|^2} + \tfrac{1}{2}\sqrt{(1-\alpha)^2\,\mathbf{ber}^2(R) + \|X\|^2}$$

for $0 \le \alpha \le 1$.

In the final part of the article we want to provide generalization of Berezin number of an operator. For our goal we need to the following inequalities, which were obtained in [11]:

(3-7) $$a^\nu b^{1-\nu} \le \nu a + (1-\nu)b \le (\nu a^r + (1-\nu)b^r)^{\frac{1}{r}},$$

and

(3-8) $$ab \le \frac{a^p}{p} + \frac{b^q}{q} \le \left(\frac{a^{pr}}{p} + \frac{b^{qr}}{q}\right)^{\frac{1}{r}},$$

where $a, b \ge 0$, $\nu \in [0, 1]$, $r \ge 1$ and $p, q > 1$ such that $1/p + 1/q = 1$.

In the next theorem, we obtain upper bound for powers of the Berezin number.

Theorem 3.7. *Assume that* $T \in \mathscr{B}(\mathscr{H})$ *and* f, g *are nonnegative continuous functions on* $[0, \infty)$ *satisfying the relation* $f(t)g(t) = t$ $(t \in [0, \infty))$. *Then*

(3-9) $$\mathbf{ber}^{2r}(T) \le \tfrac{1}{2}\Big(\|T\|^{2r} + \mathbf{ber}\Big(\frac{1}{p} f^{pr}(|T^2|) + \frac{1}{q} g^{qr}(|(T^2)^*|)\Big)\Big),$$

where $r \ge 1$, $p \ge q > 1$ *with* $1/p + 1/q = 1$ *and* $qr \ge 2$.

Proof. Let $k_\lambda \in \mathscr{H}$ be unit vector. We need the following refinement of Schwarz's inequality:

(3-10) $$\|a\|\|b\| \ge |\langle a, b\rangle - \langle a, e\rangle\langle e, b\rangle| + |\langle a, e\rangle\langle e, b\rangle| \ge |\langle a, b\rangle|,$$

where a, b, e are vectors in $\mathscr{H}$ and $\|e\| = 1$. Using (3-10), we have

(3-11) $$\tfrac{1}{2}\big(\|a\|\|b\| + |\langle a, b\rangle|\big) \ge |\langle a, e\rangle\langle e, b\rangle|.$$

By putting $e = k_\lambda$, $a = Tk_\lambda$ and $b = T^*k_\lambda$ in (3-11) and applying (3-7), we get

$$|\langle Tk_\lambda, k_\lambda\rangle|^2 \le \tfrac{1}{2}\big(\|Tk_\lambda\|\|T^*k_\lambda\| + |\langle T^2k_\lambda, k_\lambda\rangle|\big) \le \left(\frac{\|Tk_\lambda\|^r\|T^*k_\lambda\|^r + |\langle T^2k_\lambda, k_\lambda\rangle|^r}{2}\right)^{\frac{1}{r}}.$$

Hence

$$|\langle Tk_\lambda, k_\lambda\rangle|^{2r} \le \tfrac{1}{2}\big(\|Tk_\lambda\|^r\|T^*k_\lambda\|^r + |\langle T^2k_\lambda, k_\lambda\rangle|^r\big). \tag{3-12}$$

Then from Lemma 2.3, inequality (3-8) and Lemma 2.2(a), we have

$$\begin{aligned}
|\langle T^2k_\lambda, k_\lambda\rangle|^r &\le \|f(|T^2|)k_\lambda\|^r\|g(|(T^2)^*|)k_\lambda\|^r \\
&= \langle f^2(|T^2|)k_\lambda, k_\lambda\rangle^{r/2}\langle g^2(|(T^2)^*|)k_\lambda, k_\lambda\rangle^{r/2} \\
&\le \frac{1}{p}\langle f^2(|T^2|)k_\lambda, k_\lambda\rangle^{rp/2} + \frac{1}{q}\langle g^2(|(T^2)^*|)k_\lambda, k_\lambda\rangle^{rq/2} \\
&\le \frac{1}{p}\langle f^{rp}(|T^2|)k_\lambda, k_\lambda\rangle + \frac{1}{q}\langle g^{rq}(|(T^2)^*|)k_\lambda, k_\lambda\rangle \\
&= \Big\langle \frac{1}{p}f^{rp}(|T^2|) + \frac{1}{q}g^{rq}(|(T^2)^*|)k_\lambda, k_\lambda\Big\rangle.
\end{aligned}$$

By using inequality (3-12), we get

$$\begin{aligned}
|\langle Tk_\lambda, k_\lambda\rangle|^{2r} &\le \tfrac{1}{2}\Big(\|Tk_\lambda\|^r\|T^*k_\lambda\|^r + \Big\langle \frac{1}{p}f^{rp}(|T^2|) + \frac{1}{q}g^{rq}(|(T^2)^*|)k_\lambda, k_\lambda\Big\rangle\Big) \\
&\le \tfrac{1}{2}\Big(\|T\|^r\|T^*\|^r + \mathbf{ber}\Big(\frac{1}{p}f^{pr}(|T^2|) + \frac{1}{q}g^{qr}(|(T^2)^*|)\Big)\Big) \\
&= \tfrac{1}{2}\Big(\|T\|^{2r} + \mathbf{ber}\Big(\frac{1}{p}f^{pr}(|T^2|) + \frac{1}{q}g^{qr}(|(T^2)^*|)\Big)\Big).
\end{aligned}$$

By taking supremum over unit vector k_λ, we get the desired inequality. □

Remark 3.8. From inequality (3-12), we have

$$\mathbf{ber}^{2r}(T) \le \tfrac{1}{2}(\mathbf{ber}^r(T^2) + \|T\|^{2r}) \le \tfrac{1}{2}(\|T\|^{2r} + \|T\|^{2r}) \le \|T\|^{2r}.$$

Theorem 3.9. *Assume that $T \in \mathscr{B}(\mathscr{H})$. Then for any $\nu \in [0, 1]$ and $t \in \mathbb{R}$,*

$$\|T\|^2 \le ((1-\nu)^2 + \nu^2)\,\mathbf{ber}^2(T) + \nu\|T - tI\|^2 + (1-\nu)\|T - itI\|^2.$$

Proof. We use the following inequality, which obtained in [7]

$$(\nu\|tb - a\|^2 + (1-\nu)\|itb - a\|^2)\|b\|^2 \ge \|a\|^2\|b\|^2 - ((1-\nu)\mathrm{Im}\langle a, b\rangle + \nu\mathrm{Re}\langle a, b\rangle)^2 \quad (\ge 0)$$

for any $a, b \in \mathscr{H}$, $\nu \in [0, 1]$ and $t \in \mathbb{R}$. So,

$$\begin{aligned}
\|a\|^2\|b\|^2 &\le ((1-\nu)\mathrm{Im}\langle a, b\rangle + \nu\mathrm{Re}\langle a, b\rangle)^2 + (\nu\|tb - a\|^2 + (1-\nu)\|itb - a\|^2)\|b\|^2 \\
&\le ((1-\nu)^2 + \nu^2)|\langle a, b\rangle|^2 + (\nu\|tb - a\|^2 + (1-\nu)\|itb - a\|^2)\|b\|^2.
\end{aligned}$$

Choosing $a = Tk_\lambda$, $b = k_\lambda$ with $\|k_\lambda\| = 1$, we get

$$\begin{aligned}
\|Tk_\lambda\|^2\|k_\lambda\|^2 &\le ((1-\nu)\mathrm{Im}\langle Tk_\lambda, k_\lambda\rangle + \nu\mathrm{Re}\langle Tk_\lambda, k_\lambda\rangle)^2 \\
&\quad + (\nu\|tk_\lambda - Tk_\lambda\|^2 + (1-\nu)\|itk_\lambda - Tk_\lambda\|^2)\|k_\lambda\|^2 \\
&\le ((1-\nu)^2 + \nu^2)|\langle Tk_\lambda, k_\lambda\rangle|^2 \\
&\quad + (\nu\|tk_\lambda - Tk_\lambda\|^2 + (1-\nu)\|itk_\lambda - Tk_\lambda\|^2)\|k_\lambda\|^2 \\
&= ((1-\nu)^2 + \nu^2)|\langle Tk_\lambda, k_\lambda\rangle|^2 \\
&\quad + (\nu\|(t-T)k_\lambda\|^2 + (1-\nu)\|(it-T)k_\lambda\|^2)\|k_\lambda\|^2 \\
&\le ((1-\nu)^2 + \nu^2)\,\mathbf{ber}^2(T) + \nu\|T - tI\|^2 + (1-\nu)\|T - itI\|^2.
\end{aligned}$$

Taking the supremum over all unit vectors k_λ, we get the desired result. □

References

[1] Y. Al-Manasrah and F. Kittaneh, "A generalization of two refined Young inequalities", *Positivity* **19**:4 (2015), 757–768.

[2] M. Bakherad, "Some Berezin number inequalities for operator matrices", *Czechoslovak Math. J.* **68(143)**:4 (2018), 997–1009.

[3] M. Bakherad and K. Shebrawi, "Upper bounds for numerical radius inequalities involving off-diagonal operator matrices", *Ann. Funct. Anal.* **9**:3 (2018), 297–309.

[4] M. Bakherad, R. Lashkaripour, M. Hajmohamadi, and U. Yamancı, "Complete refinements of the Berezin number inequalities", *J. Math. Inequal.* **13**:4 (2019), 1117–1128.

[5] F. A. Berezin, "Covariant and contravariant symbols of operators", *Izv. Akad. Nauk SSSR Ser. Mat.* **36**:5 (1972), 1134–1167. In Russian; translated in *Math. USSR-Izv.* **6**:5 (1972), 1117–1151.

[6] F. A. Berezin, "Quantization", *Izv. Akad. Nauk SSSR Ser. Mat.* **38**:5 (1974), 1116–1175. In Russian; translated in *Math. USSR-Izv.* **8**:5 (1974), 1109–1165.

[7] S. S. Dragomir, "Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces", *Tamkang J. Math.* **39**:1 (2008), 1–7.

[8] M. T. Garayev, M. Gürdal, and A. Okudan, "Hardy–Hilbert's inequality and power inequalities for Berezin numbers of operators", *Math. Inequal. Appl.* **19**:3 (2016), 883–891.

[9] M. T. Garayev, M. Gürdal, and S. Saltan, "Hardy type inequaltiy for reproducing kernel Hilbert space operators and related problems", *Positivity* **21**:4 (2017), 1615–1623.

[10] P. R. Halmos, *A Hilbert space problem book*, 2nd ed., Encyclopedia of Mathematics and its Applications **17**, Springer, 1982.

[11] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, 1988.

[12] M. T. Karaev, "Berezin symbol and invertibility of operators on the functional Hilbert spaces", *J. Funct. Anal.* **238**:1 (2006), 181–192.

[13] M. T. Karaev, "Reproducing kernels and Berezin symbols techniques in various questions of operator theory", *Complex Anal. Oper. Theory* **7**:4 (2013), 983–1018.

[14] F. Kittaneh, "Notes on some inequalities for Hilbert space operators", *Publ. Res. Inst. Math. Sci.* **24**:2 (1988), 283–293.

[15] T. Kosem, "Inequalities between $\|f(A+B)\|$ and $\|f(A)+f(B)\|$", *Linear Algebra Appl.* **418**:1 (2006), 153–160.

[16] U. Yamanci and M. Garayev, "Some results related to the Berezin number inequalities", *Turkish J. Math.* **43**:4 (2019), 1940–1952.

[17] U. Yamancı, R. Tunç, and M. Gürdal, "Berezin number, Grüss-type inequalities and their applications", *Bull. Malays. Math. Sci. Soc.* **43**:3 (2020), 2287–2296.

MOJTABA BAKHERAD: mojtaba.bakherad@yahoo.com
Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

MONIRE HAJMOHAMADI: monire.hajmohamadi@yahoo.com
Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

RAHMATOLLAH LASHKARIPOUR: lashkari@hamoon.usb.ac.ir
Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

SATYAJIT SAHOO: satyajitsahoo2010@gmail.com
P.G. Department of Mathematics, Utkal University, Vanivihar, Bhubaneswar, India

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Rocky Mountain Mathematics Consortium