Chapter 1

Measures of dependence in bivariate distributions families

1.1 Measures of dependence

In this section we explore ways in which copulas can be used in the study of dependence or association between random variables.Jogdeo(1982): *Dependence relations between random variables is one of the most widely studied subjects in probability and statistics. The nature of the dependence can take a variety of forms and unless some specific assumptions are made about the dependence, no meaningful statistical model can be contemplated.*

1.1.1 Kendall's tau

The sample version of the measure of association known as Kendall's tau is defined in terms of concordance as follows (Kruskal 1958; Hollander and Wolfe 1973; Lehmann 1975): Let (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed random vectors each with joint distribution function H. Then the population version of Kendall's tau is defined as the probability of concordance minus the probability of discordance:

$$\tau = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0]$$

Theorem 1(Nelsen 2007).Let (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed random vectors each with joint distribution function H_1 and H_2 respectively, with common margins F (of X_1 and X_2) and G (of Y_1 and Y_2). Let C_1 and C_2 denoted the difference between the probabilities of concordance and discordance of (X_1, Y_1) and (X_2, Y_2), i.e. let

$$Q = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

Then

$$Q = Q(C_1, C_2) = 4 \int_0^1 \int_0^1 C_2(u, v) dC_1(u, v) - 1.$$

Corollary 1 Under the assumption of the Theorem

i- $Q(C_1, C_2) = Q(C_2, C_1)$.

ii-Q is non decreasing in each argument:

$$C_1 < {}'C_1,\ C_2 < {}'C_2 \Rightarrow Q(C_1, C_2) \leq Q({}'C_1, {}'C_2).$$

iii- Copulas can be replaced by survival copulas in Q

$$Q(C_1, C_2) = Q(\hat{C}_1, \hat{C}_2)$$

Theorem 2(Nelsen 2007).Let (X, Y) be Joint df F with copula C then

$$\tau = Q(C, C) = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1 = 4E(C(U, V)) - 1.$$

Example Let Π, M and W are multiple , And Frechet's bounds respectively Then

$$Q(M, M) = 1, Q(M, \Pi) = 1/3, Q(M, W) = 0 Q(W, \Pi) = -1/3, Q(W, W) = -1, Q(\Pi, \Pi) = 0$$

Theorem 3(Nelsen 2007).Let (X, Y) be Joint df F with Archimedean copula C_φ generated by φ, then

$$\tau = 1 + 4 \int_0^1 \frac{\varphi(t)}{{}'\varphi(t)} dt.$$

Theorem 4(Li et.al.2002). Let C_1 and C_2 be copulas. Then

$$\int\int_{I^2} C_(u,v)dC_2(u,v) = \frac{1}{2} - \int\int_{I^2} \frac{\partial}{\partial u}C_1(u,v)\frac{\partial}{\partial v}C_2(u,v)dudv.$$

Remark The form of τ_C given in Theorem 2 is often not amenable to computation, especially when C is singular or if C has both an absolutely continuous and a singular component. For many such copulas, the expression the following for which is a consequence of Theorem 4 is more tractable.

$$\tau_C = 1 - 4\int\int_{I^2} \frac{\partial}{\partial u}C(u,v)\frac{\partial}{\partial v}C(u,v)dudv.$$

Example Let $C_{\alpha,\beta}$ be a member of the Marshall-Olkin family of copulas for $0 < \alpha, \beta < 1$.

Then

$$\tau_{\alpha,\beta} = \frac{\alpha\beta}{\alpha - \alpha\beta + \beta}$$

In the next Theorem, we relate Likelihood ration dependence to Kendall τ, in the following theorem we see that Kendall's τ can be interpreted as a measure of average likelihood ration dependence. :

Theorem 5 Let X and Y be random variable with copula density function c, then

$$\tau = 2\int_0^1\int_0^1\int_0^s\int_0^t [c(u,v)c(t,s) - c(u,s)c(t,v)]dudvdtds$$

Corollary 2 Under the assumption of the Theorem 5,
PLRD(X,Y)$\Rightarrow \tau_C \geq 0$ and NLRD(X,Y) $\Rightarrow \tau \leq 0$.

1.1.2 Spearman's rho

As with Kendall's tau, the population version of the measure of association known as Spearman's rho is based on cocordance and discordance. To obtain the population version of this measure (Kruskal 1958; Lehmann 1966), we now let $(X_1, Y_1), (X_2, Y_2)$ and (X_3, Y_3) be three independent random vectors with a common joint df H (whose margins are given F and G) and copula C. Spearman's rho is defined as following:

$$\rho_{X,Y} = 3(P[(X_1 - X_2)(Y_1 - Y_3) > 0] - P[(X_1 - X_2)(Y_1 - Y_3) < 0]).$$

Theorem 1(Nelsen,2007) Let X and Y be continuous random variables whose copula is C. Then the population version of Spearman's rho for X and Y is given by

$$\rho_{X,Y} = \rho_C = 3Q(C, \Pi) = 12 \int\int_{I^2} uv dC(u,v) - 3 = 12 \int\int_{I^2} C(u,v) du dv - 3.$$

Remark 1 Since support of M is the diagonal $v = u$ in I^2, then

$$\int\int_{I^2} g(u,v) dM(u,v) = \int_0^1 g(u,u) du$$

where g is an integrable function whose domain is I^2. Similarly, since the support of W is the secondary diagonal $v = 1 - u$, hence

$$\int\int_{I^2} g(u,v) dW(u,v) = \int_0^1 g(u, 1-u) du.$$

So

i-

$$Q(W, \Pi) = -1/3, Q(W, W) = -1, Q(\Pi, \Pi) = 0,$$

and

$$Q(C, C) \in [-1, 1], Q(C, M) \in [0, 1], Q(C, W) \in [-1, 0], Q(C, \Pi) \in [-1/3, 1/3].$$

Corollary 1 Let X and Y be continuous random variables whose copula is C, then

$$\rho_S = 12 \int\int_{I^2} [C(u,v) - uv] du dv$$

Thus ρ_S is a measure of average distance between the distribution of X and Y and independence. In fact Spearman's rho can be interpreted as a measure of average quadrant dependence.

Definition 1 A numeric measure K of association between two continuous random variables X and Y whose copula is C is a measure of concordance if it satisfies the following properties.

1- K is defined for every pair X, Y of continuous random variables;

2- $-1 \leq K \leq 1,\ K_{X,X} = 1, K_{X,-X} = -1,\ and\ K_{X,Y} = K_{Y,X}$;

3- If X and Y are independent, then $K_{X,Y} = K_{\Pi} = 0$;

4-$K_{-X,Y} = K_{X,-Y} = -K_{X,Y}$;

5- If C_1 and C_2 are copulas such that $C_1 < C_2$, then $K_{C_1} \leq K_{C_2}$;

7- If $\{(X_n, Y_n)\}$ is a sequence of continuous random variables with copulas C_n, and if $\{C_n\}$ converges pointwise to C, then $\lim_{n\to\infty} K_{C_n} = K_C$.

As a consequence of Definition we have the following theorem.

Theorem 2(Nelsen, 2007).Let K be a measure of concordance for continuous random variables X and Y:

1-If Y is almost surely an increasing function of X, then $K_{X,Y} = K_M = 1$;

2-If Y is almost surely an decreasing function of X, then $K_{X,Y} = K_W = -1$;

3-If α and β are almost surely strictly monotone functions on $Ran(X)$ and $Ran(Y)$, respectively, then $K_{\alpha(X)\beta(Y)} = K_{X,Y}$.

The next Theorem show that both Kendall's tau and Spearman's rho are measures of concordance according to the Definition.

Theorem 2(Nelsen, 2007) If X and Y are continuous random variables whose copula is C, then the population versions of Kendall's tau and Spearman's rho satisfy the properties in Definition and Theorem 2 for a measure of concordance.

Corollary 2 If X and Y are continuous random variables whose copula is C, then,

$$\rho_S = 12 \int\int_{I^2} uvdC(u,v) - 3 = \frac{Cov(U,V)}{\sqrt{Var(U).Var(V)}}.$$

1.1.3 The relationship between ρ_s and τ

Although both Kendall's tau and Spearman's rho measure the probability of concordance between random variables with a given copula, the values of ρ and τ are often quit different. In this section, we will determine just how different ρ

and τ can be.The following Theorem due to Daniels (1950), for proof see Nelsen (2007).

Theorem 1 Let X and Y be continuous random variables,then

$$-1 \leq 3\tau - 2\rho_S \leq 1.$$

The next Theorem gives a second set of universal inequalities relating τ and ρ_S. It is due to Durbin and Sauart (1951); Proof in Nelsen (2007).

Theorem 2 Let X and Y be continuous random variables,then

$$\frac{1+\rho_S}{2} \geq (\frac{1+\tau}{2})^2$$

and

$$\frac{1-\rho_S}{2} \geq (\frac{1-\tau}{2})^2$$

The inequalities in the preceding two theorems combine to yield.

Corollary 1 under the assumptions of Theorems 1,2 we have,

$$\frac{3\tau-1}{2} \leq \rho_S \leq \frac{1+2\tau-\tau^2}{2}, \quad \tau \geq 0,$$

and

$$\frac{\tau^2+2\tau-1}{2} \leq \rho_S \leq \frac{1+3\tau}{2}, \quad \tau \leq 0.$$

Theorem 3 Let X and Y be continuous random variables with joint distribution function H, margins F and G, respectively, and copula C.

1- If X and Y are PQD, then

$$\tau \geq Q(C,C) \geq Q(C,\Pi) \geq Q(\Pi,\Pi),$$

and

$$\tau \geq 0, \ \rho_S \geq 0, \quad and \ \ 3\tau \geq \rho_S \geq 0.$$

2-If X and Y are NQD, then

$$\tau \leq Q(C,C) \leq Q(C,\Pi) \leq Q(\Pi,\Pi),$$

and

$$\tau \leq 0, \ \rho_S \leq 0, \quad and \ \ 3\tau \leq \rho_S \leq 0.$$

Theorem 4 Let X and Y be continuous random variables.
1-(Caperaa and Genest, 1993) If $LTD(Y|X)$ and $RTI(Y|X)$ (or $LTD(X|Y)$ and $RTI(X|Y)$), then

$$1 \geq \rho_S \geq \tau \geq 0.$$

2-(Fredricks and Nelsen,2007)If $LTI(Y|X)$ and $RTD(Y|X)$ (or $LTI(X|Y)$ and $RTD(X|Y)$), then

$$-1 \leq \rho_S \leq \tau \leq 0.$$

1.1.4 The Blomqvist medial coefficient

This coefficient, also known as quadrant test of Blomqvist, evaluates the dependence at the center of a distribution. This measure, often called the medial correlation coefficient, will be denoted β, and given by

$$\beta = \beta_{X,Y} = P[(X - \tilde{x})(Y - \tilde{y}) > 0] - P[(X - \tilde{x})(Y - \tilde{y})]$$

where $\tilde{x}$ and $\tilde{y}$ are medians of X and Y, respectively. But if X and Y are continuous with joint df H and margins F and G respectively, and Copula C, then $F(\tilde{x}) = G(\tilde{y}) = 1/2$ and we have $\beta = 4H(\tilde{x}, \tilde{y}) - 1$.But $H(\tilde{x}, \tilde{y}) = C(1/2, 1/2)$, and thus

$$\beta = 4C(\frac{1}{2}, \frac{1}{2}) - 1.$$

If X and Y are independent then in particular $C(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$, then

$$\beta = 4C(\frac{1}{2}, \frac{1}{2}) - 1 = 0$$

in FGM family $\beta = \frac{1}{4} + \frac{\theta}{16}$.

1.1.5 Gini's γ coefficient

Let X and Y be continuous r.v.'s with joint distribution function H and margins F and G and copula C. The Gini's γ coefficient, defined as the following:

$$\gamma_C = 2\int_0^1\int_0^1 (|u+v-1| - |u-v|)dC(u,v).$$

The following Theorem show that γ, like ρ_S and τ, is a measure of association based upon concordance.

Theorem 1(Nelsen,2007) Let X and Y be continuous random variables whose copula is C. Then the population version of Gini's measure of association for X and Y is given by,

$$\gamma_{X,Y} = \gamma_C = Q(C, M) + Q(C, W).$$

Corollary 1 Under the assumptions of Theorem 1, we have another form for Gini's γ is given by

$$\gamma_C = 4\{\int_0^1 C(u, 1-u)du - \int_0^1 [u - C(u,u)]du\}.$$

In FGM family we have $\gamma_c = \frac{4\theta}{15}$. Like Kendall's tau and Spearman's rho, both γ and β are also measures of concordance according Definition.The following Theorem show that this. **Theorem 2**(Nelsen, 2007) If X and Y are continuous random variables whose copula is C, then the population versions of Gini's γ and Blomqvist's β satisfy the properties in Definition and Theorem 2 for a measure of concordance.

1.1.6 Schweizer-Wolff's index of dependence

An index closely related to Spearman's ρ_s is the index σ_{XY} introduced by Scweizer and Wolff (1981). Instead of considering the difference $C(u, v) - uv$ Definition ρ_s they use its absolute value to define:

$$\sigma_C = 12\int_0^1\int_0^1 |C(u,v) - uv|dudv.$$

σ_{XY} is a measure of the volume between the surfaces $C(u, v)$ and uv. Since $\int_0^1 \int_0^1 |\min\{u, v\} - uv| dudv = \frac{1}{12}$, we have the two equivalences:

$$\sigma_{XY} = 0 \Leftrightarrow (X, Y) independent$$

$$\sigma_{XY} = 1 \Leftrightarrow X is a monotone function of Y.$$

1.1.7 Mutual information, relative entropy and derivatives measures

If X is a random variable, with density $f_1(x)$, then the entropy or the measure of uncertainty is defined by:

$$E_X = -\int f_1(x) \log(f_1(x) dx.$$

If (X, Y) is a pair of random variables with the density $f(x, y)$, and the marginal densities $f_1(x)$ and $f_2(y)$, then the entropy for this pair is:

$$E_{XY} = -\int \int f(x, y) \log(f(x, y)) dxdy.$$

This entropy is maximum, when X and Y are independent. This definition can be generalized with an n-vector $(X_1, X_2, ..., X_n)$ in place of (X, Y).The mutual information of relative entropy is then defined as the following

$$\delta_{X,Y} = \int \int f(x, y) \log(\frac{f(x, y)}{f_1(x) f_2(y)}) dxdy.$$

If the components of (X, Y) are independent, then $\delta_{X,Y}$ is zero and conversely when the dependence is maximal, $\delta_{X,Y}$ approaches infinity. To normalize this index, Joe(1989) defines:

$$\delta^* = \sqrt{(1 - \exp(-2\delta))}$$

The index δ^* is confined to the interval $[0, 1]$, and in the case when the pair (X, Y) is bivariate normal is equal to the absolute value of the linear correlation coefficient $|\rho|$. In FGM family $\delta =?$.

1.1.8 The quadratic Mutual information measure

The matual information between two random variables can be measured either by Renyi's divergence measure or by Kullback-Leibler divergence between their joint pdf and the factorized marginal pdfs. Unfortunately, non of them can be integrated with the Parzen window method to produce a simple from for mutual information estimation. Based on the Cauchy-Schwartz inequality Principe and Xu (—-) proposed a new mutual information measure between two random variables Y_1 and Y_2 called the Quadratic mutual information defined as:

$$C(Y_1, Y_2) = \log \frac{(\int\int f_{12}^2(y_1, y_2)dy_1dy_2)(\int\int f_1^2(y_1)f_2^2(y_2)dy_1dy_2)}{(\int\int f_{12}(y_1, y_2)f_1(y_1)f_2(y_2)dy_1dy_2)^2}$$

where $f_{12}(x, y)$ is the joint pdf and $f_i(x)$ are the marginal pdfs. All the quantities are non-negative and equal to zero if and only if Y_1 and Y_2 are statistically independent.

Let

$$f(u, v) = uv[1 + \theta(1 - u)(1 - v)], \quad u, v \in [0, 1], \quad -1 \leq \theta \leq 1.$$

then

$$C(U, V) = \log(1 + \frac{\theta^2}{9}) = 0 \Leftrightarrow \theta = 0 \Leftrightarrow (U, V) is independent.$$

1.1.9 Kochar and Gupta's dependence measure

Let k be a fixed integer and consider $d_k(x, y) = F^k(x, y) - F_1^k(x).F_2^k(y), \forall x, y \in R$ (according to notations of Kochar (1987)[3]).
Then it is obvious that for all $k \geq 1$, if $H_0 : F(x, y) = F_1(x).F_2(y)$ true then $d_k(x, y) = 0$, if $H_1 : F(x, y) < F_1(x).F_2(y)$ true then $d_k(x, y) < 0$, and if $H_2 : F(x, y) > F_1(x).F_2(y)$ true then $d_k(x, y) > 0$, that means significants of independence , NQD and PQD respectively. The following measure of deviation between H_0 and H_1 or (H_2) considered,

$$D_k = \int\int_{R^2} d_k(x, y)dF(x, y) = D_{1k} - D_{2k}$$

where,

$$\begin{aligned} P[\max_{1\le i\le k} X_i \le X_{k+1}, \max_{1\le i\le k} Y_i \le Y_{k+1}] &= \\ &= \int\int_{R^2} P[\max_{1\le i\le k} X_i \le X_{k+1}, \max_{1\le i\le k} Y_i \le Y_{k+1} | X_{k+1} = x, Y_{k+1} = \\ &= \int\int_{R^2} P[\max_{1\le i\le k} X_i \le x, \max_{1\le i\le k} Y_i \le y | X_{k+1} = x, Y_{k+1} = y] dF(\\ &= \int\int_{R^2} P[\bigcap_{i=1}^{k} \{(X_i \le x, Y_i \le y)\}] dF(x,y) \\ &= \int\int_{R^2} F^k(x,y) dF(x,y) = D_{1k} \end{aligned}$$

and

$$D_{2k} = \int\int_{R^2} F_1^k(x) F_2^k(y) dF(x,y) = \int\int_{R^2} \bar{F}(x,y) dF_1^k(y) dF_2^k(y), \quad k \ge 1.$$

It is obvious that the equality of the right hand of D_{2k} obtain via the following

$$\begin{aligned} \int\int_{R^2} F_1^k(x) F_2^k(y) dF(x,y) &= \int\int_{R^2} \{\int_{-\infty}^{y} \int_{-\infty}^{x} dF_1^k(t) dF_2^k(s)\} dF(x,y) \\ &= \int\int_{R^2} \{\int_{s}^{\infty} \int_{t}^{\infty} dF(x,y)\} dF_1^k(x) dF_2^k(y) (by \ \ Fubini's \ \ theorem) \\ &= \int\int_{R^2} \bar{F}(x,y) dF_1^k(y) dF_2^k(y). \end{aligned}$$

Note that in here $(X_1, Y_1), \cdots, (X_{k+1}, Y_{k+1})$ is a random sample of (X, Y) with common joint distribution function $F(x, y)$. Now it follows that under H_0 $D_{1k} = D_{2k} = \frac{1}{(k+1)^2}$. and under H_1 we get

$$\begin{aligned} D_{1k} &= \int\int_{R^2} F^k(x,y) dF(x,y) \\ &\le \int\int_{R^2} F_1^k(x) F_2^k(y) dF(x,y) \\ &= \int\int_{R^2} \bar{F}(x,y) dF_1^k(y) dF_2^k(y) \\ &< \int\int_{R^2} \bar{F}_1(x).\bar{F}_2 y dF_1^k(y) dF_2^k(y) = \frac{1}{(k+1)^2}. \end{aligned}$$

Similarly under H_2, we get $D_{1k} > \frac{1}{(k+1)^2}$. So, if H_1 true then $D_{1k} < D_{2k} < \frac{1}{(k+1)^2}$, and if H_2 true then $D_{1k} > D_{2k} > \frac{1}{(k+1)^2}$. for all $k \ge 1$, respectively.

Lemma Let (X, Y) be a random vector with FGM distribution function.Then for all $k \geq 1$

i)

$$\begin{aligned} D_{1k} &= \sum_{i=0}^{k} \{\theta^i (Bet(k+1, i+1))^2 + \theta^{i+1} (Bet(k+1, i+1))^2\} \\ &- 4 \sum_{i=0}^{k} \{\theta^{i+1} Bet(k+2, i+1) Bet(k+1, i+1) + 4\theta^{i+1} (Bet(k+2, i+1))^2\} \end{aligned}$$

ii)

$$D_{2k} = \frac{1}{(k+1)^2} + \frac{k^2 \theta}{(k+1)^2 (k+2)^2}.$$

Corollary Under the assumptions of above Lemma, for all $k \geq 1$ we get

$$\theta = 0 \Leftrightarrow D_k = \frac{1}{(k+1)^2} - \frac{1}{(k+1)^2} = 0 \Leftrightarrow (X, Y) \ \ is \ \ independent.$$

1.1.10 Tail dependence coefficients

Let X and Y be the r.v.'s with distribution functions F and G respectively. Coles, et.al. (2000) have proposed two indices to measure tail dependence and a diagnosis of such a dependence. The lower tail dependence coefficient is defined as:

$$\lambda_L = \lim_{t \to 0^+} P[Y \leq G^{-1}(t) | X \leq F^{-1}(t)]$$

and the upper tail dependence coefficient is defined as:

$$\lambda_U = \lim_{t \to 1^-} P[Y > G^{-1}(t) | X > F^{-1}(t)]$$

If the above limits existed. In FGM family It can be checked that $\lambda_L = \lambda_U = 0$. means that the FGM families are independence in tails.

Theorem Let $C(u, v)$ be the copula of X and Y. If the limit in above exist then

$$\lambda_L = \lim_{t \to 0^+} \frac{C(u, v)}{u}$$

and

$$\lambda_L = \lim_{t \to 1^-} \frac{\bar{C}(u, v)}{1 - u} = 2 - \lim_{t \to 1^-} \frac{1 - C(u, v)}{1 - u}$$

Corollary Let C_φ be an archimedean copula with generator φ. Then

$$\lambda_L = \lim_{x\to\infty} \frac{\varphi^{-1}(2x)}{\varphi^{-1}(x)},$$

and

$$\lambda_U = 2 - \lim_{x\to 0^+} \frac{1-\varphi^{-1}(2x)}{1-\varphi^{-1}(x)}.$$

1.1.11 Extremal tail dependence coefficients

(Frahm (2006)) Let ($X_1, X_2, \ldots, X_n$) be a random vector with joint distribution function $F(x_1, x_2, \ldots, x_n)$ and marginal distribution functions F_1 , . . . , F_n . Moreover, $F_{\min} = \min\{F_1(X_1), \ldots, F_2(X_n)\}$ and $F_{\max} = \max\{F_1(X_1), \ldots, F_2(X_n)\}$. The Lower extremal dependence coefficient (LEDC) of $(X_1, X_2, \ldots, X_n)$ is defined as $E_L = \lim_{t\to 0^+} P\left(F_{\max} \le t | F_{\min} \le t\right)$, whereas the upper extremal dependence coefficient(UEDC)of$(X_1, X_2, \ldots, X_n)$is defined as $E_u = \lim_{t\to 1^-} P\left(F_{\min} > t | F_{\max} > t\right)$, provided the corresponding limits exist.

Proposition (Frahm (2006)) we can derive E_L and E_U via the quantities λ_l and λ_u as follows.

$$E_L = \frac{\lambda_l}{2-\lambda_l} \quad and \quad E_U = \frac{\lambda_u}{2-\lambda_u}.$$

1.1.12 Clayton-Oakes association measure.

Clayton(1978) and Oakes (1989) defined the following association measure

$$\theta(x,y) = \frac{\bar{F}(x,y) D_{12}\bar{F}(x,y)}{D_1\bar{F}(x,y) D_2\bar{F}(x,y)}$$

where $D_{12}\bar{F}(x,y) = \frac{\partial^2}{\partial x \partial y}\bar{F}(x,y)$, $D_1\bar{F}(x,y) = \frac{\partial}{\partial x}\bar{F}(x,y)$ and $D_2\bar{F}(x,y) = \frac{\partial}{\partial y}\bar{F}(x,y)$. The parameter $\theta(x,y)$ measures the degree of association between X and Y , independence being implied by $\theta(x,y) = 1$, positive dependence by $\theta(x,y) > 1$ and negative dependence by $\theta(x,y) < 1$ (Gupta (2003)).

1.2 example and measures of dependence

In this section, we first discuss three local dependence measures, such as γ- measure, the Clayton-Oakes association measure(θ-measure) and ψ- measure and drive the relationship of these measures with hazard negative dependence,then we give some examples.

▷ **γ-Measure:**

Holland and Wang [9] defined, the local dependence function $\gamma_h(x, y)$ as follows;

$$\gamma_h(x,y) = \frac{\partial^2 Log h(x,y)}{\partial x \partial y} = \frac{1}{h(x,y)}\{h^{11}(x,y) - \frac{h^{10}(x,y)h^{01}(x,y)}{h(x,y)}\}, \tag{1.1}$$

where $h(x, y) \geq 0$, $h^{ij} = \frac{\partial^{i+j}h(x,y)}{\partial x^i \partial y^j}$, $i, j = 0, 1$, the mixed partial derivative of h(x,y) exists and h is defined on a Cartesian product set. They show that this measure is symmetric and $\gamma = 0$ if and only if X and Y are independent. Also, Jones [11] and [12] studied dependence properties of this measure and proved that γ is an appropriate index for measuring local likelihood ration dependence.

Remark 1.2.1. Let X and Y be continuous random variables with bivariate distribution function $F(x, y)$ and survival function $\bar{F}(x, y)$. Then, it is easy to show that,

$$\gamma_F(x,y) = \frac{f(x,y)F(x,y) - \int_{-\infty}^{x} f(u,y)du \int_{-\infty}^{y} f(x,v)dv}{F^2(x,y)},$$

and

$$\gamma_{\bar{F}}(x,y) = \frac{f(x,y)\bar{F}(x,y) - \int_{x}^{\infty} f(u,y)du \int_{y}^{\infty} f(x,v)dv}{\bar{F}^2(x,y)}.$$

Therefore,

- Lemma 4.2 in Holland and Wang [9] implies that X and Y are independent if and only if $\gamma_F(x, y) = 0$ ($\gamma_{\bar{F}}(x, y) = 0$) or, equivalently equality occur in (1) or (2).

- Moreover, it is easy to show that the following implications hold

$$HND(X,Y)(HPD(X,Y)) \Leftrightarrow \gamma_{\bar{F}}(x,y) \leq (\geq) 0,$$

and

$$LND(X,Y)(LPD(X,Y)) \Leftrightarrow \gamma_{F}(x,y) \leq (\geq) 0.$$

▷ **Θ-Measure**

Clayton[4] and Oakes[19] defined the following associated measure:

$$\Theta(x,y) = \frac{\bar{F}(x,y) D_{12}\bar{F}(x,y)}{D_1\bar{F}(x,y) D_2\bar{F}(x,y)} \ , \tag{1.2}$$

where $D_{12}\bar{F}(x,y) = \frac{\partial^2}{\partial x \partial y}\bar{F}(x,y)$, $D_1\bar{F}(x,y) = \frac{\partial}{\partial x}\bar{F}(x,y)$ and $D_2\bar{F}(x,y) = \frac{\partial}{\partial y}\bar{F}(x,y)$. The function $\Theta(x,y)$ measures the degree of association between X and Y, and has direct relation to local dependence function, $\gamma_{\bar{F}}(x,y)$.

- $\Theta(x,y) = 1$ if and only if $\gamma_{\bar{F}}(x,y) = 0$ *i.e* X and Y are independent,
- $\Theta(x,y) > 1$ if and only if $\gamma_{\bar{F}}(x,y) > 0$ *i.e* X and Y are positively dependent,
- $\Theta(x,y) < 1$ if and only if $\gamma_{\bar{F}}(x,y) < 0$ or equivalently X and Y are negatively dependent.

According to Gupta [7] we have the following quantities to formulate $\Theta(x,y)$.

$$r_1(x,y) := -\frac{\partial}{\partial x}[\log \bar{F}(x,y)] = -\frac{D_1\bar{F}(x,y)}{\bar{F}(x,y)}, r_2(x,y) := -\frac{\partial}{\partial y}[\log \bar{F}(x,y)] = -\frac{D_2\bar{F}(x,y)}{\bar{F}(x,y)}$$

and

$$\frac{\partial^2}{\partial x \partial y} \log \bar{F}(x,y) = r_1(x,y) r_2(x,y)(\Theta(x,y) - 1). \tag{1.3}$$

So,

$$r(x,y) = r_1(x,y) r_2(x,y) \Theta(x,y), \tag{1.4}$$

where $r(x,y) = \frac{f(x,y)}{\bar{F}(x,y)}$ is Basu's failure rate. We observe that,

$$\Theta(x,y) < 1 \Leftrightarrow \frac{\partial^2}{\partial x \partial y} \log \bar{F}(x,y) < 0 \Leftrightarrow RCSD(X,Y) \Leftrightarrow r(x,y) < r_1(x,y) r_2(x,y).$$

▷ **ψ- Measure**

The following associated measure (known as ψ- measure) defined by Anderson et al.[2];

$$\psi(x,y) = \frac{P(X > x | Y > y)}{P(X > x)} = \frac{\bar{F}(x,y)}{\bar{F}_1(x)\bar{F}_2(y)} \tag{1.5}$$

Under the some regular conditions, the following statements are valid for ψ-measure in (15);

- $\psi(x, y) = 1 \Leftrightarrow X$ and Y are independent.
- $\frac{\partial^2}{\partial x \partial y}\psi(x, y) = \gamma_{\bar{F}}(x, y)$.
- If $\psi(x, y) > 1$ then (X, Y) is PQD.
- If $\psi(x, y) < 1$ then (X, Y) is NQD.
- If $\Theta(x, y) < (>)1$ then $\psi(x, y) < (>)1$ (the converse is not true).
  For more details, see Gupta [7].

The following proposition gives relationship between the mentioned local dependence measures.

Proposition 1.2.2. *Let (X, Y) be an absolutely continuous random vector having survival function $\bar{F}(x, y)$. The following statements are equivalent*

- $\Theta(x, y) < 1$,
- $\gamma_{\bar{F}}(x, y) < 0$,
- $\frac{\partial^2}{\partial x \partial y}\psi(x, y) < 0$,
- $r(x, y) < r_1(x, y)r_2(x, y)$,
- (X, Y) *is* HND.

Proof. Combining (6), (11), (12), (13) and (14) the proposition proved immediately. □

Example 1.2.3. *(Farlie-Gumble-Morganstern distribution (FGM) [6])* Consider the family of bivariate distribution functions

$$F(x, y) = F_1(x)F_2(y)[1 + \alpha(1 - F_1(x))(1 - F_2(y))]$$

where $|\alpha| \leq 1$ and $F_1(x)$ and $F_2(y)$ are continuous distribution functions. It can be shown that,

$$\gamma_F(x,y)\frac{\alpha f_1(x)f_2(y)}{[1+\alpha\bar{F}_1(x)\bar{F}_2(y)]^2} \leq (\geq)0 \quad \Leftrightarrow \quad \alpha \quad -1 \leq \alpha \leq 0(0 \leq \alpha \leq 1).$$

Therefore, $LND(X,Y)(LPD(X,Y))$ if and only if $-1 \leq \alpha \leq 0(0 \leq \alpha \leq 1)$.
In terms of survival functions $\bar{F}(x,y) = P[X > x, Y > y]$; $\bar{F}_i(x_i) = P[X_i > x_i]$; $i = 1,2$ the FGM family equivalent to

$$\bar{F}(x,y) = \bar{F}_1(x)\bar{F}_2(y)[1+\alpha\,F_1(x)F_2(y)], \quad |\alpha| \leq 1.$$

It follows from simple calculations that

$$\gamma_{\bar{F}}(x,y) = \frac{\alpha\,f_1(x)f_2(y)}{[1+\alpha\,F_1(x)F_2(y)]^2} \leq (\geq)0 \quad \Leftrightarrow \quad -1 \leq \alpha \leq 0(0 \leq \alpha \leq 1),$$

so $HND(X,Y)(HPD(X,Y))$ if and only if $-1 \leq \alpha \leq 0(0 \leq \alpha \leq 1)$.
For more details about FGM family see Mari and Kotz [15].

Example 1.2.4. *(Gumbel's bivariate exponential distribution)* The survival function of Gumbel's bivariate distribution is

$$\bar{F}(x,y) = \exp\{-\alpha_1 x - \alpha_2 y - \beta xy\}, \qquad \alpha_1, \alpha_2 > 0 \quad \text{and} \quad 0 \leq \beta \leq \alpha_1\alpha_2.$$

For $x < x'$ and $y < y'$;

$$\begin{aligned}\bar{F}(x,y)\bar{F}(x',y') \;&-\bar{F}(x,y')\bar{F}(x',y) \\ &= \exp\{-\alpha_1(x+x') - \alpha_2(y+y')\} \\ &\qquad \times\Big[\exp\{-\beta(xy+x'y')\} - \exp\{-\beta(xy'+x'y)\}\Big] \quad \leq 0.\end{aligned}$$

Since $xy + x'y' \geq xy' + x'y$, hence $\bar{F}$ is RR_2, and this implies that (X,Y) is HND.

Example 1.2.5. *(Ali-Mikhail-Haq distribution [1])* Consider Ali-Mikhail-Haq family of bivariate distribution functions

$$F(x,y) = \frac{F_1(x)F_2(y)}{1-\beta\,\bar{F}_1(x)\bar{F}_2(y)}, \quad |\beta| \leq 1$$

where F_1 and F_2 are continuous distribution functions and $\bar{F}_i = 1 - F_i \quad$ i $= 1, 2$. by simple calculation, we obtain

$$\gamma_F(x, y) = \frac{\beta f_1(x) f_2(y)}{[1 - \beta \bar{F}_1(x)\bar{F}_2(y)]^2} \leq 0(\geq 0) \quad \Leftrightarrow \quad -1 \leq \beta \leq 0(0 \leq \beta \leq 1).$$

So, LND(X, Y)(LPD(X, Y)) if and only if $-1 \leq \beta \leq 0(0 \leq \beta \leq 1)$.

Remark 1.2.6. In the Example 1.2.4 we can use the Proposition 1.2.2 and obtain

$$\begin{aligned}
r_1(x, y) &= -\frac{\partial}{\partial x}[\log \bar{F}(x, y)] = \alpha_1 + \beta y \\
r_2(x, y) &= -\frac{\partial}{\partial y}[\log \bar{F}(x, y)] = \alpha_2 + \beta x \\
r(x, y) &= \frac{f(x, y)}{\bar{F}(x, y)} = (\alpha_1 + \beta y)(\alpha_2 + \beta x) - \beta \\
\Theta(x, y) &= \frac{r(x, y)}{r_1(x, y) r_2(x, y)} = \frac{(\alpha_1 + \beta y)(\alpha_2 + \beta x) - \beta}{(\alpha_1 + \beta y)(\alpha_2 + \beta x)}
\end{aligned}$$

since $\alpha_i > 0, \quad i = 1, 2$ and $\beta \geq 0$, therefore Proposition (3.1) implies that (X, Y) is HND.

1.2.1 measure of dependence based on copula

The copula function $C(u, v)$ is a bivariate distribution function with uniform marginals on $[0, 1]$, such that

$$F(x, y) = C_F(F_1(x), F_2(y))$$

By Sklar's Theorem (Sklar, 1959), this copula exists and is unique if F_1 and F_2 are continuous. Thus we can construct bivariate distributions $F(x, y) = C_F(F_1(x), F_2(y))$ with given univariate marginals F_1 and F_2 by using copula C_F,(Nelsen, 2006). Then we have the following properties:

- (Nelsen, [16]) Let X and Y be continuous random variables with joint distribution function $F(x, y)$ and marginals $F_1(x)$ and $F_2(y)$ respectively, then *i*) The copula $C(u,v)$ and survival copula which refer to $\hat{C}(u,v)$ are given by

$$C_F(u,v) = F(F_1^{-1}(u), F_2^{-1}(v)), \quad \forall u,v \in [0,1],$$

and

$$\hat{C}(u,v) = \bar{F}(\bar{F}_1^{-1}(u), \bar{F}_2^{-1}(v)), \quad \forall u,v \in [0,1]$$

Where, F_i^{-1} and $\bar{F}_i^{-1}$ are quasi-inverses of F_i and $\bar{F}_i$, $i = 1,2$ respectively. Note that;

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v), \quad \forall u,v \in [0,1]$$

ii) The partial derivatives $\frac{\partial C_F(u,v)}{\partial u}$ and $\frac{\partial C_F(u,v)}{\partial v}$ exist and $c(u,v) = \frac{\partial^2 C_F(u,v)}{\partial u \partial v}$ is density function of $C_F(u,v)$.

- The Sklar's theorem implies that in FGM family for $-1 \leq \alpha \leq 1$

$$C(u,v) = \hat{C}(u,v) = uv(1 + \alpha(1-u)(1-v)), \tag{1.6}$$

and

$$c(u,v) = 1 + \alpha(1-2u)(1-2v). \tag{1.7}$$

Also in Gumbel family for $\alpha_1 = \alpha_2 = 1$, the survival copula is

$$\hat{C}(u,v) = uv.\exp(-\beta \ln(u)\ln(v)), \quad \forall 0 \leq \beta \leq 1. \tag{1.8}$$

Proposition 1.2.7. *Let (X, Y) be a random vector with FGM distribution function and copula function given in (16), then*

i) $\psi(u,v) = \frac{\hat{C}(u,v)}{uv} = 1 + \alpha(1-u)(1-v),$

ii) $\gamma_C(u,v) = \gamma_{\hat{C}}(u,v) = \frac{\partial^2 \log(C(u,v))}{\partial u.\partial v} \frac{\alpha}{[1+\alpha(1-u)(1-v)]^2},$

iii) $\Theta(u,v) = \frac{\hat{C}(u,v)\frac{\partial^2 \hat{C}(u,v)}{\partial u.\partial v}}{\frac{\partial \hat{C}(u,v)}{\partial u}\frac{\partial \hat{C}(u,v)}{\partial v}} = \frac{(1+\alpha(1-u)(1-v)).(1+\alpha(1-2u)(1-2v))}{(1+\alpha(1-u)(1-2v))(1+\alpha(1-v)(1-2u))}.$

- Figure 1 shows the surface of $\gamma_{\alpha_2}(u,v) - \gamma_{\alpha_1}(u,v)$ for some values of α_1, α_2 such that $\alpha_1 < \alpha_2$ in FGM family with uniform marginals on $(0,1)$. These surfaces, show that $\gamma_\alpha(u,v)$ increases in α.

- Figure 2 shows the surface of $\Theta_{\alpha_2}(u,v) - \Theta_{\alpha_1}(u,v)$ for some values of α_1, α_2 such that $\alpha_1 < \alpha_2$ in FGM family with uniform marginals on $(0,1)$. These surfaces, show that $\Theta_\alpha(u,v)$ increases in α.

Proposition 1.2.8. *Let (X,Y) be a random vector with Gumbel distribution function with $\alpha_1 = \alpha_2 = 1$ and survival copula given in (18), then*

i) $\psi(u,v) = \exp(-\beta \ln(u)\ln(v))$,

ii) $\gamma_{\hat{C}}(u,v) = \frac{\beta uv \ln(uv)(1+\beta) - \beta uv - \beta u^2v^2 \ln(u) - \beta^2 \ln(u)\ln(v)}{u^3v^3}$,

iii) $\Theta(u,v) = \frac{u^2v^2 - \beta uv - \beta u^2v^2 \ln(u) + \beta^2 uv \ln(u)\ln(v)}{u^2v^2 - \beta uv \ln(u) - \beta uv \ln(v)) + \beta^2 \ln(u)\ln(v)}$.

- Figure 3 shows the surface of $\gamma_{\beta_2}(u,v) - \gamma_{\beta_1}(u,v)$ for some values of β_1, β_2 such that $\beta_1 < \beta_2$ in Gumbel family. These surfaces, show that $\gamma_\beta(u,v)$ decreases in β.

- Figure 4 shows the surface of $\Theta_{\beta_2}(u,v) - \Theta_{\beta_1}(u,v)$ for some values of β_1, β_2 such that $\beta_1 < \beta_2$ in Gumbel family. These surfaces, show that $\Theta_\beta(u,v)$ is not monotone in β.

Remark 1.2.9. It is clear that $\psi_\alpha(u,v)$ in FGM family is increasing in α and $\psi_\beta(u,v)$ in Gumbel family is decreasing in β.