

## Some matrix power and Karcher means inequalities involving positive linear maps

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**Abstract.** In this paper, we generalize some matrix inequalities involving the matrix power means and Karcher mean of positive definite matrices. Among other inequalities, it is shown that if  $\mathbb{A} = (A_1, \dots, A_n)$  is an  $n$ -tuple of positive definite matrices such that  $0 < m \leq A_i \leq M$  ( $i = 1, \dots, n$ ) for some scalars  $m < M$  and  $\omega = (w_1, \dots, w_n)$  is a weight vector with  $w_i \geq 0$  and  $\sum_{i=1}^n w_i = 1$ , then

$$\Phi^p\left(\sum_{i=1}^n w_i A_i\right) \leq \alpha^p \Phi^p(P_t(\omega; \mathbb{A})) \quad \text{and} \quad \Phi^p\left(\sum_{i=1}^n w_i A_i\right) \leq \alpha^p \Phi^p(\Lambda(\omega; \mathbb{A})),$$

where  $p > 0$ ,  $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}} Mm}\right\}$ ,  $\Phi$  is a positive unital linear map and  $t \in [-1, 1] \setminus \{0\}$ .

### 1. Introduction and preliminaries

Let  $\mathcal{M}_k$  be the  $C^*$ -algebra of all  $k \times k$  complex matrices with the identity  $I$ , and  $\langle \cdot, \cdot \rangle$  be the standard scalar product in  $\mathbb{C}^k$ . For Hermitian matrices  $A, B \in \mathcal{M}_k$ , we write  $A \geq 0$  if  $A$  is positive semidefinite,  $A > 0$  if  $A$  is positive definite, and  $A \geq B$  if  $A - B \geq 0$ . If  $m, M$  are real scalars, then we mean  $m \leq A \leq M$  whenever  $mI \leq A \leq MI$ .

The Gelfand map  $f(t) \mapsto f(A)$  is an isometrical  $*$ -isomorphism between the  $C^*$ -algebra  $C(\text{sp}(A))$  of continuous functions on the spectrum  $\text{sp}(A)$  of a Hermitian matrix  $A$  and the  $C^*$ -algebra generated by  $A$  and  $I$ . If  $f, g \in C(\text{sp}(A))$ , then  $f(t) \geq g(t)$  ( $t \in \text{sp}(A)$ ) implies that  $f(A) \geq g(A)$ . A linear map  $\Phi$  on  $\mathcal{M}_k$  is positive if  $\Phi(A) \geq 0$  whenever  $A \geq 0$ . It is said to be unital if  $\Phi(I) = I$ . Let  $A, B \in \mathcal{M}_k$  be two positive definite and  $t \in [0, 1]$ . The operator  $t$ -weighted arithmetic, geometric, and harmonic means of  $A, B$  are defined by  $A\nabla_t B = (1-t)A + tB$ ,  $A\sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$  and  $A!_t B = ((1-t)A^{-1} + tB^{-1})^{-1}$ , respectively, in which  $A!_t B \leq A\sharp_t B \leq A\nabla_t B$ . In particular, for  $t = \frac{1}{2}$  we get the usual operator arithmetic mean  $\nabla$ , the geometric mean  $\sharp$  and the harmonic mean  $!$ .

Throughout the paper, let  $\mathbb{A} = (A_1, \dots, A_n)$  be an  $n$ -tuple of positive definite matrices  $A_i$  ( $i = 1, \dots, n$ ) and  $\omega = (w_1, \dots, w_n)$  be a positive probability weight vector (we simply write the weight vector), where  $w_i \geq 0$  ( $i = 1, \dots, n$ ) and  $\sum_{i=1}^n w_i = 1$ . In [15], Lim and Palfia introduced matrix power mean of positive

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definite matrices of some fixed dimension. The matrix power mean  $P_t(\omega; \mathbb{A})$  is defined to be the unique positive definite solution of the non-linear equation:

$$X = \sum_{i=1}^n w_i (X \sharp_t A_i), \quad t \in (0, 1].$$

For  $t \in [-1, 0)$ , it is defined by  $P_t(\omega; \mathbb{A}) = P_{-t}(\omega; \mathbb{A}^{-1})^{-1}$ , where  $\mathbb{A}^{-1} = (A_1^{-1}, \dots, A_n^{-1})$ . For further information we refer the reader to [14, 20] and references therein. We denote by  $P_1(\omega; \mathbb{A}) = \sum_{i=1}^n w_i A_i$  and  $P_{-1}(\omega; \mathbb{A}) = (\sum_{i=1}^n w_i A_i^{-1})^{-1}$ , the weighted arithmetic and harmonic means of  $A_1, \dots, A_n$ , respectively.

There is one of the important properties of matrix power mean  $P_t(\omega; \mathbb{A})$ , that  $P_t(\omega; \mathbb{A})$  interpolates between the weighted harmonic and arithmetic means:

$$\left( \sum_{i=1}^n w_i A_i^{-1} \right)^{-1} \leq P_t(\omega; \mathbb{A}) \leq \sum_{i=1}^n w_i A_i \tag{1}$$

for all  $t \in [-1, 1] \setminus \{0\}$ . The Karcher mean of  $n$  positive definite matrices  $A_1, \dots, A_n$  is defined as the unique minimizer of the sum of squares of the Riemannian trace metric distances to each of the  $A_i$ , i.e.,  $\Lambda(\omega; A_1, \dots, A_n) = \arg \min_{X \in \mathbb{P}} \sum_{i=1}^n w_i \delta^2(X, A_i)$  (Recall that the trace metric distance between two positive definite matrices is given by  $\delta(A, B) = (\sum_{i=1}^n \log(\lambda_i(A^{-1}B)))^{1/2}$ , where  $\lambda_i(X)$  denotes the  $i$ -th eigenvalue of  $X$  in ascending order.). In fact, the Karcher mean coincides with the unique positive definite solution of the Karcher equation:

$$\sum_{i=1}^n w_i \log \left( X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}} \right) = 0. \tag{2}$$

The Karcher mean satisfies from (2) that  $\Lambda(\omega; \mathbb{A}^{-1})^{-1} = \Lambda(\omega; \mathbb{A})$ . It is well known that (see [15])

$$\lim_{t \rightarrow 0} P_t(\omega; \mathbb{A}) = \Lambda(\omega; \mathbb{A}) \tag{3}$$

and

$$\left( \sum_{i=1}^n w_i A_i^{-1} \right)^{-1} \leq \Lambda(\omega; \mathbb{A}) \leq \sum_{i=1}^n w_i A_i.$$

For further information about the matrix power mean, Karcher mean, operator mean and their properties, we refer the readers to [4, 5, 14–16] and references therein.

It is well known that for two positive definite matrices  $A$  and  $B$ , if  $A \geq B$ , then

$$A^p \geq B^p \quad (0 \leq p \leq 1). \tag{4}$$

In general (4) is not true for  $p > 1$ . Let  $\Phi$  be a unital positive linear map. The following inequality is known as Choi’s inequality(see [7, 12]):

$$\Phi(A)^{-1} \leq \Phi(A^{-1}). \tag{5}$$

Marshal and Olkin [19] proved a counterpart of Choi’s inequality (5) as follows:

$$\Phi(A^{-1}) \leq \frac{(M+m)^2}{4Mm} \Phi(A)^{-1} \tag{6}$$

for positive definite  $A$  with  $0 < m \leq A \leq M$ . In addition, Lin [17] and Fu [9] improved inequality (6) for  $p \geq 2$  to the form  $\Phi^p(A^{-1}) \leq \left( \frac{(M+m)^2}{4^{\frac{p}{2}} Mm} \right)^p \Phi(A)^{-p}$ .

The matrix power means satisfy the following inequality(see [8, 15]): For each  $t \in (0, 1]$

$$\Phi(P_t(\omega; \mathbb{A})) \leq P_t(\omega; \Phi(\mathbb{A})), \tag{7}$$

where  $\Phi$  is a unital positive linear map,  $\mathbb{A} = (A_1, \dots, A_n)$  is a  $n$ -tuple of positive definite matrices and  $\Phi(\mathbb{A}) = (\Phi(A_1), \dots, \Phi(A_n))$ . Ando [1] proved that if  $\Phi$  is a positive linear map, then for positive definite matrices  $A, B \in \mathcal{M}_k$ , we have

$$\Phi(A\sharp B) \leq \Phi(A)\sharp\Phi(B). \tag{8}$$

A reverse of Ando's inequality (8) states that [12, Remark 5.3]: If  $A, B \in \mathcal{M}_k$  and  $0 < m \leq A, B \leq M$ , then

$$\Phi(A)\sharp\Phi(B) \leq \frac{M+m}{2\sqrt{Mm}}\Phi(A\sharp B).$$

By inequality (4) we get

$$(\Phi(A)\sharp\Phi(B))^p \leq \left(\frac{M+m}{2\sqrt{Mm}}\right)^p \Phi^p(A\sharp B), \quad (0 < p \leq 1). \tag{9}$$

In [10], Fujii et al. obtained a reverse of inequality (1) as following:

$$\sum_{i=1}^n w_i A_i \leq \frac{(M+m)^2}{4Mm} P_t(\omega; \mathbb{A}).$$

Applying (7), we can obtain the following operator inequality:

$$\begin{aligned} \Phi\left(\sum_{i=1}^n w_i A_i\right) &\leq \frac{(M+m)^2}{4Mm} \Phi(P_t(\omega; \mathbb{A})) \\ &\leq \frac{(M+m)^2}{4Mm} P_t(\omega; \Phi(\mathbb{A})). \end{aligned}$$

Now, using inequality (4), we get

$$\Phi^p\left(\sum_{i=1}^n w_i A_i\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^p P_t^p(\omega; \Phi(\mathbb{A})) \quad (0 \leq p \leq 1). \tag{10}$$

Dehghani et al. [8] established counterparts of (7) involving matrix power means as follows:

$$P_t^2(\omega; \Phi(\mathbb{A})) \leq \left(\frac{(m+M)^2}{4mM}\right)^2 \Phi^2(P_t(\omega; \mathbb{A}))$$

for all  $t \in [-1, 1] \setminus \{0\}$  and  $0 < m \leq A_i \leq M$  ( $1 \leq i \leq n$ ). Applying inequality (4), we get

$$P_t^p(\omega; \Phi(\mathbb{A})) \leq \left(\frac{(m+M)^2}{4mM}\right)^p \Phi^p(P_t(\omega; \mathbb{A})), \quad (0 \leq p \leq 2). \tag{11}$$

It is interesting to ask whether inequality (11) is true for  $p \geq 2$ . This is the first motivation of this paper. Moreover, we improve inequality (9) for  $p \geq 2$ . We also obtain some reverses of (1). In the last section, we establish several refinements of obtained inequalities.

## 2. Main results

To prove our first result, we need the following lemmas.

**Lemma 2.1.** [2, 3, 6, 11] Let  $A, B \in \mathcal{M}_k$  be positive definite matrices and  $\alpha > 0$ . Then

- (i)  $\|AB\| \leq \frac{1}{4}\|A + B\|^2$ .
- (ii) For  $\beta \geq 1$ ,  $\|A^\beta + B^\beta\| \leq \|(A + B)^\beta\|$ .
- (iii)  $A \leq \alpha B$  if and only if  $\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\| \leq \alpha^{\frac{1}{2}}$ .
- (iv) If  $0 \leq A \leq B$  and  $0 < m \leq A \leq M$ , then  $A^2 \leq \frac{(M+m)^2}{4Mm}B^2$ .

**Lemma 2.2.** [13] Let  $A \in \mathcal{M}_k$  and  $t$  be a positive number. Then  $|A| \leq tI$  if and only if  $\|A\| \leq t$  if and only if  $\begin{bmatrix} tI & A \\ A^* & tI \end{bmatrix}$  is positive.

**Theorem 2.3.** Let  $\mathbb{A} = (A_1, \dots, A_n)$  be an  $n$ -tuple of positive definite matrices with  $0 < m \leq A_i \leq M$  ( $i = 1, \dots, n$ ) for some scalars  $m < M$  and  $\omega = (\omega_1, \dots, \omega_n)$  be a weight vector. If  $\Phi$  is a unital positive linear map, then

$$P_t^p(\omega; \Phi(\mathbb{A})) \leq \left(\frac{(m+M)^2}{4^{\frac{p}{2}}mM}\right)^p \Phi^p(P_t(\omega; \mathbb{A})) \tag{12}$$

for every  $p \geq 2$  and  $t \in [-1, 1] \setminus \{0\}$ .

*Proof.* Applying Lemma 2.1(iii), inequality (12) is equivalent to

$$\left\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A}))\Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\right\| \leq \frac{(m+M)^p}{4M^{\frac{p}{2}}m^{\frac{p}{2}}}. \tag{13}$$

Hence, it is enough to prove inequality (13). So

$$\begin{aligned} M^{\frac{p}{2}}m^{\frac{p}{2}}\left\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A}))\Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\right\| &= \left\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A}))M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\right\| \\ &\leq \frac{1}{4}\left\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A})) + M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\right\|^2 \\ &\hspace{10em} \text{(by Lemma 2.1(i))} \\ &\leq \frac{1}{4}\left\|(P_t(\omega; \Phi(\mathbb{A})) + Mm\Phi^{-1}(P_t(\omega; \mathbb{A})))^{\frac{p}{2}}\right\|^2 \\ &\hspace{10em} \text{(by Lemma 2.1(ii))} \\ &= \frac{1}{4}\|(P_t(\omega; \Phi(\mathbb{A})) + Mm\Phi^{-1}(P_t(\omega; \mathbb{A})))\|^p \\ &\leq \frac{1}{4}\|(P_t(\omega; \Phi(\mathbb{A})) + Mm\Phi(P_t(\omega; \mathbb{A})^{-1}))\|^p \\ &\hspace{10em} \text{(by (5))} \\ &\leq \frac{1}{4}\left\|\sum_{i=1}^n \omega_i \Phi(A_i) + Mm\Phi\left(\sum_{i=1}^n \omega_i A_i^{-1}\right)\right\|^p \\ &\hspace{10em} \text{(by (1))} \\ &= \frac{1}{4}\left\|\sum_{i=1}^n \omega_i (\Phi(A_i) + Mm\Phi(A_i^{-1}))\right\|^p. \end{aligned} \tag{14}$$

It follows from  $0 < m \leq A_i \leq M$  that  $(M - A_i)(m - A_i)A_i^{-1} \leq 0$  ( $i = 1, 2, \dots, n$ ). Hence

$$Mm\Phi(A_i^{-1}) + \Phi(A_i) \leq M + m \quad (i = 1, 2, \dots, n). \tag{15}$$

Applying inequalities (14) and (15), we get

$$\left\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A}))\Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\right\| \leq \frac{(m+M)^p}{4M^{\frac{p}{2}}m^{\frac{p}{2}}}.$$

This completes the proof.  $\square$

**Corollary 2.4.** Let  $\mathbb{A} = (A_1, \dots, A_n)$  be an  $n$ -tuple of positive definite matrices with  $0 < m \leq A_i \leq M$  ( $i = 1, \dots, n$ ) for some scalars  $m < M$  and  $\omega = (w_1, \dots, w_n)$  be a weight vector. If  $\Phi$  is a unital positive linear map, then

$$\Lambda^p(\omega; \Phi(\mathbb{A})) \leq \left(\frac{(m+M)^2}{4^{\frac{p}{2}}mM}\right)^p \Phi^p(\Lambda(\omega; \mathbb{A})) \tag{16}$$

for every  $p \geq 2$ .

*Proof.* The proof follows from Theorem 2.3 and relation (3).  $\square$

We would like to state the following lemma which use in the next result (see [21, page 582]).

**Lemma 2.5.** Let  $A, B \in \mathcal{M}_k$  be positive definite. Then

$$\Lambda(1-\alpha, \alpha; A, B) = A \sharp_{\alpha} B$$

for  $\alpha \in (0, 1)$ .

*Proof.* Using the definition of Karcher mean for two positive definite matrices  $A, B$  and  $\omega = (1-\alpha, \alpha)$  we have

$$(1-\alpha) \log(X^{-\frac{1}{2}} A X^{-\frac{1}{2}}) + \alpha \log(X^{-\frac{1}{2}} B X^{-\frac{1}{2}}) = 0. \tag{17}$$

Let  $X$  be the positive solution of (17). We assert that  $X = A \sharp_{\alpha} B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}}$ . First, we shall show that the Karcher mean of two matrices  $I$  and  $B$  is the operator  $B^{\alpha}$ . Let  $X$  be the solution of  $(1-\alpha) \log X^{-1} + \alpha \log(X^{-\frac{1}{2}} B X^{-\frac{1}{2}}) = 0$ , which is equivalent to  $X^{\frac{1-\alpha}{\alpha}} = X^{-\frac{1}{2}} B X^{-\frac{1}{2}}$ . Hence  $X = B^{\alpha}$  or equivalently  $\Lambda(\omega; I, B) = B^{\alpha}$ . Hence by the properties of Karcher mean (see [15, Corollary 4.5]), we have

$$\begin{aligned} \Lambda(\omega; A, B) &= A^{\frac{1}{2}} \Lambda(\omega; I, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} = A \sharp_{\alpha} B. \end{aligned}$$

$\square$

**Corollary 2.6.** Let  $A, B \in \mathcal{M}_n$  be positive definite matrices such that  $0 < m \leq A, B \leq M$  for some scalars  $m < M$  and  $\alpha \in [0, 1]$ . Then

$$(\Phi(A) \sharp_{\alpha} \Phi(B))^p \leq \left(\frac{(m+M)^2}{4^{\frac{p}{2}}mM}\right)^p \Phi^p(A \sharp_{\alpha} B)$$

for any  $p \geq 2$  and unital positive linear map  $\Phi$ .

*Proof.* Applying Lemma 2.5, we have  $\Lambda(1-\alpha, \alpha; A, B) = A \sharp_{\alpha} B$ , ( $\alpha \in [0, 1]$ ). If we put  $n = 2, w_1 = 1-\alpha$  and  $w_2 = \alpha$  in inequality (16), then we get the desired result.  $\square$

In the next theorem, we show an extension of inequality (10) for  $p > 1$ .

**Theorem 2.7.** Let  $\mathbb{A} = (A_1, \dots, A_n)$  be an  $n$ -tuple of positive definite matrices with  $0 < m \leq A_i \leq M$  ( $i = 1, \dots, n$ ) for some scalars  $m < M$  and  $\omega = (w_1, \dots, w_n)$  be a weight vector. Then

$$\Phi^p\left(\sum_{i=1}^n w_i A_i\right) \leq \alpha^p \Phi^p(P_t(\omega; \mathbb{A})), \tag{18}$$

where  $t \in [-1, 1] \setminus \{0\}$ ,  $p > 1$  and  $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{p}{2}}Mm}\right\}$ .

*Proof.* First we show inequality (18) for  $p = 2$ . We have

$$\begin{aligned} Mm \left\| \Phi \left( \sum_{i=1}^n w_i A_i \right) \Phi^{-1} (P_t(\omega; \mathbb{A})) \right\| &= \left\| \Phi \left( \sum_{i=1}^n w_i A_i \right) Mm \Phi^{-1} (P_t(\omega; \mathbb{A})) \right\| \\ &\leq \frac{1}{4} \left\| \Phi \left( \sum_{i=1}^n w_i A_i \right) + Mm \Phi^{-1} (P_t(\omega; \mathbb{A})) \right\|^2 \\ &\quad \text{(by Lemma 2.1)} \\ &\leq \frac{1}{4} \left\| \Phi \left( \sum_{i=1}^n w_i A_i \right) + Mm \Phi \left( \sum_{i=1}^n w_i A_i^{-1} \right) \right\|^2 \\ &\leq \frac{1}{4} (M + m)^2, \end{aligned}$$

whence

$$\left\| \Phi \left( \sum_{i=1}^n w_i A_i \right) \Phi^{-1} (P_t(\omega; \mathbb{A})) \right\| \leq \frac{(M + m)^2}{4Mm}.$$

Hence

$$\Phi^2 \left( \sum_{i=1}^n w_i A_i \right) \leq \left( \frac{(M + m)^2}{4Mm} \right)^2 \Phi^2 (P_t(\omega; \mathbb{A})).$$

Therefore

$$\Phi^p \left( \sum_{i=1}^n w_i A_i \right) \leq \left( \frac{(M + m)^2}{4Mm} \right)^p \Phi^p (P_t(\omega; \mathbb{A})) \quad (0 \leq p \leq 2). \tag{19}$$

Now, we prove inequality (18) for  $p > 2$ . In this case we have

$$\begin{aligned} \left\| \Phi^{\frac{p}{2}} \left( \sum_{i=1}^n w_i A_i \right) M^{\frac{p}{2}} m^{\frac{p}{2}} \Phi^{-\frac{p}{2}} (P_t(\omega; \mathbb{A})) \right\| &\leq \frac{1}{4} \left\| \Phi^{\frac{p}{2}} \left( \sum_{i=1}^n w_i A_i \right) + M^{\frac{p}{2}} m^{\frac{p}{2}} \Phi^{-\frac{p}{2}} (P_t(\omega; \mathbb{A})) \right\|^2 \\ &\quad \text{(by Lemma 2.1(i))} \\ &\leq \frac{1}{4} \left\| \left( \Phi \left( \sum_{i=1}^n w_i A_i \right) + Mm \Phi^{-1} (P_t(\omega; \mathbb{A})) \right)^{\frac{p}{2}} \right\|^2 \\ &\quad \text{(by Lemma 2.1(ii))} \\ &= \frac{1}{4} \left\| \Phi \left( \sum_{i=1}^n w_i A_i \right) + Mm \Phi^{-1} (P_t(\omega; \mathbb{A})) \right\|^p \\ &\leq \frac{(M + m)^p}{4}. \end{aligned}$$

Hence

$$\left\| \Phi^{\frac{p}{2}} \left( \sum_{i=1}^n w_i A_i \right) \Phi^{-\frac{p}{2}} (P_t(\omega; \mathbb{A})) \right\| \leq \frac{1}{4} \left( \frac{(M + m)^p}{M^{\frac{p}{2}} m^{\frac{p}{2}}} \right).$$

Thus

$$\Phi^p \left( \sum_{i=1}^n w_i A_i \right) \leq \left( \frac{(M + m)^2}{4^{\frac{2}{p}} Mm} \right)^p \Phi^p (P_t(\omega; \mathbb{A})). \tag{20}$$

Now, if we take  $\alpha = \max \left\{ \frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}} Mm} \right\}$ , then applying (19) and (20) we get the desired result.  $\square$

**Corollary 2.8.** Let  $\mathbb{A} = (A_1, \dots, A_n)$  be an  $n$ -tuple of positive definite matrices with  $0 < m \leq A_i \leq M$  ( $i = 1, \dots, n$ ) for some scalars  $m \leq M$  and  $\omega = (w_1, \dots, w_n)$  be a weight vector. Then

$$\Phi^p\left(\sum_{i=1}^n w_i A_i\right) \leq \alpha^p \Phi^p(\Lambda(\omega; \mathbb{A})),$$

where  $p \geq 1$  and  $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{1}{p}} Mm}\right\}$ .

**Remark 2.9.** By letting  $\mathbb{A} = (A, B)$  and  $\omega = (w_1, w_2)$  with  $w_1 = w_2 = \frac{1}{2}$  in Theorem 2.7, the following inequality holds:

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \alpha^p \Phi^p(A \sharp B)$$

for  $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{1}{p}} Mm}\right\}$ , which is appeared in [12, Theorem 4].

In the next result we extend inequalities (12) and (18) to the following form:

**Theorem 2.10.** Let  $\mathbb{A} = (A_1, \dots, A_n)$  be an  $n$ -tuple of positive definite matrices with  $0 < m \leq A_i \leq M$  ( $i = 1, \dots, n$ ) for some scalars  $m \leq M$  and  $\omega = (w_1, \dots, w_n)$  be a weight vector, let  $t \in [-1, 1] \setminus \{0\}$  and also  $\Phi$  be a positive unital linear map. Then

$$P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A})) + \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \Phi(\mathbb{A})) \leq 2\alpha^p$$

and

$$\Phi^p\left(\sum_{i=1}^n w_i A_i\right)\Phi^{-p}(P_t(\omega; \mathbb{A})) + \Phi^{-p}(P_t(\omega; \mathbb{A}))\Phi^p\left(\sum_{i=1}^n w_i A_i\right) \leq 2\alpha^p, \tag{21}$$

where  $p > 0$  and  $\alpha = \max\left\{\frac{(m+M)^2}{4mM}, \frac{(m+M)^2}{4^{\frac{1}{p}} mM}\right\}$ .

*Proof.* Applying inequality (11) and Lemma 2.1(iii) for  $0 < p \leq 1$ , we have

$$\|P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A}))\| \leq \left(\frac{(m+M)^2}{4mM}\right)^p.$$

We put  $\alpha = \frac{(m+M)^2}{4mM}$ . Applying Lemma 2.2,

$$\begin{bmatrix} \alpha^p I & P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A})) \\ \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \Phi(\mathbb{A})) & \alpha^p I \end{bmatrix}$$

and

$$\begin{bmatrix} \alpha^p I & \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \Phi(\mathbb{A})) \\ P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A})) & \alpha^p I \end{bmatrix}$$

are positive. Hence

$$\begin{bmatrix} 2\alpha^p I & P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A})) + \Phi^{-p}(P_t(\omega; \Phi(\mathbb{A}))P_t^p(\omega; \Phi(\mathbb{A})) \\ \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \Phi(\mathbb{A})) + P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A})) & 2\alpha^p I \end{bmatrix}$$

is positive. Applying Lemma 2.2, we get

$$P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A})) + \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \Phi(\mathbb{A})) \leq 2\alpha^p.$$

For  $p > 1$ , applying inequality (12) with the same argument, we get the desired inequality.

Applying Theorem 2.7 and a similar method we have inequality (21).  $\square$

**Corollary 2.11.** Let  $\mathbb{A} = (A_1, \dots, A_n)$  be a  $n$ -tuple of positive definite matrices with  $0 < m \leq A_i \leq M$  ( $i = 1, \dots, n$ ) for some scalars  $m \leq M$  and  $\omega = (w_1, \dots, w_n)$  be a weight vector, and also  $\Phi$  be a positive unital linear map. Then

$$\Lambda^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(\Lambda(\omega; \mathbb{A})) + \Phi^{-p}(\Lambda(\omega; \mathbb{A}))\Lambda^p(\omega; \Phi(\mathbb{A})) \leq 2\alpha^p$$

and

$$\Phi^p\left(\sum_{i=1}^n w_i A_i\right)\Phi^{-p}(\Lambda(\omega; \mathbb{A})) + \Phi^{-p}(\Lambda(\omega; \mathbb{A}))\Phi^p\left(\sum_{i=1}^n w_i A_i\right) \leq 2\alpha^p,$$

where  $p > 0$  and  $\alpha = \max\left\{\frac{(m+M)^2}{4mM}, \frac{(m+M)^2}{4^{\frac{1}{p}}mM}\right\}$ .

### 3. Some refinements

In this section, we give a refinement of inequality (18). This inequality can be refined by a similar method that known in [22].

**Theorem 3.1.** Let  $\mathbb{A} = (A_1, \dots, A_n)$  be an  $n$ -tuple of positive definite matrices with  $0 < m \leq A_i \leq M$  ( $i = 1, \dots, n$ ) for some scalars  $m \leq M$  and  $\omega = (w_1, \dots, w_n)$  be a weight vector, and also let  $t \in [-1, 1] \setminus \{0\}$ . Then for every positive unital linear map  $\Phi$

$$\Phi^{2p}\left(\sum_{i=1}^n w_i A_i\right) \leq \frac{(K(M^2 + m^2))^{2p}}{16M^{2p}m^{2p}}\Phi^{2p}(P_t(\omega; \mathbb{A})), \tag{22}$$

where  $p \geq 2$  and  $K = \frac{(M+m)^2}{4mM}$ .

*Proof.* For  $p \geq 2$ , we have

$$\begin{aligned} \left\| \Phi^p\left(\sum_{i=1}^n w_i A_i\right)M^p m^p \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\| &\leq \frac{1}{4} \left\| K^{\frac{p}{2}} \Phi^p\left(\sum_{i=1}^n w_i A_i\right) + \left(\frac{M^2 m^2}{K}\right)^{\frac{p}{2}} \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\|^2 \\ &\hspace{15em} \text{(by Lemma 2.1(i))} \\ &\leq \frac{1}{4} \left\| \left( K \Phi^2\left(\sum_{i=1}^n w_i A_i\right) + \frac{M^2 m^2}{K} \Phi^{-2}(P_t(\omega; \mathbb{A})) \right)^{\frac{p}{2}} \right\|^2 \\ &\hspace{15em} \text{(by Lemma 2.1(ii))} \\ &= \frac{1}{4} \left\| \left( K \Phi^2\left(\sum_{i=1}^n w_i A_i\right) + \frac{M^2 m^2}{K} \Phi^{-2}(P_t(\omega; \mathbb{A})) \right) \right\|^p. \end{aligned}$$

Now, It follows from inequality (18) and operator reverse monotonicity of the inverse that

$$\Phi^{-2}(P_t(\omega; \mathbb{A})) \leq K^2 \Phi^{-2}\left(\sum_{i=1}^n w_i A_i\right).$$

So

$$\begin{aligned} \left\| \Phi^p\left(\sum_{i=1}^n w_i A_i\right)M^p m^p \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\| &\leq \frac{1}{4} \left\| \left( K \Phi^2\left(\sum_{i=1}^n w_i A_i\right) + KM^2 m^2 \Phi^{-2}\left(\sum_{i=1}^n w_i A_i\right) \right) \right\| \\ &\leq \frac{1}{4} (K(M^2 + m^2))^p \hspace{10em} \text{(by [18, (4.7)])}. \end{aligned}$$



Hence

$$\left\| \Phi^p \left( \sum_{i=1}^n w_i A_i \right) \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\| \leq \frac{1}{4} \left( \frac{K(M^2 + m^2)}{Mm} \right)^p. \tag{23}$$

Since (23) is equivalent to (22), thus inequality (22) holds.  $\square$

**Corollary 3.2.** Let  $\mathbb{A} = (A_1, \dots, A_n)$  be an  $n$ -tuple of positive definite matrices with  $0 < m \leq A_i \leq M$  ( $i = 1, \dots, n$ ) for some scalars  $m \leq M$  and  $\omega = (w_1, \dots, w_n)$  be a weight vector. Then for every positive unital linear map  $\Phi$

$$\Phi^{2p} \left( \sum_{i=1}^n w_i A_i \right) \leq \frac{(K(M^2 + m^2))^{2p}}{16M^{2p}m^{2p}} \Phi^{2p}(\Lambda(\omega; \mathbb{A})),$$

where  $p \geq 2$  and  $K = \frac{(M+m)^2}{4mM}$ .

**Remark 3.3.** If we put  $\mathbb{A} = (A, B)$  and  $\omega = (w_1, w_2)$  with  $w_1 = w_2 = \frac{1}{2}$  in Corollary 3.2, then we get [22, Theorem 2.6] as follows:

$$\Phi^{2p} \left( \frac{A + B}{2} \right) \leq \frac{(K(M^2 + m^2))^{2p}}{16M^{2p}m^{2p}} \Phi^{2p}(A \sharp B).$$

**Theorem 3.4.** Let  $\mathbb{A} = (A_1, \dots, A_n)$  be an  $n$ -tuple of positive definite matrices with  $0 < m \leq A_i \leq M$  ( $i = 1, \dots, n$ ) for some scalars  $m \leq M$  and  $\omega = (w_1, \dots, w_n)$  be a weight vector, and also let  $t \in [-1, 1] \setminus \{0\}$ . Then for every positive unital linear map  $\Phi$

$$P_t^{2p}(\omega; \Phi(\mathbb{A})) \leq \frac{(K(M^2 + m^2))^{2p}}{16M^{2p}m^{2p}} \Phi^{2p}(P_t(\omega; \mathbb{A})), \tag{24}$$

where  $p \geq 2$  and  $K = \frac{(M+m)^2}{4mM}$ .

*Proof.* For  $p \geq 2$ , we have

$$\begin{aligned} \left\| P_t^p(\omega; \Phi(\mathbb{A})) M^p m^p \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\| &\leq \frac{1}{4} \left\| \frac{1}{K^{\frac{p}{2}}} P_t^p(\omega; \Phi(\mathbb{A})) + (KM^2m^2)^{\frac{p}{2}} \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\|^2 \\ &\hspace{15em} \text{(by Lemma 2.1(i))} \\ &\leq \frac{1}{4} \left\| \left( \frac{1}{K} P_t^p(\omega; \Phi(\mathbb{A})) + KM^2m^2 \Phi^{-2}(P_t(\omega; \mathbb{A})) \right)^{\frac{p}{2}} \right\|^2 \\ &\hspace{15em} \text{(by Lemma 2.1(ii))} \\ &= \frac{1}{4} \left\| \left( \frac{1}{K} P_t^2(\omega; \Phi(\mathbb{A})) + KM^2m^2 \Phi^{-2}(P_t(\omega; \mathbb{A})) \right) \right\|^p \\ &\leq \frac{1}{4} \left\| \left( K \Phi^2(P_t(\omega; \mathbb{A})) + KM^2m^2 \Phi^{-2}(P_t(\omega; \mathbb{A})) \right) \right\|^p \\ &\hspace{15em} \text{(by (12))} \\ &\leq \frac{1}{4} (K(M^2 + m^2))^p. \hspace{5em} \text{(by [18, (4.7)])} \end{aligned}$$

Therefore

$$\left\| P_t^p(\omega; \Phi(\mathbb{A})) \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\| \leq \frac{1}{4} \left( \frac{K(M^2 + m^2)}{Mm} \right)^p.$$

Since the last inequality is equivalent to (24), this completes the proof.  $\square$

**Corollary 3.5.** Let  $\mathbb{A} = (A_1, \dots, A_n)$  be an  $n$ -tuple of positive definite matrices with  $0 < m \leq A_i \leq M$  ( $i = 1, \dots, n$ ) for some scalars  $m \leq M$  and  $\omega = (\omega_1, \dots, \omega_n)$  be a weight vector. Then for every positive unital linear map  $\Phi$

$$\Lambda^{2p}(\omega; \Phi(\mathbb{A})) \leq \frac{(K(M^2 + m^2))^{2p}}{16M^{2p}m^{2p}} \Phi^{2p}(\Lambda(\omega; \mathbb{A})),$$

where  $p \geq 2$  and  $K = \frac{(M+m)^2}{4mM}$ .

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