Journal of Mathematical Inequalities
Volume 16, Number 1 (2022), 307–326 doi:10.7153/jmi-2022-16-22

FURTHER NUMERICAL RADIUS INEQUALITIES

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(*Communicated by M. Krnić*)

Abstract. In this work, some numerical radius inequalities for Hilbert space operators are introduced. Namely, we applying the Hermite-Hadamard inequality and some other recent results by using the concept of operator convexity and superquadracity.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$ with the identity operator $1_{\mathscr{H}}$ in $\mathscr{B}(\mathscr{H})$. An operator $A\in\mathscr{B}(\mathscr{H})$ is called positive if $\langle Ax,x\rangle\geqslant 0$ for all $x\in\mathscr{H}$, and we then write $A\geqslant 0$. The absolute value of A is denoted by $|A|$, that is $|A|=(A^*A)^{\frac{1}{2}}$. The Gelfand map $f(t)\mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a self-adjoint operator A and the C^*-algebra generated by A and I. If $f,g\in C(\mathrm{sp}(A))$, then $f(t)\geqslant g(t)$ $(t\in\mathrm{sp}(A))$ implies that $f(A)\geqslant g(A)$. A linear map Φ on $\mathscr{B}(\mathscr{H})$ is positive if $\Phi(A)\geqslant 0$ whenever $A\geqslant 0$. It is said to be normalized if $\Phi(1_{\mathscr{H}})=1_{\mathscr{H}}$.

For a bounded linear operator A on a Hilbert space $\mathscr{H}$, the numerical range $W(A)$ is the image of the unit sphere of $\mathscr{H}$ under the quadratic form $x\to\langle Ax,x\rangle$ associated with the operator. More precisely, $W(A)=\{\langle Ax,x\rangle : x\in\mathscr{H},\|x\|=1\}$. Also, the numerical radius is defined to be

$$w(A)=\sup_{\lambda\in W(A)}|\lambda|=\sup_{\|x\|=1}|\langle Ax,x\rangle|.$$

We recall that, the usual operator norm of an operator A is defined by

$$\|A\|=\sup\{\|Ax\| : x\in\mathscr{H},\|x\|=1\},$$

and also

$$\ell(A):=\inf\{\|Ax\| : x\in\mathscr{H},\|x\|=1\}=\inf\{|\langle Ax,y\rangle| : x,y\in\mathscr{H},\|x\|=\|y\|=1\}.$$

Mathematics subject classification (2020): Primary 47A63; Secondary 15A60, 15A64.
Keywords and phrases: Numerical radius, convex function, Hermite-Hadamard inequality, superquadratic function.

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The celebrated inequality

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\| \tag{1}$$

for any $A \in \mathscr{B}(\mathscr{H})$, was generalized by Kittaneh [13] where he refined the right-hand side of (1), where he proved

$$w(A) \leqslant \frac{1}{2}\left(\|A\| + \|A^2\|^{1/2}\right) \tag{2}$$

for any $A \in \mathscr{B}(\mathscr{H})$. After that, in 2005, the same author in [12] proved

$$\frac{1}{4}\|\,|A|^2 + |A^*|^2\| \leqslant w^2(A) \leqslant \frac{1}{2}\|\,|A|^2 + |A^*|^2\| \qquad (A \in \mathscr{B}(\mathscr{H})). \tag{3}$$

These inequalities are sharp. These inequalities were also reformulated and generalized in [8] but in terms of the Cartesian decomposition. A generalization of (2) was given in [26] as follows:

$$w(A) \leqslant \frac{1}{2}\left(\|A\| + w\left(\widetilde{A}\right)\right) \leqslant \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{1/2}\right) \tag{4}$$

for $\widetilde{A} = |A|^{1/2}U|A|^{1/2}$ where $A = U|A|$ is the polar decomposition of A and U is the appropriate partial isometry satisfying $\ker(U) = \ker(A)$. Moreover in [25], the author extended inequality (1) by using an extension of the Aluthge transform of operators. In 2008, Dragomir [6] used the Buzano inequality to improve (4), where he showed

$$w^2(A) \leqslant \frac{1}{2}\left(\|A\| + w\left(A^2\right)\right).$$

This result was also generalized in [22] and recently in [3]. We refer the reader to the recent articles [1, 9, 10, 17, 18, 19, 20, 21, 23, 24, 27, 28, 29] for different generalizations, refinements and applications of numerical radius inequalities.

A function $f : J \to \mathscr{R}$ on the interval J is convex if

$$f(\omega s + (1-\omega)t) \leqslant \omega f(s) + (1-\omega) f(t)$$

for all $s,t \in J$ and $\omega \in [0,1]$. The Hermite-Hadamard inequality for a convex function f on an interval J asserts that

$$f\left(\frac{s+t}{2}\right) \leqslant \int_0^1 f(\omega s + (1-\omega)t)\, d\omega \leqslant \frac{f(s)+f(t)}{2},$$

in which $s,t \in J$. There are several refinements and generalizations of the Hermite-Hadamard inequality for convex functions; see [7] and references therein. We say that a function $f : J \to \mathscr{R}$ is an operator convex function on the interval J, if

$$f(\omega A + (1-\omega)B) \leqslant \omega f(A) + (1-\omega) f(B),$$

where $A,B \in \mathscr{B}(\mathscr{H})$ are selfadjoint operators with spectra in J and $\omega \in [0,1]$.

A real valued function $f:[0,\infty) \to \mathscr{R}$ is called superquadratic if for all $s \geqslant 0$ there exists a constant $C_s \in \mathscr{R}$ such that

$$f(t) \geqslant f(s) + C_s(t-s) + f(|t-s|)$$

for all $t \geqslant 0$. We say that f is subquadratic if $-f$ is superquadratic. Thus, for a superquadratic function we require that f lie above its tangent line plus a translation of f itself [2]. Prima facie, superquadratic function looks to be stronger than convex function itself but if f takes negative values then it may be considered as a weaker function. Actually, if f is superquadratic and non-negative, then f is convex and increasing [2].

Moreover, we have the following lemma for superquadratic functions.

LEMMA 1. [2] *Let f be a superquadratic function. Then*

(i) *If f is differentiable and $f(0) = f'(0) = 0$, then $C_t = f'(t)$ for all $t \geqslant 0$.*

(ii) *If $f(t) \geqslant 0$ for all $t \geqslant 0$, then f is convex and $f(0) = f'(0) = 0$.*

Abramovich and et al. in [2] showed that a real valued continuous function f on $[0,\infty)$ is superquadratic if and only if

$$f(\omega s + (1-\omega)t) \leqslant \omega\left[f(s) - f((1-\omega)|s-t|)\right] + (1-\omega)\left[f(t) - f(\omega|s-t|)\right] \quad (5)$$

holds for all $\omega \in [0,1]$ and for all $s,t \in [0,\infty)$.

Banić and et al. [4] proved the following version of the Hermite-Hadamard inequality for superquadratic functions

$$\begin{aligned} & f\left(\frac{s+t}{2}\right) + \frac{1}{t-s}\int_s^t f\left(\left|\omega - \frac{s+t}{2}\right|\right) d\omega \\ & \leqslant \int_0^1 f(\omega s + (1-\omega)t)\, d\omega \qquad (6) \\ & \leqslant \frac{f(s)+f(t)}{2} - \frac{1}{(t-s)^2}\int_s^t \left[(t-s)f(\omega - s) + (\omega - s)f(t-\omega)\right] d\omega, \end{aligned}$$

in which $f:[0,\infty) \to \mathscr{R}$ is an integrable superquadratic function and $0 \leqslant s \leqslant t$. We note that, in the left hand side of (6) we have

$$\begin{aligned} \frac{1}{b-a}\int_a^b f\left(\left|t - \frac{a+b}{2}\right|\right) dt &= \int_0^1 f\left(\left|ua + (1-u)b - \frac{a+b}{2}\right|\right) du \\ &= \int_0^1 f\left(\left|\frac{1}{2} - u\right| |a-b|\right) du. \end{aligned}$$

For recent results concerning supequadricity the reader may refer to the comprehensive book [16].

Recently, Kian [14] and jointly in [15], proved the following operator version of Jensen's inequality for superquadratic functions.

THEOREM 1. *Let $A \in \mathscr{B}(\mathscr{H})$ be a positive operator and $x \in \mathscr{H}$ with $\|x\| = 1$. If $f : [0,\infty) \to \mathscr{R}$ is superquadratic and $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ is a normalized positive linear map, then*

$$\langle f(A)x,x\rangle \geqslant f(\langle Ax,x\rangle) + \langle f(|A - \langle Ax,x\rangle 1_{\mathscr{H}}|)x,x\rangle . \tag{7}$$

Moreover, we have

$$\langle \Phi(f(A))x,x\rangle \geqslant f(\langle \Phi(A)x,x\rangle) + \langle \Phi(f(|A - \langle \Phi(A)x,x\rangle 1_{\mathscr{H}}|))x,x\rangle . \tag{8}$$

In this work, some numerical radius inequalities for Hilbert space operators are proven. Namely, some refinements of Kittaneh's inequality (3), the result of Dragomir [7] regarding the Hermite-Hadamard inequality, and the result of [23] are proved by using the concept of operator convexity and superquadracity.

2. Numerical radius inequalities via operator convex functions

In the first section, we prove some numerical radius inequalities via operator convex functions. For this propose we need the following lemmas.

LEMMA 2. [11, Theorem 1] *Let $T \in \mathscr{B}(\mathscr{H})$ and $x,y \in \mathscr{H}$ be any vectors. If g, h are nonnegative continuous functions on $[0,\infty)$ which are satisfying the relation $g(t)h(t) = t$ $(t \in [0,\infty))$, then*

$$|\langle Tx,y\rangle|^2 \leqslant \langle g(|T|)x, g(|T|)x\rangle \, \langle h(|T^*|)y, h(|T^*|)y\rangle .$$

In particular, $|\langle Tx,y\rangle|^2 \leqslant \langle |T|^{\nu}x, |T|^{\nu}x\rangle \left\langle |T^*|^{(1-\nu)}y, |T^*|^{(1-\nu)}y\right\rangle$ $(0 \leqslant \nu \leqslant 1)$.

LEMMA 3. [7, Theorem 4] *Let $f : J \to \mathscr{R}$ be an operator convex function on the interval J. Then*

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leqslant (1-\omega)f\left[\frac{(1-\omega)A+(1+\omega)B}{2}\right] + \omega f\left[\frac{(2-\omega)A+\omega B}{2}\right] \\ &\leqslant \int_0^1 f((1-t)A+tB)dt, \\ &\leqslant \frac{f(A)+f(B)}{2}, \end{aligned}$$

where $A,B \in \mathscr{B}(\mathscr{H})$ are selfadjoint operators with spectra in J and $\omega \in [0,1]$.

The next result is a further refinement and generalization of [23, Theorem 2.1].

THEOREM 2. *Let $A \in \mathscr{B}(\mathscr{H})$. If $f : [0,+\infty) \to \mathscr{R}$ is an increasing operator convex function and $\omega \in [0,1]$, then we have*

$$\begin{aligned} f(w(A)) &\leqslant \Bigg\| (1-\omega) f\left(\frac{(1-\omega)g^2(|A|)+(1+\omega)h^2(|A^*|)}{2}\right) \\ &\qquad + \omega f\left(\frac{(2-\omega)g^2(|A|)+\omega h^2(|A^*|)}{2}\right)\Bigg\| \\ &\leqslant \left\| \int_0^1 f\big(tg^2(|A|)+(1-t)h^2(|A^*|)\big)dt \right\| \\ &\leqslant \left\| \frac{f\left(g^2(|A|)\right)+f\left(h^2(|A^*|)\right)}{2} \right\|, \end{aligned}$$

where g and h are non-negative continuous functions on $[0,\infty)$ which are satisfying the relation $g(t)h(t) = t\ (t \in [0,\infty))$.

Proof. Let $x \in \mathscr{H}$ be a unit vector. Since f is increasing on $[0,\infty)$, then

$$\begin{aligned} f(|\langle Ax,x\rangle|) &\leqslant f\left(\sqrt{\langle g^2(|A|)x,x\rangle\,\langle h^2(|A^*|)x,x\rangle}\right) \quad \text{(by Lemma 2)} \\ &\leqslant f\left(\frac{\langle g^2(|A|)x,x\rangle+\langle h^2(|A^*|)x,x\rangle}{2}\right) \\ &\qquad\qquad \text{(by the arithmetic-geometric mean inequality)} \\ &= f\left(\left\langle \left(\frac{g^2(|A|)+h^2(|A^*|)}{2}\right)x,x\right\rangle\right) \\ &\leqslant (1-\omega)\left[\left\langle f\left(\frac{(1-\omega)g^2(|A|)+(1+\omega)h^2(|A^*|)}{2}\right)x,x\right\rangle\right] \\ &\qquad + \omega\left[\left\langle f\left(\frac{(2-\omega)g^2(|A|)+\omega h^2(|A^*|)}{2}\right)x,x\right\rangle\right] \\ &= \Bigg\langle \Bigg[(1-\omega) f\left(\frac{(1-\omega)g^2(|A|)+(1+\omega)h^2(|A^*|)}{2}\right) \\ &\qquad + \omega f\left(\frac{(2-\omega)g^2(|A|)+\omega h^2(|A^*|)}{2}\right)\Bigg]x,x\Bigg\rangle \\ &\leqslant \int_0^1 \left\langle f\big(tg^2(|A|)+(1-t)h^2(|A^*|)\big)x,x\right\rangle dt \\ &= \left\langle \left(\int_0^1 f\big(tg^2(|A|)+(1-t)h^2(|A^*|)\big)dt\right)x,x\right\rangle. \\ &\leqslant \left\langle \left(\frac{f\left(g^2(|A|)\right)+f\left(h^2(|A^*|)\right)}{2}\right)x,x\right\rangle. \end{aligned}$$

Hence,

$$
\begin{aligned}
f(|\langle Ax,x\rangle|) &\leqslant \left\langle \left[(1-\omega)f\left(\frac{(1-\omega)g^2(|A|)+(1+\omega)h^2(|A^*|)}{2}\right)\right.\right.\\
&\qquad\left.\left. +\,\omega f\left(\frac{(2-\omega)g^2(|A|)+\omega h^2(|A^*|)}{2}\right)\right]x,x\right\rangle\\
&\leqslant \left\langle\left(\int_0^1 f\big(tg^2(|A|)+(1-t)h^2(|A^*|)\big)dt\right)x,x\right\rangle\\
&\leqslant \left\langle\left(\frac{f\left(g^2(|A|)\right)+f\left(h^2(|A^*|)\right)}{2}\right)x,x\right\rangle.
\end{aligned}
$$

Taking supremum over $x \in \mathscr{H}$ with $\|x\| = 1$, we have

$$
\begin{aligned}
f(w(A)) &\leqslant \left\|(1-\omega)f\left(\frac{(1-\omega)g^2(|A|)+(1+\omega)h^2(|A^*|)}{2}\right)\right.\\
&\qquad\left. +\,\omega f\left(\frac{(2-\omega)g^2(|A|)+\omega h^2(|A^*|)}{2}\right)\right\|\\
&\leqslant \left\|\int_0^1 f\big(tg^2(|A|)+(1-t)h^2(|A^*|)\big)dt\right\|\\
&\leqslant \left\|\frac{f\left(g^2(|A|)\right)+f\left(h^2(|A^*|)\right)}{2}\right\|. \quad \square
\end{aligned}
$$

REMARK 1. If we put $g(t) = t^{\nu}$, $h(t) = t^{1-\nu}$ in Theorem 2, we have

$$
\begin{aligned}
f(w(A)) &\leqslant \left\|(1-\omega)f\left(\frac{(1-\omega)|A|^{2\nu}+(1+\omega)|A^*|^{2(1-\nu)}}{2}\right)\right.\\
&\qquad\left. +\,\omega f\left(\frac{(2-\omega)|A|^{2\nu}+\omega|A^*|^{2(1-\nu)}}{2}\right)\right\|\\
&\leqslant \left\|\int_0^1 f\left(t|A|^{2\nu}+(1-t)|A^*|^{2(1-\nu)}\right)dt\right\|\\
&\leqslant \left\|\frac{f\left(|A|^{2\nu}\right)+f\left(|A^*|^{2(1-\nu)}\right)}{2}\right\|.
\end{aligned}
$$

REMARK 2. For $g(t) = t^{\frac{1}{2}}$, $h(t) = t^{\frac{1}{2}}$, we have

$$
\begin{aligned}
f(w(A)) &\leqslant \left\|(1-\omega)f\left(\frac{(1-\omega)|A|+(1+\omega)|A^*|}{2}\right)+\omega f\left(\frac{(2-\omega)|A|+\omega|A^*|}{2}\right)\right\|\\
&\leqslant \left\|\int_0^1 f\left(t|A|+(1-t)|A^*|\right)dt\right\|\\
&\leqslant \left\|\frac{f(|A|)+f(|A^*|)}{2}\right\|. \qquad (9)
\end{aligned}
$$

In particular for $\omega = \frac{1}{2}$;

$$f(w(A)) \leqslant \frac{1}{2}\left\| f\left(\frac{|A|+3|A^*|}{4}\right) + f\left(\frac{3|A|+|A^*|}{4}\right)\right\| \leqslant \left\| \int_0^1 f\left(t|A|+(1-t)|A^*|\right)dt \right\| \leqslant \left\| \frac{f(|A|)+f(|A^*|)}{2} \right\|. \tag{10}$$

It should mention here that the inequality (9) is refinement of [23, Theorem 2.1] and inequality (10) is already in [24]. Based on these results one can observe that Theorem 2 is more general than [23, Theorem 2.1].

The function $f(t) = t^r$ $(1 \leqslant r \leqslant 2)$ is an increasing operator convex function. The following result is a refinement and generalization of (3).

COROLLARY 1. *Let $A \in \mathscr{B}(\mathscr{H})$. Then for any $\omega \in [0,1]$ and $1 \leqslant r \leqslant 2$, we have*

$$\begin{aligned} w^r(A) &\leqslant \left\| (1-\omega)\left(\frac{(1-\omega)g^2(|A|)+(1+\omega)h^2(|A^*|)}{2}\right)^r \right. \\ &\qquad \left. + \omega\left(\frac{(2-\omega)g^2(|A|)+\omega h^2(|A^*|)}{2}\right)^r \right\| \\ &\leqslant \left\| \int_0^1 \left(tg^2(|A|)+(1-t)h^2(|A^*|)\right)^r dt \right\| \\ &\leqslant \left\| \frac{\left(g^2(|A|)\right)^r + \left(h^2(|A^*|)\right)^r}{2} \right\|. \end{aligned}$$

In particular

$$\begin{aligned} &w^r(A) \\ &\leqslant \left\| (1-\omega)\left(\frac{(1-\omega)|A|^{2\nu}+(1+\omega)|A^*|^{2(1-\nu)}}{2}\right)^r \right. \\ &\qquad \left. + \omega\left(\frac{(2-\omega)|A|^{2\nu}+\omega|A^*|^{2(1-\nu)}}{2}\right)^r \right\| \\ &\leqslant \left\| \int_0^1 \left(t|A|^{2\nu}+(1-t)|A^*|^{2(1-\nu)}\right)^r dt \right\| \\ &\leqslant \left\| \frac{\left(|A|^{2\nu}\right)^r + \left(|A^*|^{2(1-\nu)}\right)^r}{2} \right\|, \end{aligned}$$

where $\nu \in [0,1]$.

REMARK 3. By putting $f(t) = t^2$ in (10) we have [24, Corollary 2.3].

In the next theorem, we show a refinement and generalization of [23, Proposition 2.2].

THEOREM 3. *Let* $A,B \in \mathscr{B}(\mathscr{H})$. *If* $f:[0,+\infty) \to \mathscr{R}$ *is an increasing operator convex function and* $\omega \in [0,1]$, *then*

$$\begin{aligned}
&f(w(B^*A)) \\
&\leqslant \left\| (1-\omega) f\left(\frac{(1-\omega)|A|^2+(1+\omega)|B|^2}{2}\right) + \omega f\left(\frac{(2-\omega)|A|^2+\omega|B|^2}{2}\right) \right\| \\
&\leqslant \left\| \int_0^1 f\big(t|A|^2+(1-t)|B|^2\big)\,dt \right\| \\
&\leqslant \left\| \frac{f\left(|A|^2\right)+f\left(|B|^2\right)}{2} \right\|.
\end{aligned}$$

In particular, for any $1 \leqslant r \leqslant 2$ *we have*

$$\begin{aligned}
&w^r(B^*A) \\
&\leqslant \left\| (1-\omega)\left(\frac{(1-\omega)|A|^2+(1+\omega)|B|^2}{2}\right)^r + \omega\left(\frac{(2-\omega)|A|^2+\omega|B|^2}{2}\right)^r \right\| \\
&\leqslant \left\| \int_0^1 \big(t|A|^2+(1-t)|B|^2\big)^r dt \right\| \\
&\leqslant \left\| \frac{|A|^{2r}+|B|^{2r}}{2} \right\|.
\end{aligned}$$

Proof. Let $x \in \mathscr{H}$ be a unit vector. Since f is increasing on $[0,\infty)$ we have

$$\begin{aligned}
f\left(|\langle B^*Ax,x\rangle|\right) &= f\left(|\langle Ax,Bx\rangle|\right) \qquad (11)\\
&\leqslant f\left(\sqrt{\langle |A|x,|A|x\rangle \langle |B|x,|B|x\rangle}\right) \\
&= f\left(\sqrt{\left\langle |A|^2x,x\right\rangle \left\langle |B|^2x,x\right\rangle}\right) \\
&\leqslant f\left(\frac{\left\langle |A|^2x,x\right\rangle+\left\langle |B|^2x,x\right\rangle}{2}\right) \\
&\qquad \text{(by the arithmetic-geometric mean inequality)} \\
&= f\left(\left\langle \left(\frac{|A|^2+|B|^2}{2}\right)x,x\right\rangle\right).
\end{aligned}$$

Applying inequalities (11) and a similar method in the proof of Theorem 2, we can obtain the desired results. □

In the following result, we prove some inequalities involving the operator norm.

PROPOSITION 1. *Let $A,B \in \mathscr{B}(\mathscr{H})$. If $f : I \to \mathscr{R}$ be an increasing operator convex function on the interval I and for any $\omega \in [0,1]$, we have*

$$f\left(\left\|\frac{A+B}{2}\right\|\right) \leqslant \left\|(1-\omega)f\left(\frac{(1-\omega)|A|+(1+\omega)|B|}{2}\right)+\omega f\left(\frac{(2-\omega)|A|+\omega|B|}{2}\right)\right\|$$
$$\leqslant \left\|\int_0^1 f\big(t|A|+(1-t)|B|\big)dt\right\|$$
$$\leqslant \frac{1}{2}\|f(|A|)+f(|B|)\|.$$

In particular for $\omega = \frac{1}{2}$, we have

$$f\left(\left\|\frac{A+B}{2}\right\|\right) \leqslant \left\|f\left(\frac{|A|+3|B|}{4}\right)+f\left(\frac{3|A|+|B|}{4}\right)\right\|$$
$$\leqslant \left\|\int_0^1 f\big(t|A|+(1-t)|B|\big)dt\right\|$$
$$\leqslant \frac{1}{2}\|f(|A|)+f(|B|)\|.$$

Proof. Let $x \in \mathscr{H}$ be a unit vector, we have

$$f\left(\left|\left\langle\left(\frac{A+B}{2}\right)x,x\right\rangle\right|\right) = f\left(\left|\frac{\langle Ax,x\rangle+\langle Bx,x\rangle}{2}\right|\right)$$
$$\leqslant f\left(\frac{|\langle Ax,x\rangle|+|\langle Bx,x\rangle|}{2}\right)$$
$$\leqslant f\left(\frac{\langle |A|x,x\rangle+\langle |B|x,x\rangle}{2}\right)$$
$$= f\left(\left\langle\frac{|A|+|B|}{2}x,x\right\rangle\right).$$

Using a similar method in the proof of Theorem 2, we get our results. □

REMARK 4. In [5] it has shown that for an increasing convex function $f : [0,\infty) \to [0,\infty)$ we have

$$f\left(\frac{|A|+|B|}{2}\right) \leqslant U\frac{f(|A|)+f(|B|)}{2}U^* \quad \text{for some unitary operator } U.$$

This inequality implies that

$$\left\|f\left(\frac{|A|+|B|}{2}\right)\right\| \leqslant \frac{1}{2}\|f(|A|)+f(|B|)\|. \tag{12}$$

If $f : [0,\infty) \to [0,\infty)$ an increasing operator convex function, then by using the functional calculus for positive operator $\frac{|A|+|B|}{2}$ we have

$$f\left(\left\|\frac{|A|+|B|}{2}\right\|\right) = \left\|f\left(\frac{|A|+|B|}{2}\right)\right\|.$$

Now, if we replace A by $|A|$ and B by $|B|$ in Proposition 1, then we get a refinement of (12) as follows

$$\begin{aligned}\left\|f\left(\frac{|A|+|B|}{2}\right)\right\| &\leqslant \left\|(1-\omega)f\left(\frac{(1-\omega)|A|+(1+\omega)|B|}{2}\right)+\omega f\left(\frac{(2-\omega)|A|+\omega|B|}{2}\right)\right\| \\ &\leqslant \left\|\int_0^1 f\big(t|A|+(1-t)|B|\big)dt\right\| \\ &\leqslant \frac{1}{2}\|f(|A|)+f(|B|)\|. \end{aligned} \tag{13}$$

For the last result of this section, we give the following refinement of [24, Theorem 2.2].

THEOREM 4. *Let $A \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $A = B + iC$. If $f : [0,\infty) \to [0,\infty)$ is an increasing operator convex function, then*

$$\begin{aligned}\left\|f\left(\frac{A^*A+AA^*}{4}\right)\right\| &\leqslant \left\|(1-\omega)f\left(\frac{(1-\omega)B^2+(1+\omega)C^2}{2}\right)+\omega f\left(\frac{(2-\omega)B^2+\omega C^2}{2}\right)\right\| \\ &\leqslant \left\|\int_0^1 f\big(tC^2+(1-t)B^2\big)dt\right\| \\ &\leqslant \frac{1}{2}\|f(B^2)+f(C^2)\| \\ &\leqslant f(w^2(A)),\end{aligned}$$

where $\omega \in [0,1]$.

Proof. If $A = B + iC$ is the Cartesian decomposition, then $B^2 + C^2 = \frac{A^*A+AA^*}{2}$. Now, if we replace $|A|$ by B^2 and $|B|$ by C^2 in inequalities (13), we have

$$\begin{aligned}\left\|f\left(\frac{A^*A+AA^*}{4}\right)\right\| &= \left\|f\left(\frac{B^2+C^2}{2}\right)\right\| \\ &\leqslant \left\|(1-\omega)f\left(\frac{(1-\omega)B^2+(1+\omega)C^2}{2}\right)+\omega f\left(\frac{(2-\omega)B^2+\omega C^2}{2}\right)\right\| \\ &\leqslant \left\|\int_0^1 f\big(tC^2+(1-t)B^2\big)dt\right\| \\ &\leqslant \frac{1}{2}\|f(B^2)+f(C^2)\|. \end{aligned} \tag{14}$$

Moreover,

$$\begin{aligned}\frac{1}{2}\|f(B^2)+f(C^2)\| &\leqslant \frac{1}{2}\big(\|f(B^2)\|+\|f(C^2)\|\big) \\ &= \frac{1}{2}\big(f(\|B^2\|)+f(\|C^2\|)\big) \quad (\text{since } \|f(|X|)\| = f(\|X\|)) \\ &\leqslant f(w^2(A)). \end{aligned} \tag{15}$$

Using (14) and (15) we get our desired result. □

Let $f(t)=t^r$, $1 \leqslant r \leqslant 2$ is an increasing operator convex function. The previous theorem implies the following extension and refinement of the first inequality in (3).

COROLLARY 2. *Let $A \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $A=B+iC$. Then for every $\omega \in [0,1]$ and $1 \leqslant r \leqslant 2$,*

$$\begin{aligned}
&\frac{1}{4^r}\left\|A^*A+AA^*\right\|^r\\
&\leqslant \frac{1}{2^r}\left\|(1-\omega)\Big((1-\omega)B^2+(1+\omega)C^2\Big)^r+\omega\Big((2-\omega)B^2+\omega C^2\Big)^r\right\|\\
&\leqslant \left\|\int_0^1 \left(tC^2+(1-t)B^2\right)^r dt\right\|\\
&\leqslant \frac{1}{2}\|B^{2r}+C^{2r}\|\\
&\leqslant w^{2r}(A).
\end{aligned}$$

In particular,

$$\begin{aligned}
\frac{1}{4}\left\|A^*A+AA^*\right\| &\leqslant \frac{1}{4\sqrt{2}}\left\|\Big(B^2+3C^2\Big)^2+\Big(3B^2+C^2\Big)^2\right\|^{\frac{1}{2}}\\
&\leqslant \left\|\int_0^1 \left(tC^2+(1-t)B^2\right)^2 dt\right\|^{\frac{1}{2}}\\
&\leqslant \frac{1}{\sqrt{2}}\|B^4+C^4\|^{\frac{1}{2}}\\
&\leqslant w^2(A).
\end{aligned}$$

3. Further numerical radius inequalities via superquadratic functions

In this section, we show some numerical radius inequalities involving superquadratic functions. Now, we demonstrate one of the results of this section.

THEOREM 5. *Let $A \in \mathscr{B}(\mathscr{H})$ and $f:[0,\infty) \to \mathscr{R}$ be superquadratic increasing on $[0,\infty)$. Then*

$$\begin{aligned}
&f(w(A))\\
&\leqslant \left\|\int_0^1 f\left(t\,|A|+(1-t)\,|A^*|\right)dt\right\| \qquad (16)\\
&-\inf_{x\in\mathscr{H},\|x\|=1}\left\langle \int_0^1 f\left(\left|\left[t\,|A|+(1-t)\,|A^*|\right]-\left\langle\left[t\,|A|+(1-t)\,|A^*|\right]x,x\right\rangle 1_{\mathscr{H}}\right|\right)dt\,x,x\right\rangle\\
&-\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1 f\left(\left|\left\langle(|A|-|A^*|)x,x\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt
\end{aligned}$$

$$\leqslant \frac{1}{2}\left\|f(|A|)+f(|A^*|)\right\| - \ell\left(\int_0^1 t f((1-t)\,||A|-|A^*||)\,dt\right)$$
$$-\ell\left(\int_0^1 (1-t) f(t\,||A|-|A^*||)\,dt\right) - \inf_{x\in\mathscr{H},\|x\|=1}\int_0^1 f\left(|\langle(|A|-|A^*|)x,x\rangle|\left|t-\frac{1}{2}\right|\right)dt$$
$$-\inf_{x\in\mathscr{H},\|x\|=1}\left\langle\int_0^1 f\left(|[t|A|+(1-t)|A^*|]-\langle[t|A|+(1-t)|A^*|]x,x\rangle 1_{\mathscr{H}}|\right)dt x,x\right\rangle,$$

where $t\in[0,1]$.

Proof. Let $x\in\mathscr{H}$ be a unit vector. Since f is increasing on $[0,\infty)$, then we have

$$\begin{aligned} f(|\langle Ax,x\rangle|) &\leqslant f\left(\sqrt{\langle|A|x,x\rangle\langle|A^*|x,x\rangle}\right)\\ &\leqslant f\left(\frac{\langle|A|x,x\rangle+\langle|A^*|x,x\rangle}{2}\right) \qquad (17)\\ &\qquad\text{(by the arithmetic-geometric mean inequality)}\\ &\leqslant \int_0^1 f(t\langle|A|x,x\rangle+(1-t)\langle|A^*|x,x\rangle)\,dt\\ &\qquad -\int_0^1 f\left(|\langle(|A|-|A^*|)x,x\rangle|\left|t-\frac{1}{2}\right|\right)dt. \qquad \text{(by (6))} \end{aligned}$$

Now, since f is superquadratic we have

$$\begin{aligned} &f(t\langle|A|x,x\rangle+(1-t)\langle|A^*|x,x\rangle)\\ &=f(\langle[t|A|+(1-t)|A^*|]x,x\rangle)\\ &\leqslant\langle f(t|A|+(1-t)|A^*|)x,x\rangle\\ &\qquad-\langle f(|[t|A|+(1-t)|A^*|]-\langle[t|A|+(1-t)|A^*|]x,x\rangle 1_{\mathscr{H}}|)x,x\rangle \qquad \text{(by (7))}\\ &\leqslant t\langle f(|A|)x,x\rangle+(1-t)\langle f(|A^*|)x,x\rangle\\ &\qquad-t\langle f((1-t)\,||A|-|A^*||)x,x\rangle-(1-t)\langle f(t\,||A|-|A^*||)x,x\rangle \qquad \text{(by (5))}\\ &\qquad-\langle f(|[t|A|+(1-t)|A^*|]-\langle[t|A|+(1-t)|A^*|]x,x\rangle 1_{\mathscr{H}}|)x,x\rangle. \end{aligned}$$

Integrating the previous inequalities over t on $[0,1]$, we get

$$\begin{aligned} &\int_0^1 f(t\langle|A|x,x\rangle+(1-t)\langle|A^*|x,x\rangle)\,dt\\ &=f\left(\left\langle\int_0^1[t|A|+(1-t)|A^*|]\,dt\,x,x\right\rangle\right)\\ &\leqslant\left\langle\int_0^1 f(t|A|+(1-t)|A^*|)\,dt\,x,x\right\rangle \qquad (18)\\ &\qquad-\left\langle\int_0^1 f(|[t|A|+(1-t)|A^*|]-\langle[t|A|+(1-t)|A^*|]x,x\rangle 1_{\mathscr{H}}|)\,dt\,x,x\right\rangle \end{aligned}$$

$$\leqslant \frac{1}{2}\left[\langle f(|A|)x,x\rangle + \langle f(|A^*|)x,x\rangle\right] - \int_0^1 t\,\langle f((1-t)\,||A|-|A^*||)x,x\rangle\,dt$$
$$- \int_0^1 (1-t)\,\langle f(t\,||A|-|A^*||)x,x\rangle\,dt$$
$$- \left\langle \int_0^1 f\left(\left|[t\,|A| + (1-t)\,|A^*|] - \langle [t\,|A| + (1-t)\,|A^*|]x,x\rangle\,1_{\mathscr{H}}\right|\right)dt\,x,x \right\rangle$$

Combining (17) and (18), we get

$$\sup_{x\in\mathscr{H},\|x\|=1} f\left(|\langle Ax,x\rangle|\right)$$
$$\leqslant \sup_{x\in\mathscr{H},\|x\|=1} \left\langle \int_0^1 f\left(t\,|A| + (1-t)\,|A^*|\right)dtx,x \right\rangle$$
$$- \inf_{x\in\mathscr{H},\|x\|=1} \left\langle \int_0^1 f\left(\left|[t\,|A| + (1-t)\,|A^*|] - \langle [t\,|A| + (1-t)\,|A^*|]x,x\rangle\,1_{\mathscr{H}}\right|\right)dtx,x \right\rangle$$
$$- \inf_{x\in\mathscr{H},\|x\|=1} \int_0^1 f\left(|\langle (|A|-|A^*|)x,x\rangle|\left|t-\frac{1}{2}\right|\right)dt$$
$$\leqslant \frac{1}{2}\left[\langle f(|A|)x,x\rangle + \langle f(|A^*|)x,x\rangle\right] - \inf_{x\in\mathscr{H},\|x\|=1} \int_0^1 t\,\langle f((1-t)\,||A|-|A^*||)x,x\rangle\,dt$$
$$- \inf_{x\in\mathscr{H},\|x\|=1} \int_0^1 (1-t)\,\langle f(t\,||A|-|A^*||)x,x\rangle\,dt$$
$$- \inf_{x\in\mathscr{H},\|x\|=1} \int_0^1 f\left(|\langle (|A|-|A^*|)x,x\rangle|\left|t-\frac{1}{2}\right|\right)dt$$
$$- \inf_{x\in\mathscr{H},\|x\|=1} \left\langle \int_0^1 f\left(\left|[t\,|A| + (1-t)\,|A^*|] - \langle [t\,|A| + (1-t)\,|A^*|]x,x\rangle\,1_{\mathscr{H}}\right|\right)dtx,x \right\rangle. \quad \square$$

The function $f(x) = x^r$ $(r \geqslant 2)$ is an increasing(non-negative) superquadratic function, so we have the next result.

COROLLARY 3. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$w^r(A)$$
$$\leqslant \left\| \int_0^1 \left(t\,|A| + (1-t)\,|A^*|\right)^r dt \right\|$$
$$- \inf_{x\in\mathscr{H},\|x\|=1} \left\langle \int_0^1 \left(\left|[t\,|A| + (1-t)\,|A^*|] - \langle [t\,|A| + (1-t)\,|A^*|]x,x\rangle\,1_{\mathscr{H}}\right|\right)^r dtx,x \right\rangle$$
$$- \inf_{x\in\mathscr{H},\|x\|=1} \int_0^1 \left(|\langle (|A|-|A^*|)x,x\rangle|\left|t-\frac{1}{2}\right|\right)^r dt$$

$$\leqslant \frac{1}{2}\left\|(|A|)^r+(|A^*|)^r\right\|-\ell\left(\int_0^1 t\left((1-t)\left||A|-|A^*|\right|\right)^r dt\right)$$
$$-\ell\left(\int_0^1 (1-t)\left(t\left||A|-|A^*|\right|\right)^r dt\right)-\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1\left(|\langle(|A|-|A^*|)x,x\rangle|\left|t-\frac{1}{2}\right|\right)^r dt$$
$$-\inf_{x\in\mathscr{H},\|x\|=1}\left\langle\int_0^1\left(|[t|A|+(1-t)|A^*|]-\langle[t|A|+(1-t)|A^*|]x,x\rangle 1_{\mathscr{H}}|\right)^r dtx,x\right\rangle$$

for all $r\geqslant 2$ *and* $t\in[0,1]$.

EXAMPLE 1. If we take $A=\begin{pmatrix}0&0\\2&0\end{pmatrix}$ and $r=2$ in Corollary 3, then a simple calculation yields that $w(A)=1$,

$$\left\|\int_0^1\left(t|A|+(1-t)|A^*|\right)^r dt\right\|=\frac{4}{3},$$

$$\inf_{x\in\mathscr{H},\|x\|=1}\left\langle\int_0^1\left(|[t|A|+(1-t)|A^*|]-\langle[t|A|+(1-t)|A^*|]x,x\rangle 1_{\mathscr{H}}|\right)^r dtx,x\right\rangle=\frac{-1}{\sqrt{2}}$$

and

$$\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1\left(|\langle(|A|-|A^*|)x,x\rangle|\left|t-\frac{1}{2}\right|\right)^r dt=\frac{1}{3}.$$

Thus

$$\left\|\int_0^1\left(t|A|+(1-t)|A^*|\right)^r dt\right\|$$
$$-\inf_{x\in\mathscr{H},\|x\|=1}\left\langle\int_0^1\left(|[t|A|+(1-t)|A^*|]-\langle[t|A|+(1-t)|A^*|]x,x\rangle 1_{\mathscr{H}}|\right)^r dtx,x\right\rangle$$
$$-\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1\left(|\langle(|A|-|A^*|)x,x\rangle|\left|t-\frac{1}{2}\right|\right)^r dt$$
$$=\frac{4}{3}+\frac{1}{\sqrt{2}}-\frac{1}{3}=1.70719.$$

For the second inequality(with $r=2$) we have

$$\frac{1}{2}\left\|(|A|)^r+(|A^*|)^r\right\|=2,$$

$$\ell\left(\int_0^1 t\left((1-t)\left||A|-|A^*|\right|\right)^r dt\right)=\ell\left(\int_0^1(1-t)\left(t\left||A|-|A^*|\right|\right)^r dt\right)=\frac{1}{3}$$

and

$$\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1\left(|\langle(|A|-|A^*|)x,x\rangle|\left|t-\frac{1}{2}\right|\right)^r dt=\frac{-1}{\sqrt{2}}.$$

Thus

$$\begin{aligned}
&\frac{1}{2}\left\|\left(|A|\right)^r+\left(|A^*|\right)^r\right\| \\
&-\ell\left(\int_0^1 t\left((1-t)\left||A|-|A^*|\right|\right)^r dt\right)-\ell\left(\int_0^1 (1-t)\left(t\left||A|-|A^*|\right|\right)^r dt\right) \\
&-\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1\left(\left|\left\langle\left(|A|-|A^*|\right)x,x\right\rangle\right|\left|t-\frac{1}{2}\right|\right)^r dt \\
&=2-\frac{1}{2}-\frac{1}{2}+\frac{1}{\sqrt{2}}=2.04044.
\end{aligned}$$

Therefore,

$$\begin{aligned}
1=&w^r(A) \\
&\lneqq\left\|\int_0^1\left(t|A|+(1-t)|A^*|\right)^r dt\right\| \\
&-\inf_{x\in\mathscr{H},\|x\|=1}\left\langle\int_0^1\left(\left|\left[t|A|+(1-t)|A^*|\right]-\left\langle\left[t|A|+(1-t)|A^*|\right]x,x\right\rangle 1_{\mathscr{H}}\right|\right)^r dt x,x\right\rangle \\
&-\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1\left(\left|\left\langle\left(|A|-|A^*|\right)x,x\right\rangle\right|\left|t-\frac{1}{2}\right|\right)^r dt \\
&=1.70719 \\
&\lneqq\frac{1}{2}\left\|\left(|A|\right)^r+\left(|A^*|\right)^r\right\|-\ell\left(\int_0^1 t\left((1-t)\left||A|-|A^*|\right|\right)^r dt\right) \\
&-\ell\left(\int_0^1 (1-t)\left(t\left||A|-|A^*|\right|\right)^r dt\right)-\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1\left(\left|\left\langle\left(|A|-|A^*|\right)x,x\right\rangle\right|\left|t-\frac{1}{2}\right|\right)^r dt \\
&=2.04044.
\end{aligned}$$

The next result provides a refinement of [23, Proposition 2.2].

THEOREM 6. *Suppose that $A,B\in\mathscr{B}(\mathscr{H})$. If $f:[0,\infty)\to\mathscr{R}$ is superquadratic increasing on $[0,\infty)$, then we have*

$$\begin{aligned}
&f\left(w\left(B^*A\right)\right) \\
&\leqslant\left\|\int_0^1 f\left(t|A|^2+(1-t)|B|^2\right)dt\right\| \\
&-\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1 f\left(\left|\left[t|A|^2+(1-t)|B|^2\right]-\left\langle\left[t|A|^2+(1-t)|B|^2\right]x,x\right\rangle 1_{\mathscr{H}}\right|\right)dt \\
&-\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1 f\left(\left|\left\langle\left(|A|^2-|B|^2\right)x,x\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt \\
&\leqslant\frac{1}{2}\left\|f\left(|A|^2\right)+f\left(|B|^2\right)\right\|-\ell\left(\int_0^1 t f\left((1-t)\left||A|^2-|B|^2\right|\right)dt\right) \\
&-\ell\left(\int_0^1 (1-t) f\left(t\left||A|^2-|B|^2\right|\right)dt\right)
\end{aligned}$$

$$-\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1 f\left(\left|\left[t|A|^2+(1-t)|B|^2\right]-\left\langle\left[t|A|^2+(1-t)|B|^2\right]x,x\right\rangle 1_{\mathscr{H}}\right|\right)dt$$
$$-\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1 f\left(\left|\left\langle\left(|A|^2-|B|^2\right)x,x\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt.$$

Proof. Let $x\in\mathscr{H}$ be a unit vector. Since f is increasing on $[0,\infty)$ by the arithmetic-geometric inequality and the Schwarz inequality, we have

$$\begin{aligned}
f(|\langle B^*Ax,x\rangle|) &= f(|\langle Ax,Bx\rangle|)\\
&\leqslant f(\|Ax\|\,\|Bx\|)\\
&\leqslant f\left(\sqrt{\left\langle|A|^2x,x\right\rangle\left\langle|B|^2x,x\right\rangle}\right)\\
&\leqslant f\left(\frac{\left\langle|A|^2x,x\right\rangle+\left\langle|B|^2x,x\right\rangle}{2}\right)\\
&\qquad\text{(by the arithmetic-geometric mean inequality)}\\
&\leqslant \int_0^1 f\left(t\left\langle|A|^2x,x\right\rangle+(1-t)\left\langle|B|^2x,x\right\rangle\right)dt\\
&\qquad -\int_0^1 f\left(\left|\left\langle\left(|A|^2-|B|^2\right)x,x\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt. \qquad \text{(by (6))}
\end{aligned}$$

Using similar argument as used in Theorem 5 we get our desired result. □

The next result we show an upper bound for $f(w(B^*A))$.

PROPOSITION 2. *Let $A,B\in\mathscr{B}(\mathscr{H})$. If $f:[0,\infty)\to\mathscr{R}$ is superquadratic increasing on $[0,\infty)$, then*

$$\begin{aligned}
f(w(B^*A)) \leqslant &\sup_{x\in\mathscr{H},\ \|x\|=1}\left(\int_0^1 f\left(\left\|\left(t|A|^2+(1-t)|B|^2\right)^{\frac{1}{2}}x\right\|^2\right)dt\right)\\
&-\inf_{x\in\mathscr{H},\|x\|=1}\int_0^1 f\left(\left|\left\langle\left(|A|^2-|B|^2\right)x,x\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt.
\end{aligned}$$

Proof. Let $x\in\mathscr{H}$ be a unit vector. Since f is increasing on $[0,\infty)$ by the arithmetic-geometric inequality and the Schwarz inequality, we have

$$\begin{aligned}
f(|\langle B^*Ax,x\rangle|) &= f(|\langle Ax,Bx\rangle|)\\
&\leqslant f(\|Ax\|\,\|Bx\|)\\
&\leqslant f\left(\sqrt{\left\langle|A|^2x,x\right\rangle\left\langle|B|^2x,x\right\rangle}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant f\left(\frac{\left\langle |A|^2 x,x\right\rangle + \left\langle |B|^2 x,x\right\rangle}{2}\right) \\
&\qquad\qquad \text{(by the arithmetic-geometric mean inequality)} \\
&\leqslant \int_0^1 f\left(t\left\langle |A|^2 x,x\right\rangle + (1-t)\left\langle |B|^2 x,x\right\rangle\right) dt \\
&\quad - \int_0^1 f\left(\left|\left\langle\left(|A|^2-|B|^2\right)x,x\right\rangle\right|\left|t-\frac{1}{2}\right|\right) dt \qquad \text{(by (6))} \\
&= \int_0^1 f\left(\left\langle t\,|A|^2+(1-t)\,|B|^2 x,x\right\rangle\right) dt \\
&\quad - \int_0^1 f\left(\left|\left\langle\left(|A|^2-|B|^2\right)x,x\right\rangle\right|\left|t-\frac{1}{2}\right|\right) dt \\
&= \int_0^1 f\left(\left\langle\left(t\,|A|^2+(1-t)\,|B|^2\right)^{\frac{1}{2}}x,\left(t\,|A|^2+(1-t)\,|B|^2\right)^{\frac{1}{2}}x\right\rangle\right) dt \\
&\quad - \int_0^1 f\left(\left|\left\langle\left(|A|^2-|B|^2\right)x,x\right\rangle\right|\left|t-\frac{1}{2}\right|\right) dt \\
&= \int_0^1 f\left(\left\|\left(t\,|A|^2+(1-t)\,|B|^2\right)^{\frac{1}{2}}x\right\|^2\right) dt \\
&\quad - \int_0^1 f\left(\left|\left\langle\left(|A|^2-|B|^2\right)x,x\right\rangle\right|\left|t-\frac{1}{2}\right|\right) dt.
\end{aligned}
$$

Taking supremum over all $x \in \mathscr{H}$ with $\|x\| = 1$, we get the desired result. $\square$

THEOREM 7. *Let $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ be a normalized positive linear map and $A \in \mathscr{B}(\mathscr{H})$. If $f : [0,\infty) \to \mathscr{R}$ is superquadratic, then*

$$
\begin{aligned}
&f(w^2(\Phi(A))) \\
&\leqslant \sup_{x\in\mathscr{H},\|x\|=1}\left(\int_0^1 f\left(\left\|\Phi^{\frac{1}{2}}(t|A|^2+(1-t)|A^*|^2)x\right\|^2\right)dt\right) \\
&- \inf_{x\in\mathscr{H},\|x\|=1}\int_0^1 f\left(\left|\langle(\Phi(A^*A)-\Phi(AA^*))x,x\rangle\right|\left|t-\frac{1}{2}\right|\right)dt \\
&\leqslant \frac{1}{2}\left\|\Phi(f(|A|^2)+f(|A^*|^2))\right\| \\
&- \inf_{x\in\mathscr{H},\|x\|=1}\left\langle\Phi\int_0^1(f(t|A|^2+(1-t)|A^*|^2-\langle\Phi(t|A|^2+(1-t)|A^*|^2)x,x\rangle 1_{\mathscr{H}}|))dt x,x\right\rangle \\
&- \inf_{x\in\mathscr{H},\|x\|=1}\int_0^1 f\left(\left|\langle(\Phi(A^*A)-\Phi(A^*A))x,x\rangle\right|\left|t-\frac{1}{2}\right|\right)dt.
\end{aligned}
$$

Proof. Let $A = B + iC$ be the Cartesian decomposition of $A \in \mathscr{B}(\mathscr{H})$, where $B = \frac{A+A^*}{2}$ and $C = \frac{A-A^*}{2i}$. Using a similar argument in the proof of Theorem 2.2 [23] we have

$$f(|\langle\Phi(A)x,x\rangle|^2) \leqslant f\left(\frac{\langle\Phi(A^*A)x,x\rangle + \langle\Phi(AA^*)x,x\rangle}{2}\right).$$

Replacing s by $\langle\Phi(A^*A)x,x\rangle$ and t by $\langle\Phi(AA^*)x,x\rangle$, where $x \in \mathscr{H}$ is a unit vector in (6) we have

$$\begin{aligned}
&f\left(\frac{\langle\Phi(A^*A)x,x\rangle + \langle\Phi(AA^*)x,x\rangle}{2}\right)\\
&\leqslant \int_0^1 f(t\langle\Phi(A^*A)x,x\rangle + (1-t)\langle\Phi(A^*A)x,x\rangle)dt\\
&\qquad\qquad - \int_0^1 f\left(\left|\langle(\Phi(A^*A) - \Phi(A^*A))x,x\rangle\right|\left|t-\frac{1}{2}\right|\right)dt\\
&= \int_0^1 f(\langle(t\Phi(A^*A) + (1-t)\Phi(AA^*))x,x\rangle)dt\\
&\qquad\qquad - \int_0^1 f\left(\left|\langle(\Phi(A^*A) - \Phi(A^*A))x,x\rangle\right|\left|t-\frac{1}{2}\right|\right)dt.
\end{aligned}$$

Hence

$$\begin{aligned}
f(|\langle\Phi(A)x,x\rangle|^2) &\leqslant \int_0^1 f(\langle\Phi(t(A^*A) + (1-t)(AA^*))x,x\rangle)dt\\
&\qquad\qquad - \int_0^1 f\left(\left|\langle(\Phi(A^*A) - \Phi(A^*A))x,x\rangle\right|\left|t-\frac{1}{2}\right|\right)dt.
\end{aligned} \tag{19}$$

Since f is superquadratic we have,

$$\begin{aligned}
&f(\langle\Phi(tA^*A + (1-t)AA^*)x,x\rangle)\\
&\qquad\leqslant \langle\Phi(f(t|A|^2 + (1-t)|A^*|^2))x,x\rangle - \langle\Phi(f(|t|A|^2 + (1-t)|A^*|^2\\
&\qquad\qquad - \langle\Phi(t|A|^2 + (1-t)|A^*|^2)x,x\rangle 1_{\mathscr{H}}|))x,x\rangle \quad \text{(by (8))}.
\end{aligned}$$

Integrating the previous inequality over t on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f(\langle\Phi(t|A|^2 + (1-t)|A^*|^2)x,x\rangle)dt\\
&\leqslant \left\langle\Phi\left(\int_0^1 f(t|A|^2 + (1-t)|A^*|^2)dt\right)x,x\right\rangle\\
&- \left\langle\Phi\int_0^1 \left(f\left(\left|t|A|^2 + (1-t)|A^*|^2 - \langle\Phi(t|A|^2 + (1-t)|A^*|^2)x,x\rangle 1_{\mathscr{H}}\right|\right)\right)dtx,x\right\rangle\\
&\leqslant \frac{1}{2}[\langle\Phi(f(|A|^2) + f(|A^*|^2))x,x\rangle]\\
&- \left\langle\Phi\int_0^1 (f(t|A|^2 + (1-t)|A^*|^2 - \langle\Phi(t|A|^2 + (1-t)|A^*|^2)x,x\rangle 1_{\mathscr{H}}|))dtx,x\right\rangle.
\end{aligned}$$

So,

$$\begin{aligned}
&f\left(|\langle\Phi(A)x,x\rangle|^2\right)\\
&\leqslant \int_0^1 f\left(\langle\Phi(t|A|^2+(1-t)|A^*|^2)x,x\rangle\right)dt\\
&\qquad -\int_0^1 f\left(\left|\langle(\Phi(A^*A)-\Phi(AA^*))x,x\rangle\right|\left|t-\frac{1}{2}\right|\right)dt\\
&\leqslant \frac{1}{2}[\langle\Phi(f(|A|^2)+f(|A^*|^2))x,x\rangle]\\
&\qquad -\left\langle\Phi\int_0^1 (f(t|A|^2+(1-t)|A^*|^2-\langle\Phi(t|A|^2+(1-t)|A^*|^2)x,x\rangle 1_{\mathscr{H}}|))dtx,x\right\rangle\\
&\qquad\qquad -\int_0^1 f\left(\left|\langle(\Phi(A^*A)-\Phi(A^*A))x,x\rangle\right|\left|t-\frac{1}{2}\right|\right)dt.
\end{aligned}$$

Taking supremum over $x\in\mathscr{H}$, with $\|x\|=1$ we get our desired result. □

Acknowledgement. The third author would like to indicate special thanks and gratitude to Professor Miguel Martin Suarez and the University of Granada, Department of Mathematical analysis to pass his sabbatical.

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(Received December 6, 2020)

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