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IMPROVEMENTS OF SOME OPERATOR INEQUALITIES INVOLVING POSITIVE LINEAR MAPS VIA THE KANTOROVICH CONSTANT

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Abstract. We present some operator inequalities for positive linear maps that generalize and improve the derived results in some recent years. For instant, if A and B are positive operators and $m, m^{'}, M, M^{'}$ are positive real numbers satisfying either one of the condition $0 < m \leq B \leq m^{'} < M^{'} \leq A \leq M$ or $0 < m \leq A \leq m^{'} < M^{'} \leq B \leq M$, then

$$\Phi^p\big(A\nabla_v B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\big) \leq \left(\frac{K(h)}{4^{\frac{2}{p}-1}K^{r_1}\left(\sqrt{h^{'}}\right)}\right)^p \Phi^p(A\sharp_\nu B)$$

and

$$\Phi^p\big(A\nabla_v B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\big) \leq \left(\frac{K(h)}{4^{\frac{2}{p}-1}K^{r_1}\left(\sqrt{h^{'}}\right)}\right)^p (\Phi(A)\sharp_\nu \Phi(B))^p,$$

where Φ is a positive unital linear map, $0 \leq \nu \leq 1$, $p \geq 2$, $r = \min\{\nu, 1-\nu\}$, $h = \frac{M}{m}$, $h^{'} = \frac{M^{'}}{m^{'}}$, $K(h) = \frac{(1+h)^2}{4h}$ and $r_1 = \min\{2r, 1-2r\}$. We also obtain a reverse of the Ando inequality for positive linear maps via the Kantorovich constant.

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1. Introduction and preliminaries

Let $\mathbb{B}(\mathbf{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathbf{H}$ whose identity is denoted by I. An operator $A \in \mathbb{B}(\mathbf{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbf{H}$ and in this case we write $A \geq 0$. We write $A > 0$ if A is a positive invertible operator. The absolute value of A is denoted by $|A|$, that is $|A| = (A^*A)^{\frac{1}{2}}$. For self-adjoint operators $A, B \in \mathbb{B}(\mathbf{H})$, we say $A \leq B$ if $B - A \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a self-adjoint operator A and the C^*-algebra generated by A and I. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \geq g(A)$. A linear map Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. If $A, B \in \mathbb{B}(\mathbf{H})$ be positive invertible, then the $\nu-$weighted arithmetic mean and geometric mean of A and B denoted by $A\nabla_\nu B$ and $A\sharp_\nu B$, respectively, which are defined by

$$A\nabla_\nu B = \nu A + (1-\nu)B \quad \text{and} \quad A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu A^{\frac{1}{2}},$$

respectively, where $0 \leq \nu \leq 1$. In case of $\nu = \frac{1}{2}$, we write $A\nabla B$ and the $A\sharp B$ for the arithmetic mean and the geometric mean, respectively. The well-known $\nu-$weighted arithmetic-geometric (AM-GM) operator inequality says that if $A, B \in \mathbb{B}(\mathbf{H})$ are positive and $0 \leq \nu \leq 1$, then $A\sharp_\nu B \leq A\nabla_\nu B$; see [7]. For $\nu = \frac{1}{2}$, we obtain the AM-GM operator inequality

$$A\sharp B \leq \frac{A+B}{2}. \tag{1}$$

For further information about the AM-GM operator inequality and positive linear maps inequalities we refer the reader to [1, 2, 3, 8, 10, 15] and references therein. Lin [12] presented a reverse of inequality (1) for a positive linear map Φ and positive operators $A, B \in \mathbb{B}(\mathbf{H})$ such that $m \leq A, B \leq M$ as follows:

$$\Phi\left(\frac{A+B}{2}\right) \leq K(h)\Phi(A\sharp B), \tag{2}$$

where $K(h) = \frac{(1+h)^2}{4h}$ and $h = \frac{M}{m}$. The constant $K(t) = \frac{(t+1)^2}{4t}(t > 0)$ is called the Kantorovich constant which satisfies the following properties:
(i) $K(1, 2) = 1$;
(ii) $K(t, 2) = K(\frac{1}{t}, 2) \geq 1$ $(t > 0)$;
(iii) $K(t, 2)$ is monotone increasing on the interval $[1, \infty)$ and monotone decreasing on the interval $(0, 1]$.

The Lowner-Heinz theorem [9] says that if $A, B \in \mathbb{B}(\mathbf{H})$ are positive, then for $0 \le p \le 1$,

$$A \le B \quad \text{implies} \quad A^p \le B^p. \tag{3}$$

In general (3) is not true for $p > 1$. In [12], the author showed that inequality (2) can be squared that is,

$$\Phi^2\left(\frac{A+B}{2}\right) \le K^2(h)\Phi^2(A\sharp B) \tag{4}$$

and

$$\Phi^2\left(\frac{A+B}{2}\right) \le K^2(h)(\Phi(A)\sharp\Phi(B))^2. \tag{5}$$

It follows (3), (4) and (5) that for $0 < p \le 2$ we have

$$\Phi^p\left(\frac{A+B}{2}\right) \le K^p(h)\Phi^p(A\sharp B) \tag{6}$$

and

$$\Phi^p\left(\frac{A+B}{2}\right) \le K^p(h)(\Phi(A)\sharp\Phi(B))^p. \tag{7}$$

It is natural to ask whether inequalities (6) and (7) are true for $p > 2$. In [6], the authors gave a positive answer to this question and proved the following theorem:

Theorem 1.1. *Let $0 < m \le A, B \le M$. Then for every positive unital linear map Φ and for every $p \ge 2$*

$$\Phi^p\left(\frac{A+B}{2}\right) \le \left(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right)^p \Phi^p(A\sharp B) \tag{8}$$

and

$$\Phi^p\left(\frac{A+B}{2}\right) \le \left(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right)^p (\Phi(A)\sharp\Phi(B))^p. \tag{9}$$

The next result is a further generalization [2]:

Theorem 1.2. [2] *Let $0 < m \le A, B \le M$. Then for every positive unital linear map Φ, $0 \le \nu \le 1$ and for every $p > 0$*

$$\Phi^p\left(A\nabla_\nu B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) \le \alpha^p\Phi^p(A\sharp_\nu B)$$
$$\Phi^p\left(A\nabla_\nu B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) \le \alpha^p(\Phi(A)\sharp_\nu\Phi(B))^p,$$

where $r = \min\{\nu, 1-\nu\}$ and $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right\}$.

The authors of [18] proved the following theorem, which is another improvement of inequalities (8) and (9).

Theorem 1.3. [18] *Let $0 < m \leq A \leq m' < M' \leq B \leq M$. Then for every positive unital linear map Φ, $0 \leq \nu \leq 1$ and for every $p \geq 2$*

$$\Phi^p(A\nabla_\nu B) \leq \left(\frac{K(h)}{4^{\frac{2}{p}-1}K^r(h')}\right)^p \Phi^p(A\sharp_\nu B)$$

$$\Phi^p(A\nabla_\nu B) \leq \left(\frac{K(h)}{4^{\frac{2}{p}-1}K^r(h')}\right)^p (\Phi(A)\sharp_\nu \Phi(B))^p,$$

where $r = \min\{\nu, 1-\nu\}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $K(h) = \frac{(1+h)^2}{4h}$.

In this article, we give some operator inequalities involving positive linear maps that generalize inequalities (8), (9) and refine some results in [2, 18]. Moreover, we obtain a reverse of Ando's inequality.

2. Some operator inequalities involving positive linear maps

We begin this section with several essential lemmas.

Lemma 2.1. [4] (*Choi's inequality*) *Let $A \in \mathbb{B}(\mathbf{H})$ be positive and Φ be a positive unital linear map. Then*

$$\Phi(A)^{-1} \leq \Phi\left(A^{-1}\right). \tag{10}$$

The next lemma, part (i) is proved for matrices but a careful investigation shows that it is true for operators on an arbitrary Hilbert space; see [14, page 79].

Lemma 2.2. [5, 9, 2] *Let $A, B \in \mathbb{B}(\mathbf{H})$ be positive and $\alpha > 0$. Then*
(i) $||\mathrm{AB}|| \leq \frac{1}{4}||\mathrm{A}+\mathrm{B}||^2$.
(ii) *If $\alpha \geq 1$, then $||A^\alpha + B^\alpha|| \leq ||(A+B)^\alpha||$.*
(iii) $\mathrm{A} \leq \alpha\mathrm{B}$ *if and only if* $||A^{\frac{1}{2}}B^{-\frac{1}{2}}|| \leq \alpha^{\frac{1}{2}}$.

To obtain our results, we need to prove the following lemma. Its proof is similar to that of [17, Theorem 3.1].

Lemma 2.3. *Suppose that $A, B \in \mathbb{B}(\mathbf{H})$ are positive and m, m', M, M' are positive real numbers satisfying either one of the following conditions:*

(1) $0 < m \leq B \leq m' < M' \leq A \leq M$;
(2) $0 < m \leq A \leq m' < M' \leq B \leq M$.

Then for every $0 \le \nu \le 1$,

$$2r\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right) + K^{r_1}\left(\sqrt{h'}\right)\left(A^{-1}\sharp_\nu B^{-1}\right) \le \left(A^{-1}\nabla_\nu B^{-1}\right), \tag{11}$$

where $r = \min\{\nu, 1-\nu\}$, $r_1 = \min\{2r, 1-2r\}$, $K(h) = \frac{(1+h)^2}{4h}$, $h = \frac{M}{m}$ *and* $h^{'} = \frac{M^{'}}{m^{'}}$.

Proof. It follows from [17, Lemma 2.3] that

$$2r\left(\frac{1+x}{2} - \sqrt{x}\right) + K^{r_1}(\sqrt{x})x^\nu \le (1-\nu) + \nu x$$

for any $x > 0$. The first condition, that is, $0 < m \le B \le m^{'} < M^{'} \le A \le M$ ensures that $1 < h^{'}I = \frac{M^{'}}{m^{'}}I \le A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}} \le \frac{M}{m}I = hI$. By setting $X = A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$, we see $\mathrm{sp}(X) \subseteq [h^{'}, h] \subset (1, +\infty)$. Now, the monotonicity principle for operator functions yields the inequality

$$\begin{aligned}(1-\nu) + \nu X &\ge 2r\left(\frac{I+X}{2} - \sqrt{X}\right) + \min_{h^{'} \le x \le h} K^{r_1}(\sqrt{x})X^\nu \\ &\ge 2r\left(\frac{I+X}{2} - \sqrt{X}\right) + K^{r_1}\left(\sqrt{h^{'}}\right)X^\nu. \qquad (12)\end{aligned}$$

The last above inequality follows by the increasing property of the function $K(t)$ on the interval $(1, +\infty)$; see [7]. Finally, multiplying the both sides of inequality (12) by $A^{-\frac{1}{2}}$, we obtain the desired result. The inequality can be proved under the second condition (2) in a similar way.

□

Our first main result is the following:

Theorem 2.4. *Suppose that* $A, B \in \mathbb{B}(\mathbf{H})$ *are positive and* $m, m^{'}, M, M^{'}$ *are positive real numbers satisfying either one of the following conditions:*

(1) $0 < m \le B \le m^{'} < M^{'} \le A \le M$;
(2) $0 < m \le A \le m^{'} < M^{'} \le B \le M$.

Then for every positive unital linear map Φ *and every* $0 \le \nu \le 1$

$$\begin{aligned}&\Phi^2\left(A\nabla_v B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) \\ &\le \left(\frac{K(h)}{K^{r_1}\left(\sqrt{h^{'}}\right)}\right)^2 \Phi^2(A\sharp_\nu B) \qquad (13)\end{aligned}$$

and

$$\Phi^2\Big(A\nabla_v B + 2rMm\big(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\big)\Big) \leq \left(\frac{K(h)}{K^{r_1}\left(\sqrt{h'}\right)}\right)^2 (\Phi(A)\sharp_\nu \Phi(B))^2, \tag{14}$$

where $r = \min\{\nu, 1-\nu\}$, $K(h) = \frac{(1+h)^2}{4h}$, $h = \frac{M}{m}$, $h^{'} = \frac{M'}{m'}$ *and* $r_1 = \min\{2r, 1-2r\}$.

Proof. We shall prove inequality (13), and leave inequality (14) to the reader because the proof is similar. By Lemma 2.3, inequality (13) is equivalent to

$$\left\|\Phi\left(A\nabla_v B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) MmK^{r_1}\left(\sqrt{h'}\right)\Phi^{-1}(A\sharp_\nu B)\right\| \leq \frac{(M+m)^2}{4}.$$

Using Lemma 2.2, inequalities (10), (11) and the linear property of Φ, we obtain

$$\begin{aligned}
&\left\|\Phi\left(A\nabla_v B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) MmK^{r_1}\left(\sqrt{h'}\right)\Phi^{-1}(A\sharp_\nu B)\right\| \\
&\leq \frac{1}{4}\Big\|\Phi\Big(A\nabla_v B + 2rMm\Big(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\Big)\Big) \\
&\quad + MmK^{r_1}\left(\sqrt{h'}\right)\Phi\Big(A^{-1}\sharp_\nu B^{-1}\Big)\Big\|^2 \\
&= \frac{1}{4}\Big\|\Phi(A\nabla_v B) + Mm\Big(\Phi\Big(2r\Big(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\Big) \\
&\quad + K^{r_1}\left(\sqrt{h'}\right)\Big(A^{-1}\sharp_\nu B^{-1}\Big)\Big)\Big)\Big\|^2 \\
&\leq \frac{1}{4}\Big\|\Phi\left(A\nabla_v B\right) + Mm\Phi\Big(A^{-1}\nabla_v B^{-1}\Big)\Big\|^2 \\
&\leq \frac{(M+m)^2}{4}.
\end{aligned}$$

The last above inequality holds since by our assumptions,

$$A + MmA^{-1} \leq Mm \qquad \text{and} \qquad B + MmB^{-1} \leq Mm.$$

By multiplying the inequalities above by $(1-\nu)$ and ν, respectively, and then summing up the derived inequalities, we get

$$A\nabla_v B + Mm\Big(A^{-1}\nabla_v B^{-1}\Big) \leq M + m.$$

Since Φ is a positive linear map, we obtain

$$\Phi\left(A\nabla_v B\right) + Mm\Phi\Big(A^{-1}\nabla_v B^{-1}\Big) \le M + m.$$

So, inequality (13) holds.

□

Corollary 2.5. *Suppose that $A, B \in \mathbb{B}(\mathbf{H})$ are positive and $m, m^{'}, M, M^{'}$ are positive real numbers satisfying either one of the following conditions:*

(1) $0 < m \le B \le m^{'} < M^{'} \le A \le M$;
(2) $0 < m \le A \le m^{'} < M^{'} \le B \le M$.

Then for every positive unital linear map Φ, $0 \le \nu \le 1$ and for every $0 < p \le 2$

$$\Phi^p\big(A\nabla_v B + 2rMm\big(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\big)\big) \le \left(\frac{K(h)}{K^{r_1}\left(\sqrt{h^{'}}\right)}\right)^p \Phi^p(A\sharp_\nu B)$$

and

$$\Phi^p\big(A\nabla_v B + 2rMm\big(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\big)\big) \le \left(\frac{K(h)}{K^{r_1}\left(\sqrt{h^{'}}\right)}\right)^p (\Phi(A)\sharp_\nu \Phi(B))^p,$$

where $r = \min\{\nu, 1-\nu\}$, $K(h) = \frac{(1+h)^2}{4h}$, $h = \frac{M}{m}$, $h^{'} = \frac{M^{'}}{m^{'}}$ *and* $r_1 = \min\{2r, 1-2r\}$.

Proof. If $0 < p \le 2$, then $0 < \frac{p}{2} \le 1$. Using inequalities (3), (13) and (14), we get the desired results. □

Theorem 2.6. *Suppose that $A, B \in \mathbb{B}(\mathbf{H})$ are positive and $m, m^{'}, M, M^{'}$ are positive real numbers satisfying either one of the following conditions:*

(1) $0 < m \le B \le m^{'} < M^{'} \le A \le M$;
(2) $0 < m \le A \le m^{'} < M^{'} \le B \le M$.

Then for every positive unital linear map Φ, $0 \le \nu \le 1$ and for every $p \ge 2$, we have

$$\Phi^p\big(A\nabla_v B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\big) \le \left(\frac{K(h)}{4^{\frac{2}{p}-1}K^{r_1}\left(\sqrt{h^{'}}\right)}\right)^p \Phi^p(A\sharp_\nu B) \tag{15}$$

and

$$\Phi^p\big(A\nabla_v B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\big) \leq \left(\frac{K(h)}{4^{\frac{2}{p}-1}K^{r_1}\left(\sqrt{h'}\right)}\right)^p (\Phi(A)\sharp_\nu\Phi(B))^p, \tag{16}$$

where $r = \min\{\nu, 1-\nu\}$, $K(h) = \frac{(1+h)^2}{4h}$, $K(h^{'}) = \frac{(1+h^{'})^2}{4h^{'}}$, $h = \frac{M}{m}$, $h^{'} = \frac{M^{'}}{m^{'}}$ *and* $r_1 = \min\{2r, 1-2r\}$.

Proof. Since the proof of inequality (16) is similar to the proof of inequality (15), we only prove inequality (15). By Lemma 2.2, inequality (15) is equivalent to

$$\left\|\Phi^{\frac{p}{2}}\left(A\nabla_v B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right)\Phi^{-\frac{p}{2}}(A\sharp_\nu B)\right\| \leq \left(\frac{K(h)}{4^{\frac{2}{p}-1}K^{r_1}\left(\sqrt{h'}\right)}\right)^{\frac{p}{2}}.$$

Using Lemma 2.2, inequalities (10), (11), and applying the same reasoning as in the last inequality of Theorem 2.4, we have

$$\begin{aligned}
& M^{\frac{p}{2}}m^{\frac{p}{2}}\left\|\Phi^{\frac{p}{2}}\left(A\nabla_v B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right)K^{\frac{r_1 p}{2}}\left(\sqrt{h'}\right)\Phi^{-\frac{p}{2}}(A\sharp_\nu B)\right\| \\
&= \left\|\Phi^{\frac{p}{2}}\left(A\nabla_v B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right)M^{\frac{p}{2}}m^{\frac{p}{2}}K^{\frac{pr_1}{2}}\left(\sqrt{h'}\right)\Phi^{-\frac{p}{2}}(A\sharp_\nu B)\right\| \\
&\leq \frac{1}{4}\left\|\Phi^{\frac{p}{2}}\left(A\nabla_v B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right)\right. \\
&\quad \left. + M^{\frac{p}{2}}m^{\frac{p}{2}}K^{\frac{pr_1}{2}}\left(\sqrt{h'}\right)\Phi^{-\frac{p}{2}}(A\sharp_\nu B)\right\|^2 \\
&\leq \frac{1}{4}\left\|\Big(\Phi\left(A\nabla_v B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right)\right. \\
&\quad \left. + MmK^{r_1}\left(\sqrt{h'}\right)\Phi^{-1}(A\sharp_\nu B)\Big)^{\frac{p}{2}}\right\|^2 \\
&= \frac{1}{4}\left\|\Phi\left(A\nabla_v B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) + MmK^{r_1}\left(\sqrt{h'}\right)\Phi^{-1}(A\sharp_\nu B)\right\|^p \\
&\leq \frac{1}{4}\left\|\Phi\left(A\nabla_v B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) + MmK^{r_1}\left(\sqrt{h'}\right)\Phi(A^{-1}\sharp_\nu B^{-1})\right\|^p \\
&= \frac{1}{4}\left\|\Phi(A\nabla_v B) + Mm\Big(\Phi\Big(2r\Big(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\Big) + K^{r_1}\left(\sqrt{h'}\right)\Big(A^{-1}\sharp_\nu B^{-1}\Big)\Big)\Big)\right\|^p \\
&\leq \frac{1}{4}\left\|\Phi(A\nabla_v B) + Mm\Phi\Big(A^{-1}\nabla B^{-1}\Big)\right\|^p \\
&\leq \frac{1}{4}(M+m)^p.
\end{aligned}$$

Thus we get the desired result. □

Remark. For $p \geq 1$, we have

$$\Phi^p(A\nabla_v B) \leq \Phi^p(A\nabla_v B) + (2rMm)^p\Phi^p\Big(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\Big).$$

On the other hand, Lemma 2.2 yields that

$$\begin{aligned}\|\Phi^p(A\nabla_v B)\| &\leq \left\|\Phi^p(A\nabla_v B) + (2rMm)^p\Phi^p\Big(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\Big)\right\| \\ &\leq \left\|\Phi^p\Big(A\nabla_v B + 2rMm\Big(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\Big)\Big)\right\|.\end{aligned}$$

Therefore, Theorem 2.6 is a refinement of Theorem 1.3 for the operator norm and $p \geq 2$.

Remark. Since the Kantorovich constant $K(h)$ is an increasing function on the interval $(1, +\infty)$ and also $K(h) \geq 1$ for every $h > 0$, so Theorem 2.6 is a refinement of Theorem 1.2; see [7].

Zhang [19] obtained the following inequalities for $p \geq 4$:

$$\Phi^p(A\nabla B) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{4^{\frac{2}{p}}Mm}\right)^p \Phi^p(A\sharp B);$$

$$\Phi^p(A\nabla B) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{4^{\frac{2}{p}}Mm}\right)^p (\Phi(A)\sharp\Phi(B))^p.$$

Recently, the authors of [18] improved the above inequalities as follows:

$$\Phi^p(A\nabla_\nu B) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{4^{\frac{2}{p}}MmK^r(h')}\right)^p \Phi^p(A\sharp_\nu B); \tag{17}$$

$$\Phi^p(A\nabla_\nu B) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{4^{\frac{2}{p}}MmK^r(h')}\right)^p (\Phi(A)\sharp_\nu\Phi(B))^p. \tag{18}$$

In the following theorem, we show some refinements of inequalities (17) and (18).

Theorem 2.7. *Let $A, B \in \mathbb{B}(\mathbf{H})$ are positive and $m, m^{'}, M, M^{'}$ are positive real numbers satisfying either one of the following conditions:*

(1) $0 < m \leq B \leq m^{'} < M^{'} \leq A \leq M$;
(2) $0 < m \leq A \leq m^{'} < M^{'} \leq B \leq M$.

Then for every positive unital linear map Φ, $0 \leq \nu \leq 1$ and for every $p \geq 4$

$$\begin{aligned}&\Phi^p\big(A\nabla_v B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\big) \\ &\qquad \leq \left(\frac{K(h)\left(M^2+m^2\right)}{4^{\frac{2}{p}}MmK^r(h')}\right)^p \Phi^p(A\sharp_\nu B)\end{aligned} \tag{19}$$

and

$$\Phi^p\big(A\nabla_v B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\big) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{4^{\frac{2}{p}}MmK^r(h')}\right)^p(\Phi(A)\sharp_\nu\Phi(B))^p, \tag{20}$$

where $r=\min\{\nu,1-\nu\}$, $K(h)=\frac{(1+h)^2}{4h}$, $h=\frac{M}{m}$, $h^{'}=\frac{M^{'}}{m^{'}}$ *and* $r_1=\min\{2r,1-2r\}$.

Proof. It follows from Lemma 2.2 and Theorem 2.4 that

$$\begin{aligned}
& M^{\frac{p}{2}}m^{\frac{p}{2}}\left\|\Phi^{\frac{p}{2}}\left(A\nabla_v B+2rMm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right)\right)\Phi^{-\frac{p}{2}}(A\sharp_\nu B)\right\| \\
= & \left\|\Phi^{\frac{p}{2}}\left(A\nabla_v B+2rMm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right)\right)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\nu B)\right\| \\
\leq & \frac{1}{4}\Big\|\frac{K^{\frac{r_1p}{4}}\left(\sqrt{h'}\right)}{K^{\frac{p}{4}}(h)}\Phi^{\frac{p}{2}}\left(A\nabla_v B+2rMm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right)\right) \\
& +\left(\frac{M^2m^2K(h)}{K^{r_1}\left(\sqrt{h'}\right)}\right)^{\frac{p}{4}}\Phi^{-\frac{p}{2}}(A\sharp_\nu B)\Big\|^2 \\
\leq & \frac{1}{4}\Big\|\Big(\frac{K^{r_1}\left(\sqrt{h'}\right)}{K(h)}\Phi^{2}\left(A\nabla_v B+2rMm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right)\right) \\
& +\frac{M^2m^2K(h)}{K^{r_1}\left(\sqrt{h'}\right)}\Phi^{-2}(A\sharp_\nu B)\Big)^{\frac{p}{4}}\Big\|^2 \\
= & \frac{1}{4}\Big\|\frac{K^{r_1}\left(\sqrt{h'}\right)}{K(h)}\Phi^{2}\left(A\nabla_v B+2rMm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right)\right) \\
& +\frac{M^2m^2K(h)}{K^{r_1}\left(\sqrt{h'}\right)}\Phi^{-2}(A\sharp_\nu B)\Big\|^{\frac{p}{2}} \\
\leq & \frac{1}{4}\Big\|\frac{K(h)}{K^{r_1}\left(\sqrt{h'}\right)}\Big(\Phi^2\left(A\sharp_v B\right)+M^2m^2\Phi^{-2}\left(A\sharp_v B\right)\Big)\Big\|^{\frac{p}{2}} \\
\leq & \frac{1}{4}\left(\frac{K(h)(M^2+m^2)}{K^{r_1}\left(\sqrt{h'}\right)}\right)^{\frac{p}{2}}.
\end{aligned}$$

It follows from $0<m\leq A\sharp_\nu B\leq M$ and the linearity Φ that $0<m\leq \Phi\left(A\sharp_\nu B\right)\leq M$. In addition, for every $T\in\mathbb{B}(\mathbf{H})$ such that $0<m\leq T\leq M$, we

have

$$M^2m^2T^{-2}+T^2\leq M^2+m^2.$$

Now, by putting $\Phi\left(A\sharp_\nu B\right)$ in the latter inequality, we obtain the last inequality. Hence,

$$M^{\frac{p}{2}}m^{\frac{p}{2}}\left\|\Phi^{\frac{p}{2}}\left(A\nabla_v B+2rMm\big(A^{-1}\nabla B^{-1}A^{-1}\sharp B^{-1}\big)\right)\Phi^{-\frac{p}{2}}(A\sharp_\nu B)\right\| \leq\left(\frac{K(h)\left(M^2+m^2\right)}{4^{\frac{2}{p}}MmK^r(h')}\right)^{\frac{p}{2}}.$$

By Lemma 2.2, the last inequality implies inequality (19). Analogously, we can prove inequality (20). □

Remark. Note that inequalities (19) and (20) are refinements of (17) and (18) for the operator norm, respectively.

Theorem 2.8. *Let $A,B\in\mathbb{B}(\mathbf{H})$ are positive and m,m',M,M' positive real numbers satisfying either one of the following conditions:*

(1) $0<m\leq B\leq m'<M'\leq A\leq M$.
(2) $0<m\leq A\leq m'<M'\leq B\leq M$.

Then for every positive unital linear map Φ and $0\leq\nu\leq 1$

$$\Phi^p\big(A\nabla_v B+2rMm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right)\big) \leq\frac{\left(K^{-\frac{r_1\alpha}{2}}\left(\sqrt{h'}\right)K^{\frac{\alpha}{2}}(h)(M^\alpha+m^\alpha)\right)^{\frac{2p}{\alpha}}}{16M^pm^p}\Phi^p(A\sharp_\nu B), \tag{21}$$

$$\Phi^p\big(A\nabla_v B+2rMm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right)\big) \leq\frac{\left(K^{-\frac{r_1\alpha}{2}}\left(\sqrt{h'}\right)K^{\frac{\alpha}{2}}(h)(M^\alpha+m^\alpha)\right)^{\frac{2p}{\alpha}}}{16M^pm^p}(\Phi(A)\sharp_\nu\Phi(B))^p, \tag{22}$$

where $1\leq\alpha\leq 2$, $K(h)=\frac{(1+h)^2}{4h}$, and $p\geq 2\alpha$.

PROOF. By Lemma 2.2, inequality (21) is equivalent to the following inequality

$$\left\|\Phi^{\frac{p}{2}}\left(A\nabla_v B+2rMm\big(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\big)\right)\Phi^{-\frac{p}{2}}(A\sharp_\nu B)\right\| \leq\frac{\left(K^{-\frac{r_1\alpha}{2}}\left(\sqrt{h'}\right)K^{\frac{\alpha}{2}}(h)(M^\alpha+m^\alpha)\right)^{\frac{p}{\alpha}}}{4M^{\frac{p}{2}}m^{\frac{p}{2}}}.$$

By using Lemma 2.2 and Theorem 2.4, one can obtain

$$\begin{aligned}
& M^{\frac{p}{2}} m^{\frac{p}{2}} \left\| \Phi^{\frac{p}{2}} \left(A\nabla_v B + 2rMm \left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1} \right) \right) \Phi^{-\frac{p}{2}} (A\sharp_\nu B) \right\| \\
= & \left\| \Phi^{\frac{p}{2}} \left(A\nabla_v B + 2rMm \left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1} \right) \right) M^{\frac{p}{2}} m^{\frac{p}{2}} \Phi^{-\frac{p}{2}} (A\sharp_\nu B) \right\| \\
\leq & \frac{1}{4} \Bigg\| \frac{K^{\frac{r_1 p}{4}}\left(\sqrt{h'}\right)}{K^{\frac{p}{4}}(h)} \Phi^{\frac{p}{2}} \left(A\nabla_v B + 2rMm \left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1} \right) \right) \\
& + \left(\frac{M^2 m^2 K(h)}{K^{r_1}\left(\sqrt{h'}\right)} \right)^{\frac{p}{4}} \Phi^{-\frac{p}{2}} (A\sharp_\nu B) \Bigg\|^2 \\
\leq & \frac{1}{4} \Bigg\| \Bigg(\frac{K^{\frac{r_1 \alpha}{2}}\left(\sqrt{h'}\right)}{K^{\frac{\alpha}{2}}(h)} \Phi^{\alpha} \left(A\nabla_v B + 2rMm \left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1} \right) \right) \\
& + \frac{M^\alpha m^\alpha K^{\frac{\alpha}{2}}(h)}{K^{\frac{r_1 \alpha}{2}}\left(\sqrt{h'}\right)} \Phi^{-\alpha} (A\sharp_\nu B) \Bigg)^{\frac{p}{2\alpha}} \Bigg\|^2 \\
= & \frac{1}{4} \Bigg\| \frac{K^{\frac{r_1 \alpha}{2}}\left(\sqrt{h'}\right)}{K^{\frac{\alpha}{2}}(h)} \Phi^{\alpha} \left(A\nabla_v B + 2rMm \left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1} \right) \right) \\
& + \frac{M^\alpha m^\alpha K^{\frac{\alpha}{2}}(h)}{K^{\frac{r_1 \alpha}{2}}\left(\sqrt{h'}\right)} \Phi^{-\alpha} (A\sharp_\nu B) \Bigg\|^{\frac{p}{\alpha}} \\
\leq & \frac{1}{4} \Bigg\| \frac{K^{\frac{\alpha}{2}}(h)}{K^{\frac{r_1 \alpha}{2}}\left(\sqrt{h'}\right)} \left(\Phi^{\alpha} \left(A\sharp_v B \right) + M^\alpha m^\alpha \Phi^{-\alpha} \left(A\sharp_v B \right) \right) \Bigg\|^{\frac{p}{\alpha}} \\
\leq & \frac{1}{4} \left(\frac{K^{\frac{\alpha}{2}}(h)(M^\alpha + m^\alpha)}{K^{\frac{r_1 \alpha}{2}}\left(\sqrt{h'}\right)} \right)^{\frac{p}{\alpha}}.
\end{aligned}$$

By the property of the arithmetic mean (see [7])

$$m = m\sharp_\nu m \leq A\sharp_\nu B \leq M\sharp_\nu M = M.$$

Since Φ is linear, we have

$$0 < m \leq \Phi\left(A\sharp_\nu B\right) \leq M.$$

On the other hand, for every $T \in \mathbb{B}(\mathbf{H})$ such that $0 < m \leq T \leq M$, we have $0 < T^\alpha - m^\alpha$ and $0 < T^{-\alpha} - M^{-\alpha}$, whence $0 < (T^\alpha - m^\alpha)(T^{-\alpha} - M^{-\alpha})$ or equivalently

$$M^\alpha m^\alpha T^{-\alpha} + T^\alpha \leq M^\alpha + m^\alpha.$$

Now, by setting $\Phi\left(A\sharp_\nu B\right)$ in the latter inequality we obtain the last inequality. This proves inequality (21). By utilizing the same ideas as in the proof of inequality (21), we can reach inequality (22). □

Remark. If we take $\alpha = 1, 2$, then Theorem 2.8 reduces to Theorem 2.7 and Theorem 2.6, respectively.

3. Reverse of Ando's inequality

For positive operators $A, B \in \mathbb{B}(\mathbf{H})$, we know [4] that for every positive unital linear map Φ

$$\Phi(A\sharp B) \leq \Phi(A)\sharp\Phi(B). \tag{23}$$

Ando's inequality says that if A, B be positive operators and Φ be a positive unital linear map, then

$$\Phi(A\sharp_\nu B) \leq \Phi(A)\sharp_\nu\Phi(B). \tag{24}$$

The author [11] presented the following theorem that can be viewed as a reversed version of (23).

Theorem 3.1. *If $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$, then for every positive linear map Φ and some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$*

$$\Phi(A)\sharp\Phi(B) \leq \frac{\sqrt{M} + \sqrt{m}}{2\sqrt{Mm}}\Phi(A\sharp B), \tag{25}$$

where $m = \frac{m_2}{M_1}$ and $M = \frac{M_2}{m_1}$.
Seo [16] improved inequality above and obtained the following inequality:

Theorem 3.2. *Let $A, B \in \mathbb{B}(\mathbf{H})$ be positive such that $0 < m_1^2 \leq A \leq M_1^2$, $m_2^2 \leq B \leq M_2^2$, $m = \left(\frac{m_2}{M_1}\right)^2$ and $M = \left(\frac{M_2}{m_1}\right)^2$. Then for every positive unital linear map Φ and $0 \leq \nu \leq 1$*

$$\Phi(A)\sharp_\nu\Phi(B) \leq K(m, M, \nu)^{-1}\Phi(A\sharp_\nu B), \tag{26}$$

where $K(m, M, \nu) = \frac{mM^\nu - Mm^\nu}{(\nu-1)(M-m)}\left(\frac{\nu-1}{\nu}\frac{M^\nu - m^\nu}{mM^\nu - Mm^\nu}\right)^\nu$.

In this section, we give a refinement of inequality (26). To achieve this, we need the following theorem:

Theorem 3.3. [20] *Suppose that $A, B \in \mathbb{B}(\mathbf{H})$ are positive and m, m', M, M' are positive real numbers satisfying either one of the following conditions:*

(1) $0 < m \leq B \leq m' < M' \leq A \leq M$;
(2) $0 < m \leq A \leq m' < M' \leq B \leq M$.

Then for $0 \leq \nu \leq 1$

$$A\nabla_v B \geq K^r(h)(A\sharp_\nu B), \tag{27}$$

where $r = \min\{\nu, 1-\nu\}$, $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$.

Theorem 3.4. *Let* $A, B \in \mathbb{B}(\mathbf{H})$ *such that* $0 < m_1^2 \leq A \leq M_1^2$, $m_2^2 \leq B \leq M_2^2$, $m = \left(\frac{m_2}{M_1}\right)^2$ *and* $M = \left(\frac{M_2}{m_1}\right)^2$. *If* $M_1 < m_2$, *then for every positive unital linear map* Φ *and* $0 \leq \nu \leq 1$,

$$\Phi(A)\sharp_\nu \Phi(B) \leq K(m, M, \nu)^{-1} K(h)^{-r} \Phi(A\sharp_\nu B), \tag{28}$$

where $K(m, M, \nu) = \frac{mM^\nu - Mm^\nu}{(\nu-1)(M-m)} \left(\frac{\nu-1}{\nu} \frac{M^\nu - m^\nu}{mM^\nu - Mm^\nu}\right)^\nu$, $r = \min\{\nu, 1-\nu\}$, $K(h) = \frac{(1+h)^2}{4h}$ *and* $h = \frac{m_2^2}{M_1^2}$. *Similarly, one can prove the inequality for* $M_2 < m_1$ *and* $h = \frac{M_2^2}{m_1^2}$.

Proof. For $t \in [m, M]$. We put $F(t) = \nu t^{1-\nu} + (1-\nu)\lambda_0 t^{-\nu}$, where

$$\mu_0 = \frac{\nu(M-m)}{M^\nu - m^\nu} \qquad \lambda_0 = \frac{\nu}{1-\nu} \frac{M^{1-\nu} - m^{1-\nu}}{m^{-\nu} - M^{-\nu}}.$$

Easy computation shows that $\max_{t\in[m,M]} F(t) = F(M) = F(m)$ and $F(M) = F(m) = \mu_0$. Hence

$$\nu t^{1-\nu} + (1-\nu)\lambda_0 t^{-\nu} \leq \mu_0. \tag{29}$$

Using the fact that $0 < m_1^2 \leq A \leq M_1^2$ and $m_2^2 \leq B \leq M_2^2$, we get $mI \leq C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq MI$. Considering inequality (29) with $C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$, we obtain

$$\nu C + (1-\nu)\lambda_0 I \leq \mu_0 C^\nu.$$

Multiplying both sides of the latter inequality by $A^{\frac{1}{2}}$, we have

$$\nu\Phi(B) + (1-\nu)\lambda_0 \Phi(A) \leq \mu_0 \Phi(A\sharp_\nu B). \tag{30}$$

Using (27) for two operators $\lambda_0\Phi(A)$ and $\Phi(B)$ yields that

$$\lambda_0^{1-\nu} \Phi(A)\sharp_\nu \Phi(B) \leq K(h, 2)^{-r}(\nu\Phi(B) + (1-\nu)\lambda_0\Phi(A)). \tag{31}$$

From (30) and (31), we obtain inequality (28). □

Remark. Note that the right side of inequality (28) is a better bound than inequality (26), since the Kantorovich constant $K(h)$ is increasing on the interval $(1, +\infty)$.

Remark. If we put $\nu = \frac{1}{2}$ in Theorem 3.4, then we obtain a refinement of (25), since the Kantorovich constant $K(h)$ is increasing on the interval $(1, +\infty)$.

References

[1] T. Ando, *Positive map as difference of two positive completely of super-positive maps*, Adv. Oper. Theory 3 (2018), no. 1, 53–60.

[2] M. Bakherad, *Refinements of a reversed AM-GM operator inequality*, Linear Multilinear Algebra 64 (2016), no. 9, 1687–1695.

[3] M. Bakherad, M. Krnić and M.S. Moslehian, *Reverse Young-type inequalities for matrices and operators*, Rocky Mountain J. Math. 46 (2016), no. 4, 1089–1105.

[4] R. Bhatia, *Positive Definite Matrices*, Princeton University Press, Princeton, 2007.

[5] R. Bhatia and F. Kittaneh, *Notes on matrix arithmetic-geometric mean inequalities*, Linear Algebra Appl. **308** (2000), no. 1-3, 203–211.

[6] X. Fu and C. He, *Some operator inequalities for positive linear maps*, Linear Multilinear Algebra **63** (2015), no. 3, 571–577.

[7] T. Furuta, J. Mićić Hot, J. Pecarić and Y. Seo, *Mond-Pecarić method in operator inequalities*, Element, Zagreb, 2005.

[8] M. Fujii and R. Nakamoto, *Refinements of Holder-McCarthy inequality and Young inequality*, Adv. Oper. Theory 1 (2016), no. 2, 184-188.

[9] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann. 246 (1980), 205–224.

[10] M. Hajmohamadi, M. Bakherad and R. Lashkaripour, *Some matrix power and Karcher means inequalities involving positive linear maps*, Filomat 32 (2018), no. 7, 2625–2634.

[11] E-Y. Lee, *A matrix reverse Cauchy-Schwarz inequality*, Linear Algebra Appl. 430 (2009), 805–810.

[12] M. Lin, *Squaring a reverse AM-GM inequality*, Studia Math. 215 (2013), no. 2, 187–194.

[13] M. Lin, *On an operator Kantorovich inequality for positive linear maps*, J. Math. Anal. Appl. 402 (2013), no. 1, 127–132.

[14] M.S. Moslehian and X. Fu, *Squaring operator Polya-Szego and Diaz-Metcalf type inequalities*, Linear Algebra Appl. 491 (2016), 73–82.

[15] M.S. Moslehian, R. Nakamoto and Y. Seo, *A Diaz-Metcalf type inequality for positive linear maps and its applications*, Electron. J. Linear Algebra. 22 (2011), 179–190.

[16] Y. Seo, *Reverses of Ando's inequality for positive linear maps*, Math. Inequal. Appl. 14 (2011), 905–910.

[17] J.L. Wu and J. G. Zhao, *Operator inequalities and reverse inequalities related to the Kittaneh-Manasrah inequalities*, Linear Multilinear Algebra 62 (2014), no. 7, 884–894.

[18] C. Yang and D. Wang, *Some refinements of operator inequalities for positive linear maps*, J. Math. Inequal. 11 (2017), 17–26.

[19] P. Zhang, *More operator inequalities for positive linear maps*, Banach J. Math. Anal. 9 (2015), 166–172.

[20] H. Zuo, G. Shi and M. Fujii, *Refined Young inequalities with Kantorovich constant*, J. Math. Inequal. 75 (1995), 33–38.

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