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A variational iteration method for solving nonlinear Lane–Emden problems

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HIGHLIGHTS

- A semi-analytical method for nonlinear Lane–Emden problems.
- Includes adaptive step size and domain decomposition.
- Overcomes the main difficulty arising in the singularity of the equation at the origin point.
- Is simple to implement, accurate when applied to Lane–Emden type equations.

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ABSTRACT

In this paper, an explicit analytical method called the variational iteration method is presented for solving the second-order singular initial value problems of the Lane–Emden type. In addition, the local convergence of the method is discussed. It is often useful to have an approximate analytical solution to describe the Lane–Emden type equations, especially in the case that the closed-form solutions do not exist at all. This convince us that an effective improvement of the method will be useful to obtain a better approximate analytical solution. The improved method is then treated as a local algorithm in a sequence of intervals. Besides, an adaptive version is suggested for finding accurate approximate solutions of the nonlinear Lane–Emden type equations. Some examples are given to demonstrate the efficiency and accuracy of the proposed method.

1. Introduction

Recently, a lot of attention has been focused on the study of singular initial value problems (IVPs) in the second-order ordinary differential equations (ODEs). Many problems in mathematical physics and astrophysics can be modelled by the so-called IVPs of the Lane–Emden type equation (Chandrasekhar, 1967; Davis, 1962; Richardson, 1921):

$$\begin{cases} y'' + \dfrac{2}{x}y' + f(x, y) = g(x), \\ y(0) = a, \quad y'(0) = b, \end{cases} \tag{1}$$

where a and b are constants, $f(x, y)$ is a continuous real valued function, and $g(x) \in C[0, \infty]$. When $f(x, y) = K(y)$, $g(x) = 0$, Eq. (1) reduces to the classical Lane–Emden equation, which was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere, and theory of thermionic currents (Chandrasekhar, 1967; Davis, 1962; Richardson, 1921).

The Lane–Emden type equations have significant applications in many fields of scientific and technical world. Therefore various forms of $f(x, y)$ and $g(x)$ have been investigated by many researchers (e.g., Chowdhury and Hashim, 2007; Shawagfeh, 1993; Wazwaz, 2001). A discussion of the formulation of these models and the physical structure of the solutions can be found in the literature. The numerical solution of the Lane–Emden equation (1), as well as other types of linear and nonlinear singular IVPs in quantum mechanics and astrophysics (Krivec and Mandelzweig, 2001), is numerically challenging because of the singularity behavior at the origin $x = 0$. But analytical solutions are more needed to understand physical better. Recently, many analytical methods were used to solve the Lane–Emden equation (He, 2003; Liao, 2003; Yildirim and Ozis, 2007). Those methods are based on either series solutions or perturbation techniques (Bender et al., 1989; Mandelzweig and Tabakin, 2001; Ramos, 2005; 2008). However, the convergence region of the corresponding results is very small.

The variational iteration method (VIM) was first introduced by the Chinese mathematician J.H. He (He, 1999; 1997a; 1997b; 1998; He et al., 1999; He, 2000, 2006; He and Wu, 2007; He, 2007)

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and has been widely applied by many researchers to handle linear and nonlinear problems. The VIM is used in Tatari and Dehghan (2007) to solve some problems in calculus of variations. This technique is used in Ozer (2007) to solve the boundary value problems with jump discontinuities. Authors of Biazar and Ghazvini (2007) applied the variational iteration method to solve the hyperbolic differential equations. This method is employed in Odibat and Momani (2006) to solve the nonlinear differential equations of fractional order. For more and new applications of the method the interested reader is referred to Lu (2015), Lu and Ma (2016), Hu and He (2016).

The strategy being pursued in this work rests mainly on establishing a useful algorithm based on the VIM (He, 1999; Ghorbani and Momani, 2010) to find highly accurate solution of the Lane–Emden type equations, which it

- overcomes the main difficulty arising in the singularity of the equation at $x = 0$.
- is simple to implement, accurate when applied to the Lane–Emden type equations and avoid tedious computational works.

The examples analyzed in the present paper reveal that the newly developed algorithms are easy, effective and accurate to solve the singular IVPs of the Lane–Emden type equation.

2. Description of the method and its convergence

The basic idea of the VIM is constructing a correction functional by a general Lagrange multiplier where the multiplier in the functional could be identified by variational theory (He and Wu, 2007; He, 2007).

Here, the VIM is described for solving Eq. (1). This method provides the solution as a sequence of iterations. It gives convergent successive approximations of the exact solution if such a solution exists, otherwise approximations can be used for numerical purposes.

To explain the basic idea of the VIM, we first consider Eq. (1) as follows:

$$L[y(x)] + N[y(x)] = g(x), \tag{2}$$

with

$$L[y(x)] = y''(x) + \frac{2}{x}y'(x) \quad \text{and} \quad N[y(x)] = f(x, y(x)), \tag{3}$$

where L denotes the linear operator with respect to y and N is a nonlinear operator with respect to y. The basic character of the VIM is to construct a correction functional according to the variational method as:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(t)\Big(y_n''(t) + \frac{2}{t}y_n'(t) + f(t, \tilde{y}_n(t)) - g(t)\Big)dt, \tag{4}$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory, the subscript n denotes the nth approximation, and $\tilde{y}_n$ is considered as a restricted variation, namely $\delta\tilde{y}_n = 0$. Successive approximations, $y_{n+1}(x)$'s, will be obtained by applying the obtained Lagrange multiplier and a properly chosen initial approximation $y_0(x)$. Consequently, the exact solution can be obtained by using

$$y(x) = \lim_{n\longrightarrow\infty} y_n(x). \tag{5}$$

Now, if we want to determine the optimal value of $\lambda(t)$, we continue as follows:

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(t)\Big(y_n''(t) + \frac{2}{t}y_n'(t)\Big)dt, \tag{6}$$

which the stationary conditions can be achieved from the relation (6) as:

$$\begin{cases} 1 - \lambda'(x) + \frac{2}{x}\lambda(x) = 0, \\ \lambda(x) = 0, \\ \lambda''(x) - 2\frac{x\lambda'(x) - \lambda(x)}{x^2} = 0, \end{cases} \tag{7}$$

and the Lagrange multiplier is gained via the relation

$$\lambda(t) = -\left(t - \frac{t^2}{x}\right). \tag{8}$$

Finally, the iteration formula can be given as:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(t - \frac{t^2}{x}\right) \Big(y_n''(t) + \frac{2}{t}y_n'(t) + f(t, y_n(t)) - g(t)\Big)dt. \tag{9}$$

It is interesting to note that for linear Lane–Emden type equations, its exact solution can be obtained easily by only one iteration step due to the fact that the multiplier can be suitably identified, as will be shown in this paper later.

Now we give the following lemma for the iteration formula (9).

Lemma 1. *If $y(x) \in C^2[0, T]$, then, for $x \le T$*

$$\int_0^x \left(t - \frac{t^2}{x}\right)\Big(y''(t) + \frac{2}{t}y'(t)\Big)dt = y(x) - y(0). \tag{10}$$

Proof. The left hand side of the relation (10) can be written as below:

$$\int_0^x \left(t - \frac{t^2}{x}\right)(y''(t))dt + \int_0^x \left(2 - \frac{2t}{x}\right)(y'(t))dt. \tag{11}$$

Now integrating by parts first integral (11) yields

$$y'(t)\left[t - \frac{t^2}{x}\right]_{t=0}^{t=x} - \int_0^x \left(1 - \frac{2t}{x}\right)(y'(t))dt + \int_0^x \left(2 - \frac{2t}{x}\right)(y'(t))dt = \int_0^x y'(t)dt = y(x) - y(0), \tag{12}$$

this completes the proof of (10). □

Using (9) and (10), we have the following simple variational iteration formula:

$$y_{n+1}(x) = y(0) - \int_0^x \left(t - \frac{t^2}{x}\right)(f(t, y_n(t)) - g(t))dt. \tag{13}$$

The VIM (13) makes a recurrence sequence $\{y_n(x)\}$ for $x \in [0, T]$. Obviously, the limit of this sequence is the solution of (1) if this sequence is convergent.

Theorem 2. *If $N[y(x)] = f(x, y)$ is Lipschitz-continuous in $[0, T]$ and $g(x) \in C[0, T]$, then the sequence $\{y_n(x)\}$ produced by (13) is convergent for $x \in [0, T]$.*

Proof. In order to prove the sequence $\{y_n(x)\}$ is uniformly convergent to the solution $y(x)$ of (1), we first note that $y_n(x)$ can be written as

$$\begin{aligned} y_n(x) &= y_0(x) + y_1(x) - y_0(x) + \cdots + y_n(x) - y_{n-1}(x) \\ &= y_0(x) + \sum_{j=0}^{n-1}[y_{j+1}(x) - y_j(x)]. \end{aligned} \tag{14}$$

Shortly we show that

$$|y_n(x) - y_{n-1}(x)| \le \frac{N}{L}\frac{(MLx)^n}{n!}, \tag{15}$$

which this allows us to conclude that the series

$$\sum_{n=0}^{\infty}[y_{n+1}(x) - y_n(x)], \tag{16}$$

is absolutely and uniformly convergent on the interval [0, T] since it is dominated by an infinite series $\sum_{n=0}^{\infty}\frac{N}{L}\frac{(MLx)^n}{n!}$ which obviously does converge uniformly on the interval [0, T] with no restriction on M.

According to (13), note that

$$|y_1(x) - y_0(x)| = \left|\int_0^x \left(t - \frac{t^2}{x}\right)(f(t, y_0(t)) - g(t))dt\right| \le MNx, \tag{17}$$

where

$$M = \max_{0\le t\le x\le T}\left|t - \frac{t^2}{x}\right| \quad \text{and} \quad N = \max_{0\le t\le x\le T}|f(t, y_0(t)) - g(t)|. \tag{18}$$

From (13) and (17), and the assumption that $|f(t, y_n) - f(t, y_{n-1})| \le L|y_n - y_{n-1}|$ where L denotes the Lipschitz constant of $f(x, y)$, it follows that

$$|y_2(x) - y_1(x)| \le ML\left|\int_0^x |y_1(t) - y_0(t)|dt\right| \le \frac{N}{L}\frac{(MLx)^2}{2!}, \tag{19}$$

$$|y_3(x) - y_2(x)| \le ML\left|\int_0^x |y_2(t) - y_1(t)|dt\right| \le \frac{N}{L}\frac{(MLx)^3}{3!}, \tag{20}$$

$$\vdots$$

$$|y_n(x) - y_{n-1}(x)| \le \frac{N}{L}\frac{(MLx)^n}{n!}. \tag{21}$$

So with the uniform convergence of the series (16) which we have here on the right side of (14), we can take the limit of both sides of (14) as $n \to \infty$ to conclude that $\lim_{n\to\infty} y_n(x)$ exists for all $x \in [0, T]$. But this is exactly the sequence $y_n(x)$ in (13), and since we assumed that $f(t, y_n(t))$ is continuous in $y_n(t)$, we can take the limit as $n \to \infty$ on both sides of (13), allowing $\lim_{n\to\infty} f(t, y_n(t)) = f(t, y(t))$, to conclude that $\lim_{n\to\infty} y_n(t) = y(t)$, the solution to (1). □

2.1. A truncated VIM

The successive iterations of the VIM may be very complex, so that the resulting integrals in the relation (4) may not be performed analytically. Also, the implementation of the VIM generally leads to calculation of unneeded terms, which more time is consumed in repeated calculations for series solutions. Here, an effective modification of the VIM is applied to eliminate these repeated calculations. To completely stop these repeats in each step, provided that the integrand of (4) in each of iterations is expanded in multivariate Taylor series around 0, we propose the following improvement of the VIM (4), which is called the truncated VIM (TV):

$$y_{n+1}(x) = y_n(x) - \int_0^x F_n(x, t)dt, \tag{22}$$

where

$$\left(t - \frac{t^2}{x}\right)\left(y_n''(t) + \frac{2}{t}y_n'(t) + f(t, y_n(t)) - g(t)\right) = F_n(x, t) + O(x^{n+1}) + O(t^{n+1}). \tag{23}$$

It is noteworthy to point out that the TV formula (22) can cancel all the repeated calculations and terms that are not needed as will be shown below. Furthermore, it can reduce the size of calculations. Most importantly, however, it is the fact that the TV algorithm (22) will solve a Lane–Emden equation exactly if its solution is an algebraic polynomial up to some degree.

2.2. A local VIM

In general, by using the TV formula (22), we obtain a series solution, which in practice is a truncated series solution. This series solution gives a good approximation to the exact solution in a small region of x. An easy and reliable way of ensuring validity of the approximations (22) for large x is to determine the solution in a sequence of equal subintervals of x, i.e. $I_i = [x_i, x_{i+1}]$ where $h_i = x_{i+1} - x_i$, $i = 0, 1, \cdots, N-1$, with $x_0 = 0$ and $x_N = T$. According to the relation (22), we can construct the following piecewise TV approximations (PTV) in the subintervals I_i. On [x_0, x_1], let

$$\begin{cases} y_{1,m+1}(x) = y_{1,m}(x) - \int_{x_0}^x F_{1,m}(x, t)dt, \quad m = 0, 1, \cdots, n_1 - 1, \\ y_{1,0}(x) = y(0) + y'(0)(x - x_0) = c_0 + c_0'(x - x_0), \\ \left(t - \frac{t^2}{x}\right)\left(y_{1,m}''(t) + \frac{2}{t}y_{1,m}'(t) + f(t, y_{1,m}(t)) - g(t)\right) = F_{1,m}(x, t) \\ \qquad + O((x - x_0)^{n+1}) + O((t - x_0)^{n+1}). \end{cases} \tag{24}$$

Then we can obtain the n_1-order approximation $y_{1,n_1}(x)$ on [x_0, x_1]. On [x_1, x_2], let

$$\begin{cases} y_{2,m+1}(x) = y_{2,m}(x) - \int_{x_1}^x F_{2,m}(x, t)dt, \quad m = 0, 1, \cdots, n_2 - 1, \\ y_{2,0}(x) = y_{1,n_1}(x_1) + y_{1,n_1}'(x_1)(x - x_1) = c_1 + c_1'(x - x_1), \\ \left(t - \frac{t^2}{x}\right)\left(y_{2,m}''(t) + \frac{2}{t}y_{2,m}'(t) + f(t, y_{2,m}(t)) - g(t)\right) = F_{2,m}(x, t) \\ \qquad + O((x - x_1)^{n+1}) + O((t - x_1)^{n+1}). \end{cases} \tag{25}$$

Then we can obtain the n_2-order approximation $y_{2,n_2}(x)$ on [x_1, x_2]. In a similar way, on $[x_i, x_{i+1}]$, $i = 2, 3, \cdots, N-1$, we set

$$\begin{cases} y_{i+1,m+1}(x) = y_{i+1,m}(x) - \int_{x_i}^x F_{i+1,m}(x, t)dt, \quad m = 0, 1, \cdots, n_{i+1} - 1, \\ y_{i+1,0}(x) = y_{i,n_i}(x_i) + y_{i,n_i}'(x_i)(x - x_i) = c_i + c_i'(x - x_i), \\ \left(t - \frac{t^2}{x}\right)\left(y_{i+1,m}''(t) + \frac{2}{t}y_{i+1,m}'(t) + f(t, y_{i+1,m}(t)) - g(t)\right) \\ \quad = F_{i+1,m}(x, t) + O((x - x_i)^{n+1}) + O((t - x_i)^{n+1}), \end{cases} \tag{26}$$

to obtain the n_{i+1}-order approximation $y_{i+1,n_{i+1}}(x)$ on $[x_i, x_{i+1}]$.

Therefore, according to (24)–(26), the approximation of Eq. (1) on the entire interval [0, T] can be obtained. It should be emphasized that the VIM and TV algorithms provide analytical solutions in [0, T], while the PTV technique provides analytical solutions in $[x_i, x_{i+1}]$, which are continuous at the end points of each interval, i.e., $y_{i,n_i}(x_i) = c_i = y_{i+1,n_{i+1}}(x_i)$ and $y_{i,n_i}'(x_i) = c_i' = y_{i+1,n_{i+1}}'(x_i)$, $i = 1, 2, \cdots, N-1$.

It is obvious that the best PTV method of (26) can be achieved by using a variable order of n_{i+1} and a variable step size h_i in the solution to obtain a specified tolerance. Therefore, the following adaptive strategy based on the variable step size is proposed for the PTV method, which we summarize it as the APTV (see, e.g., Hairer and Wanner, 1991 and the references therein). This technique simplifies computation, and saves time and work, as will be observed later in this paper.

Let $\mathbf{y}_{i+1,k}$ be the solution of the fixed k-order PTV formula with the step size h_i and $\widehat{\mathbf{y}}_{i+1,k}$ the solution with the step size $h_i/2$. Taking the difference of $\mathbf{y}_{i+1,k}$ and $\widehat{\mathbf{y}}_{i+1,k}$, the local error estimator of

$$\mathbf{y}_{i+1,k}$$
$$Est = \widehat{\mathbf{y}}_{i+1,k} - \mathbf{y}_{i+1,k}, \tag{27}$$

is defined. This value is an estimation of the main part of the local discretization error of the method. Additionally, let r be the dimension of the ODE system, and $Atol$ and $Rtol$ the user-specified absolute and relative error tolerances, respectively. The tolerances occurring in each step are denoted by

$$Tol_j = Atol + Rtol.|y^j_{i+1,k}|, \qquad j = 1, \dots, r. \tag{28}$$

Taking

$$err = \sqrt{\frac{1}{r}\sum_{j=1}^{r}\left(\frac{Est}{Tol_j}\right)^2}, \tag{29}$$

as a measure we find an optimal step size h_{opt} by comparing err to 1. Thus we obtain the optimal step size as

$$h_{opt} = h_i.\left(\frac{1}{err}\right)^{\alpha}, \tag{30}$$

where for $err \le fac_{err}\,(fac_{err} \in (0, 1])$, we use $\alpha = \frac{1}{k+1}$, and for $err > fac_{err}$, $\alpha = \frac{1}{k}$. This is, of course, not the best choice for all problems. The new step size

$$h_{new} = h_{i+1} = h_i.min\left\{fac_{max}, max\left\{fac_{min}, fac.\left(\frac{1}{err}\right)^{\alpha}\right\}\right\}, \tag{31}$$

is obtained by using err with k as order of the approximation, instead of order of consistency. The integration of the growth factors fac_{max} and fac_{min} to relation (31) prevents for too large step increase and contribute to the safety of the code. Additionally, using the safety factor fac makes sure that err will be accepted in the next step with high probability. The step is accepted, in the case when $err \le fac_{err}$, otherwise it is rejected and then the procedure is redone. In both cases the new solution is computed with h_{new} as step size.

3. Implementations

To give a clear overview of the content of this study, several Lane–Emden type equations will be studied. These equations will be tested by the above-mentioned algorithms, which will ultimately show the usefulness and accuracy of these methods. Moreover, the numerical results indicate that the approach is easy to implement. All the results here are calculated by using the symbolic calculus software Maple 17. Also, all calculations are carried out in a Toshiba Tecra A8 (Windows 8.1 Professional): Intel(R) Core(TM)2 Duo Processor T7200 (2.00 GHz, 4 MB Cache, 997 MHz, 0.99 GB of RAM).

Example 1. As a first example, we consider the following linear, non-homogeneous Lane–Emden equation, i.e., Eq. (1) with $f(x, y) = y$ and $g(x) = 6 + 12x + x^2 + x^3$ (see, e.g., Parand et al., 2010):

$$y'' + \frac{2}{x}y' + y = 6 + 12x + x^2 + x^3, \tag{32}$$

subject to the initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0.$$

The VIM has a very simple approach. Its concepts begin with dividing the left hand side of (32) into two parts, i.e., the linear operator L and the nonlinear operator N as:

$$L[y(x)] = y'' + \frac{2}{x}y' + y \qquad \text{and} \qquad N[y(x)] \equiv 0. \tag{33}$$

This allows us to construct a variational iteration relation for Eq. (32) as follows:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(\frac{t}{x}\sin(x-t)\right)\left(y''_n(t) + \frac{2}{t}y'_n(t) + y_n(t) - 6 - 12x - x^2 - x^3\right)dt. \tag{34}$$

By using simple integration by parts, similar to Lemma 1, we have

$$\int_0^x \left(\frac{t}{x}\sin(x-t)\right)\left(y''_n(t)+\frac{2}{t}y'_n(t)+y_n(t)\right)dt = y(x) - y(0)\frac{\sin(x)}{x}. \tag{35}$$

In the light of (34) and (35), therefore, we have the following VIM:

$$y_{n+1}(x) = y_0\frac{\sin(x)}{x} + \int_0^x \left(\frac{t}{x}\sin(x-t)\right)\left(6 + 12x + x^2 + x^3\right)dt, \tag{36}$$

where $y_0 = y(0)$ and $y_0(x) = y(0) + y'(0)x$. Utilizing (36), we get the following approximations with starting the initial guess $y_0(x) = 0$:

$$y_n(x) = x^2 + x^3 \qquad \text{for all} \qquad n \ge 1, \tag{37}$$

which is the exact solution of the Lane–Emden equation (32). This proves our above-mentioned claim that the VIM could solve the linear Lane–Emden equation by only one iteration.

Example 2. As another example, we consider the nonlinear, non-homogeneous Lane–Emden equation, i.e., Eq. (1) with $f(x, y) = y^3$ and $g(x) = 6 + x^6$ (see, e.g., Parand et al., 2010):

$$y'' + \frac{2}{x}y' + y^3 = 6 + x^6, \tag{38}$$

subject to the initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0.$$

Here, we aim to solve the equation (38) by means of the TV algorithm (22). According to (22), we can easily obtain the following approximations of the TV with starting the initial approximation $y_0(x) = 0$:

$$\begin{aligned} &y_1(x) = 0, \\ &y_n(x) = x^2 \qquad \text{for all} \qquad n \ge 2, \end{aligned} \tag{39}$$

which the TV algorithm yields the exact solution. This also demonstrates our above-noted claim that the PV algorithm can solve the linear/nonlinear Lane–Emden equation exactly if its solution is an algebraic polynomial up to some degree.

Example 3. As final example, we consider the nonlinear, homogeneous Lane–Emden-type equation, i.e., Eq. (1) with $f(x, y) = e^y$ and $g(x) = 0$ (see, e.g., Parand et al., 2010):

$$y'' + \frac{2}{x}y' + e^y = 0, \tag{40}$$

subject to the initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0.$$

Here, we aim to solve the equation (40) by means of the above-proposed methods. Since the integration of the nonlinear term e^y in Eq. (40) is not easily evaluated, thus the VIM requires a large amount of computational work to obtain few iterations of the solution (we can replace the nonlinear term with a series of finite components). However, we use the modified VIM method, i.e., the

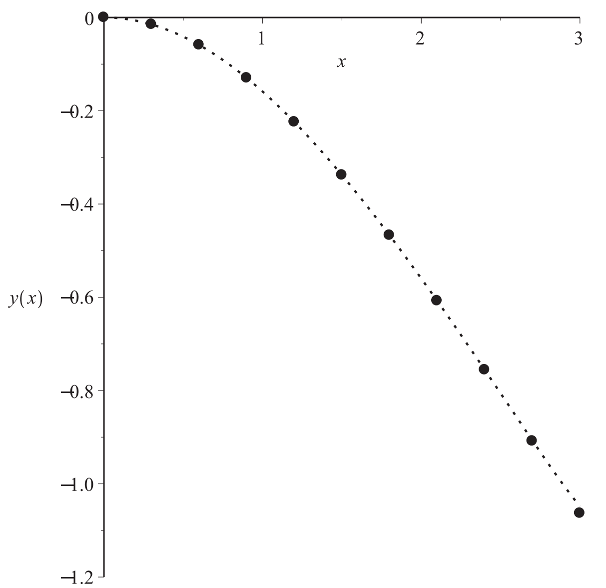

Fig. 1. Approximate solution for Example 3 using the TV algorithm where the dotted-line: the 20th-order TV algorithm and symbol: the numerical solution.

Table 1
The numerical results obtained from solving Example 3 using the 4th-order APTV algorithm when $fac_{err} = 1$, $fac = 0.9$, $fac_{min} = 0.5$ and $fac_{max} = 1.5$.

Algorithm	T	$Atol$	$Rtol$	No. of steps	CPU time (s)
APTV	1000	10^{-10}	10^{-10}	1030	4.156
APTV	1000	10^{-11}	10^{-11}	1819	7.047
APTV	1000	10^{-12}	10^{-12}	3223	12.156
APTV	1000	10^{-13}	10^{-13}	5720	21.765

TV algorithm (22). According to (22), we can easily obtain the following approximations of (40) with starting the initial approximation $y_0(x) = 0$:

$$y_2(x) = -\frac{1}{6}x^2,$$
$$y_4(x) = -\frac{1}{6}x^2 + \frac{1}{120}x^4,$$
$$y_6(x) = -\frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{1890}x^6, \tag{41}$$

and so on. Fig. 1 shows a comparison of approximation obtained using the 20th-order TV algorithm with the numerical solution of Eq. (40). As observed, the TV algorithm (22) in solving Eq. (40) gives good approximations to the exact solution in a small region of x, i.e., $x \in [0, 2.5]$. In order to enlarge the convergence region of the series solution, here we implement the PTV (26) proposed in Section 2.2. According to (26), taking $N = 4000$ and $n_{i+1} = 4$, we can obtain the approximations of (36) on [0, 1000]. Fig. 2 shows the absolute error (the difference between the approximate value and the numerical value) of the PTV solution for $n_{i+1} = 4$ and $h_i = 0.25$. From Fig. 2, it is easily found that the present approximation is efficient for a larger interval.

Now, in order to show the efficiency of the above adaptive mechanism controlling the truncation error, we solve the above equation using the before-mentioned APTV algorithm. The numerical results can be observed in Table 1. In Table 1, we listed the costed number of steps (labeled as No. of steps) for some different values of T, $Atol$ and $Rtol$, and the corresponding costed CPU elapsed time (labeled as CPU time).

Moreover, in Fig. 3, one can see the plot of the variable step size using the fourth-order APTV algorithm for $Atol = Rtol = 10^{-13}$

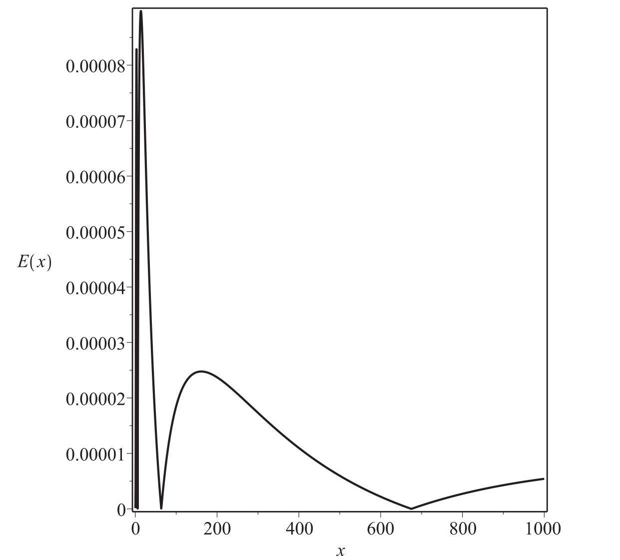

Fig. 2. Shows the absolute error ($E(x) = | y_4(x) - y_{Numeric}(x) |$) of the 4th-order PTV solution for Example 3.

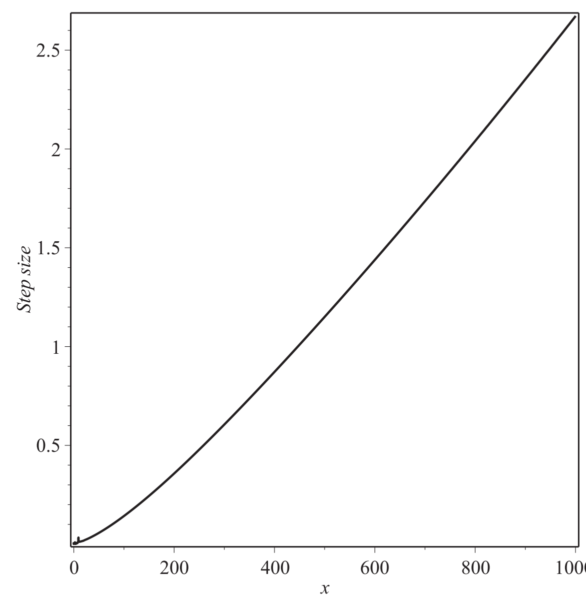

Fig. 3. Variable step size of the 4th-order APTV algorithm when $Atol = Rtol = 10^{-13}$ for Example 3.

under the assumptions of Table 1. By observing this graph we can perfectly comprehend how the developed method works.

Furthermore, the local discretization error of the APTV algorithm for the value $Atol = Rtol = 10^{-13}$ under the assumptions of Table 1, which is an estimation of the principal portion of the local error, have been given in Fig. 4.

In closing our analysis, we point out that three concreted modeling equations of second-order singular IVPs of the Lane–Emden type equation were investigated by using the algorithms proposed in this paper, and the obtained results have shown noteworthy performance.

4. Conclusion

Our application of the methods based on the VIM presented in this paper to three Lane–Emden type equations indicates that for linear Lane–Emden type equations, its exact solution can be obtained easily by only one iteration step. This is due to the fact that

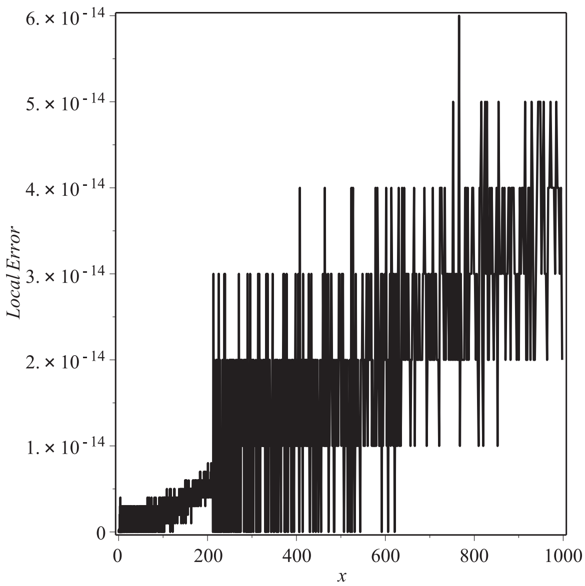

Fig. 4. Local error of the 4th-order APTV algorithm when $Rtol = Atol = 10^{-13}$ for Example 3.

the multiplier can be suitably identified. Moreover, the TP algorithm can solve a nonlinear Lane–Emden differential equation exactly if its solution is an algebraic polynomial up to some degree. For nonlinear Lane–Emden type equations, the method can be useful in general. It is well-known that the achievement of methods to solve the nonlinear IVPs of ODEs depends on the use of adaptive step size mechanisms controlling the truncation error. For this reason, an adaptive version of the VIM was proposed. The numerical results demonstrate that the VIM is a useful analytic tool for solving the Lane–Emden type equations.

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