Journal of
Mathematical
Inequalities
Volume 12, Number 2 (2018), 447–457 doi:10.7153/jmi-2018-12-33

SOME GENERALIZATIONS OF NUMERICAL RADIUS ON OFF–DIAGONAL PART OF 2×2 OPERATOR MATRICES

MONIRE HAJMOHAMADI, RAHMATOLLAH LASHKARIPOUR AND MOJTABA BAKHERAD

(*Communicated by J. Pečarić*)

Abstract. We generalize several inequalities involving powers of the numerical radius for off-diagonal part of 2×2 operator matrices of the form $T=\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, where B,C are two operators. In particular, if $T=\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, then we get

$$\frac{1}{2^{\frac{3}{2}(r-1)}}\max\{\|\mu\|,\|\eta\|\}\leqslant w^r(T)\leqslant \frac{1}{2^{r+1}}\max\{\|\mu\|,\|\eta\|\},$$

where $r\geqslant 2$, $\mu=|(C-B^*)+i(C+B^*)|^r+|(B^*-C)+i(C+B^*)|^r$ and $\eta=|(B-C^*)+i(B+C^*)|^r+|(C^*-B)+i(B+C^*)|^r$.

1. Introduction

Let $(\mathscr{H},\langle .,.\rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathscr{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathscr{H}$. In the case when $\dim\mathscr{H}=n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n\times n$ matrices with entries in the complex field. The numerical radius of $T\in\mathbb{B}(\mathscr{H})$ is defined by

$$w(T):=\sup\{|\langle Tx,x\rangle| : x\in\mathscr{H},\|x\|=1\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|.\|$. In fact, for any $T\in\mathbb{B}(\mathscr{H})$, $\frac{1}{2}\|T\|\leqslant w(T)\leqslant\|T\|$; see [11]. An important inequality for $w(A)$ is the power inequality stating that $w(A^n)\leqslant w(A)^n$ $(n=1,2,\cdots)$. It has been shown in [8], that if $T\in\mathbb{B}(\mathscr{H})$, then

$$w(T)\leqslant\frac{1}{2}\||T|+|T^*|\|, \tag{1.1}$$

where $|T|=(T^*T)^{\frac{1}{2}}$ is the absolute value of T. Recently in [12] the authors showed

$$w^{2r}(T)\leqslant\frac{1}{2}\left(\|A\|^{2r}+\left\|\frac{1}{p}f^{pr}(|A^2|)+\frac{1}{q}g^{qr}(|(A^*)^2|)\right\|\right), \tag{1.2}$$

Mathematics subject classification (2010): Primary 47A12, secondary 47A30, 47A63, 47B33.

Keywords and phrases: Cartesian decomposition, Jensen inequality, numerical radius, off-diagonal part, operator mean, operator matrix, positive operator, Young inequality.

Paper JMI-12-33

in which f, g are nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t)=t$ $(t\in[0,\infty))$, $r\geqslant 1$, $p\geqslant q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $pr\geqslant 2$.

Let $\mathscr{H}_1,\mathscr{H}_2,\cdots,\mathscr{H}_n$ be Hilbert spaces, and consider the direct sum $\mathscr{H}=\bigoplus_{j=1}^n\mathscr{H}_j$. With respect to this decomposition, every operator $T\in\mathbb{B}(\mathscr{H})$ has an $n\times n$ operator matrix represantation $T=[T_{ij}]$ with entries $T_{ij}\in\mathbb{B}(\mathscr{H}_j,\mathscr{H}_i)$, the space of all bounded linear operators from $\mathscr{H}_j$ to $\mathscr{H}_i$. Operator matrices provide a usual tool for studying Hilbert space operators, which have been extensively studied in the literatures. The classical Young inequality says that if $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then $ab\leqslant\frac{a^p}{p}+\frac{b^q}{q}$ for positive real numbers a,b. A refinement of the scalar Young inequality is presented in [3] as following $(a^{\frac{1}{p}}b^{\frac{1}{q}})^m+r_0^m(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2\leqslant(\frac{a}{p}+\frac{b}{q})^m$, where $r_0=\min\{\frac{1}{p},\frac{1}{q}\}$ and $m=1,2,\cdots$. In particular, if $p=q=2$, then

$$(a^{\frac{1}{2}}b^{\frac{1}{2}})^m+\left(\frac{1}{2}\right)^m(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2\leqslant 2^{-m}(a+b)^m. \tag{1.3}$$

Let $T_1,T_2,\cdots,T_n\in\mathbb{B}(\mathscr{H})$. The functional w_p of operators $T_1,\cdots,T_n$ for $p\geqslant 1$ is defined in [13] as following

$$w_p(T_1,\cdots,T_n):=\sup_{\|x\|=1}\Big(\sum_{i=1}^n|\langle T_ix,x\rangle|^p\Big)^{\frac{1}{p}}.$$

In [14] the authors showed the following inequality

$$w_p^p(A_1^*T_1B_1,\cdots,A_n^*T_nB_n)\leqslant\frac{1}{2}\left\|\sum_{i=1}^n\left([B_i^*f^2(|T_i|)B_i]^p+[A_i^*g^2(|T_i^*|)A_i]^p\right)\right\|-\inf_{\|X\|=1}\zeta(X),$$

where $A_i,B_i,T_i\in\mathbb{B}(\mathscr{H})$ $(i=1,2,\cdots,n)$, f, g are nonnegative continuous functions on $[0,\infty)$ such that $f(t)g(t)=t$ $(t\in[0,\infty))$, $p,r\geqslant m$, $m=1,2,\cdots$, and

$$\zeta(X)=2^{-m}\sum_{i=1}^n\left(\langle[B_i^*f^2(|T_i|)B_i]^{\frac{p}{m}}x,x\rangle^{\frac{m}{2}}-\langle[A_i^*g^2(|T_i^*|)A_i]^{\frac{p}{m}}x,x\rangle^{\frac{m}{2}}\right)^2.$$

For further information about numerical radius inequalities we refer the reader to [1, 4, 14] and references therein.

In this paper, we establish some generalizations of inequalities that is based on the off-diagonal parts of 2×2 operator matrices. We also show some inequalities involving powers of the numerical radius for the off-diagonal parts of 2×2 operator matrices.

2. Main results

To prove our first result, we need several well known lemmas.

LEMMA 2.1. [6, 15] *Let $A\in\mathbb{B}(\mathscr{H}_1)$, $B\in\mathbb{B}(\mathscr{H}_2,\mathscr{H}_1)$, $C\in\mathbb{B}(\mathscr{H}_1,\mathscr{H}_2)$ and $D\in\mathbb{B}(\mathscr{H}_2)$. Then the following statements hold:*

(a) $w\left(\begin{bmatrix}A&0\\0&D\end{bmatrix}\right)=\max\{w(A),w(D)\}$;

$(b)\ w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & C \\ B & 0 \end{bmatrix}\right);$

$(c)\ w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \frac{1}{2}\sup_{\theta\in\mathscr{R}} \| e^{i\theta}B + e^{-i\theta}C^* \|;$

$(d)\ w\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = \max\{w(A+B), w(A-B)\}.$

In particular,

$$w\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = w(B).$$

The second lemma is a simple consequence of the classical Jensen and Young inequalities; see [5].

LEMMA 2.2. *Let $a,b \geqslant 0$ and $p,q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$ab \leqslant \frac{a^p}{p}+\frac{b^q}{q} \leqslant \left(\frac{a^{pr}}{p}+\frac{b^{qr}}{q}\right)^{\frac{1}{r}}$$

for $r \geqslant 1$.

The next lemma follows from the spectral theorem for positive operators and Jensen inequality; see [7].

LEMMA 2.3. (McCarty inequality) *Let $T \in \mathbb{B}(\mathscr{H})$, $T \geqslant 0$ and $x \in \mathscr{H}$ be a unit vector. Then*

(a) $\langle Tx,x\rangle^r \leqslant \langle T^r x,x\rangle$ *for* $r \geqslant 1$;

(b) $\langle T^r x,x\rangle \leqslant \langle Tx,x\rangle^r$ *for* $0 < r \leqslant 1$.

The following lemma is a consequence of convexity of the absolute value function.

LEMMA 2.4. *Let $T \in \mathbb{B}(\mathscr{H})$ be self-adjoint and $x \in \mathscr{H}$ be a unit vector. Then*

$$| \langle Tx,x\rangle | \leqslant \langle | T | x,x\rangle.$$

LEMMA 2.5. [7, Theorem 1] *Let $T \in \mathbb{B}(\mathscr{H})$ and $x,y \in \mathscr{H}$ be any vectors.*

(a) *If f, g are nonnegative continuous functions on $[0,\infty)$ which are satisfying the relation $f(t)g(t) = t$ $(t \in [0,\infty))$, then*

$$| \langle Tx,y\rangle | \leqslant \| f(| T |)x \| \| g(| T^* |)x \|;$$

(b) *If $0 \leqslant \alpha \leqslant 1$, then*

$$| \langle Tx,y\rangle |^2 \leqslant \langle | T |^{2\alpha} x,x\rangle \langle | T^* |^{2(1-\alpha)} y,y\rangle.$$

Now we are in a position to state the main results of this section.

THEOREM 2.6. *Let* $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(\mathscr{H}_2, \mathscr{H}_1)$ *and* f, g *be nonnegative continuous functions on* $[0,\infty)$ *satisfying the relation* $f(t)g(t) = t$ $(t \in [0,\infty))$. *Then*

$$w^r(T) \leqslant \max\left\{\left\|\frac{1}{p}f^{pr}(|C|) + \frac{1}{q}g^{qr}(|B^*|)\right\|, \left\|\frac{1}{p}f^{pr}(|B|) + \frac{1}{q}g^{qr}(|C^*|)\right\|\right\}, \quad (2.1)$$

in which $r \geqslant 1$, $p \geqslant q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $pr \geqslant 2$.

Proof. For any unit vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathscr{H}_1 \oplus \mathscr{H}_2$ we have

$$\begin{aligned}
|\langle TX, X\rangle|^r &\leqslant \| f(|T|)X\|^r \| g(|T^*|)X\|^r \qquad \text{(by Lemma 2.5)} \\
&= \langle f^2(|T|)X, X\rangle^{\frac{r}{2}} \langle g^2(|T^*|)X, X\rangle^{\frac{r}{2}} \\
&\leqslant \frac{1}{p}\left\langle f^2\left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix}\right)X, X\right\rangle^{\frac{pr}{2}} + \frac{1}{q}\left\langle g^2\left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix}\right)X, X\right\rangle^{\frac{qr}{2}} \\
&\qquad \text{(by Lemma 2.2)} \\
&\leqslant \frac{1}{p}\left\langle \begin{bmatrix} f^{pr}|C| & 0 \\ 0 & f^{pr}|B| \end{bmatrix}X, X\right\rangle + \frac{1}{q}\left\langle \begin{bmatrix} g^{qr}|B^*| & 0 \\ 0 & g^{qr}|C^*| \end{bmatrix}X, X\right\rangle \\
&\qquad \text{(by Lemma 2.3(a))} \\
&= \left\langle \begin{bmatrix} \frac{1}{p}f^{pr}(|C|) + \frac{1}{q}g^{qr}(|B^*|) & 0 \\ 0 & \frac{1}{p}f^{pr}(|B|) + \frac{1}{q}g^{qr}(|C^*|) \end{bmatrix}X, X\right\rangle.
\end{aligned}$$

Then

$$|\langle TX, X\rangle|^r \leqslant \left\langle \begin{bmatrix} \frac{1}{p}f^{pr}(|C|) + \frac{1}{q}g^{qr}(|B^*|) & 0 \\ 0 & \frac{1}{p}f^{pr}(|B|) + \frac{1}{q}g^{qr}(|C^*|) \end{bmatrix}X, X\right\rangle.$$

Now, applying the definition of numerical radius and Lemma 2.1(a), we have

$$w^r(T) \leqslant \max\left\{\left\|\frac{1}{p}f^{pr}(|C|) + \frac{1}{q}g^{qr}(|B^*|)\right\|, \left\|\frac{1}{p}f^{pr}(|B|) + \frac{1}{q}g^{qr}(|C^*|)\right\|\right\}. \quad \square$$

COROLLARY 2.7. [2, Corollary 3] *Let* $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(\mathscr{H}_2, \mathscr{H}_1)$ *be a positive operator matrix and* $r \geqslant 1$. *Then*

$$w(T) = \frac{1}{2}\| B + C\|.$$

Proof. Putting $f(t) = g(t) = t^{\frac{1}{2}}$, $r = 1$ and $p = q = 2$ in inequality (2.1) and applying Lemma 2.1(c), we get the equality. $\square$

THEOREM 2.8. *Let* $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(\mathscr{H}_2, \mathscr{H}_1)$ *and* f, g *be nonnegative continuous functions on* $[0,\infty)$ *satisfying the relation* $f(t)g(t) = t$ $(t \in [0,\infty))$. *Then*

$$w^{2r}(T) \leqslant \max\left\{\left\|\frac{1}{p}f^{2pr}(|C|) + \frac{1}{q}g^{2qr}(|B^*|)\right\|, \left\|\frac{1}{p}f^{2pr}(|B|) + \frac{1}{q}g^{2qr}(|C^*|)\right\|\right\}, \tag{2.2}$$

where $r \geqslant 1$ *and* $p \geqslant q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $pr \geqslant 1$.

Proof. Assume that $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathscr{H}_1 \oplus \mathscr{H}_2$ is a unit vector. Then

$$\begin{aligned}
|\langle TX, X\rangle|^{2r} &\leqslant \| f(|T|)X\|^{2r} \| g(|T^*|X\|^{2r} \qquad \text{(by Lemma 2.5)} \\
&= \langle f^2(|T|)X, X\rangle^r \langle g^2(|T^*|)X, X\rangle^r \\
&\leqslant \frac{1}{p}\left\langle f^2\left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix}\right)X, X\right\rangle^{rp} + \frac{1}{q}\left\langle g^2\left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix}\right)X, X\right\rangle^{rq} \\
&\qquad \text{(by Lemma 2.2)} \\
&\leqslant \frac{1}{p}\left\langle \begin{bmatrix} f^{2pr}|C| & 0 \\ 0 & f^{2pr}|B| \end{bmatrix}X, X\right\rangle + \frac{1}{q}\left\langle \begin{bmatrix} g^{2qr}|B^*| & 0 \\ 0 & g^{2qr}|C^*| \end{bmatrix}X, X\right\rangle \\
&\qquad \text{(by Lemma 2.3(a))} \\
&= \left\langle \begin{bmatrix} \frac{1}{p}f^{2pr}(|C|) + \frac{1}{q}g^{2qr}(|B^*|) & 0 \\ 0 & \frac{1}{p}f^{2pr}(|B|) + \frac{1}{q}g^{2qr}(|C^*|) \end{bmatrix}X, X\right\rangle.
\end{aligned}$$

Thus

$$|\langle TX, X\rangle|^{2r} \leqslant \left\langle \begin{bmatrix} \frac{1}{p}f^{2pr}(|C|) + \frac{1}{q}g^{2qr}(|B^*|) & 0 \\ 0 & \frac{1}{p}f^{2pr}(|B|) + \frac{1}{q}g^{2qr}(|C^*|) \end{bmatrix}X, X\right\rangle.$$

Now by the definition of numerical radius and Lemma 2.1(a), we have

$$w^{2r}(T) \leqslant \max\left\{\left\|\frac{1}{p}f^{2pr}(|C|) + \frac{1}{q}g^{2qr}(|B^*|)\right\|, \left\|\frac{1}{p}f^{2pr}(|B|) + \frac{1}{q}g^{2qr}(|C^*|)\right\|\right\}. \qquad \square$$

Inequality (2.2) induces several numerical radius inequalities as follows.

COROLLARY 2.9. *Let* $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(\mathscr{H}_2, \mathscr{H}_1)$. *Then*

$$w^{2r}(T) \leqslant \frac{1}{2}\max\{\||C|^{4r\alpha} + |B^*|^{4r(1-\alpha)}\|, \||B|^{4r\alpha} + |C^*|^{4r(1-\alpha)}\|\}$$

for any $r \geqslant 1$ *and* $0 \leqslant \alpha \leqslant 1$.

Proof. Letting $f(t)=t^{\alpha}$, $g(t)=t^{1-\alpha}$ and $p=q=2$ in inequality (2.2), we get the desired inequality. □

COROLLARY 2.10. *Let* $B\in\mathbb{B}(\mathscr{H})$, $0\leqslant\alpha\leqslant 1$ *and* $r\geqslant 1$. *Then*

$$w^{2r}(B)\leqslant\frac{1}{2}\,\||\,B\,|^{4r\alpha}+|\,B^{*}\,|^{4r(1-\alpha)}\|\,. \tag{2.3}$$

Proof. We put $f(t)=t^{\alpha}$, $g(t)=t^{1-\alpha}$, $p=q=2$ and $T=\begin{bmatrix}0 & B\\ B & 0\end{bmatrix}$ and apply Lemma 2.1(d), we get the desired result. □

THEOREM 2.11. *Let* $T_i=\begin{bmatrix}0 & B_i\\ C_i & 0\end{bmatrix}\in\mathbb{B}(\mathscr{H}_2\oplus\mathscr{H}_1)$ *for any* $i=1,2,\cdots,n$. *Then*

$$\begin{aligned}&w_p^p(T_1,T_2,\cdots,T_n)\\ &\leqslant\max\left\{\left\|\sum_{i=1}^{n}\alpha\,|\,C_i\,|^p+(1-\alpha)\,|\,B_i^{*}\,|^p\right\|,\left\|\sum_{i=1}^{n}\alpha\,|\,B_i\,|^p+(1-\alpha)\,|\,C_i^{*}\,|^p\right\|\right\}\end{aligned} \tag{2.4}$$

for $0\leqslant\alpha\leqslant 1$ *and* $p\geqslant 2$.

Proof. For any unit vector $X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}\in\mathscr{H}_1\oplus\mathscr{H}_2$, we have

$$\begin{aligned}
&\sum_{i=1}^{n}|\,\langle T_iX,X\rangle\,|^p\\
&=\sum_{i=1}^{n}(|\,\langle T_iX,X\rangle\,|^2)^{\frac{p}{2}}\\
&\leqslant\sum_{i=1}^{n}(\langle|\,T_i\,|^{2\alpha}\,X,X\rangle\langle|\,T_i^{*}\,|^{2(1-\alpha)}\,X,X\rangle)^{\frac{p}{2}} \qquad \text{(by Lemma 2.5 }(b))\\
&\leqslant\sum_{i=1}^{n}\langle|\,T_i\,|^{p\alpha}\,X,X\rangle\langle|\,T_i^{*}\,|^{p(1-\alpha)}\,X,X\rangle \qquad \text{(by Lemma 2.3 }(b))\\
&\leqslant\sum_{i=1}^{n}\langle|\,T_i\,|^{p}\,X,X\rangle^{\alpha}\langle|\,T_i^{*}\,|^{p}\,X,X\rangle^{1-\alpha}\\
&\leqslant\sum_{i=1}^{n}(\alpha\langle|\,T_i\,|^{p}\,X,X\rangle+(1-\alpha)\langle|\,T_i^{*}\,|^{p}\,X,X\rangle) \qquad \text{(by Lemma 2.2)}\\
&=\sum_{i=1}^{n}\left(\alpha\left\langle\begin{bmatrix}|\,C_i\,|^p & 0\\ 0 & |\,B_i\,|^p\end{bmatrix}X,X\right\rangle+(1-\alpha)\left\langle\begin{bmatrix}|\,B_i^{*}\,|^p & 0\\ 0 & |\,C_i^{*}\,|^p\end{bmatrix}X,X\right\rangle\right)\\
&=\sum_{i=1}^{n}\left\langle\begin{bmatrix}\alpha\,|\,C_i\,|^p+(1-\alpha)\,|\,B_i^{*}\,|^p & 0\\ 0 & \alpha\,|\,B_i\,|^p+(1-\alpha)\,|\,C_i^{*}\,|^p\end{bmatrix}X,X\right\rangle\\
&=\left\langle\begin{bmatrix}\sum_{i=1}^{n}\alpha\,|\,C_i\,|^p+(1-\alpha)\,|\,B_i^{*}\,|^p & 0\\ 0 & \sum_{i=1}^{n}\alpha\,|\,B_i\,|^p+(1-\alpha)\,|\,C_i^{*}\,|^p\end{bmatrix}X,X\right\rangle.
\end{aligned}$$

By the definition of numerical radius and Lemma 2.1, we have

$$\begin{aligned} & w_p^p(T_1,T_2,\cdots,T_n) \\ & \leqslant \max\left\{\left\|\sum_{i=1}^{n}\alpha\mid C_i\mid^p+(1-\alpha)\mid B_i^*\mid^p\right\|,\left\|\sum_{i=1}^{n}\alpha\mid B_i\mid^p+(1-\alpha)\mid C_i^*\mid^p\right\|\right\}. \quad \square \end{aligned}$$

REMARK 2.12. As a special case for $\alpha=\frac{1}{2}$ and $B_i=C_i$ for any $i=1,2,\cdots,n$, we have the following inequality

$$w_p^p(B_1,B_2,\cdots,B_n)\leqslant\frac{1}{2}\ \|\sum_{i=1}^{n}\mid B_i\mid^p+\mid B_i^*\mid^p\|,$$

which already shown in [13, Proposition 3.9].

Now using a refinement of the classical Young inequality, we have the following theorem.

THEOREM 2.13. *Let* $T=\begin{bmatrix}0 & B\\ C & 0\end{bmatrix}\in\mathbb{B}(\mathscr{H}_2,\mathscr{H}_1)$ *and* f, g *be nonnegative continuous functions on* $[0,\infty)$ *satisfying the relation* $f(t)g(t)=t$ $(t\in[0,\infty))$. *Then for* $m=1,2,\cdots$ *and* $p,r\geqslant m$

$$\begin{aligned} w^r(T)\leqslant\left(\frac{1}{2}\right)^m\max\{\| f^{\frac{2r}{m}}(\mid C\mid)+g^{\frac{2r}{m}}(\mid B^*\mid)\|^m,\| f^{\frac{2r}{m}}(\mid B\mid)+g^{\frac{2r}{m}}(\mid C^*\mid)\|^m\} \\ -\inf_{\|X\|=1}\zeta(X), \quad (2.5) \end{aligned}$$

where

$$\zeta(X)=2^{-m}\left(\left\langle f^{\frac{2r}{m}}\left(\begin{bmatrix}\mid C\mid & 0\\ 0 & \mid B\mid\end{bmatrix}\right)X,X\right\rangle^{\frac{m}{2}}-\left\langle g^{\frac{2r}{m}}\left(\begin{bmatrix}\mid B^*\mid & 0\\ 0 & \mid C^*\mid\end{bmatrix}\right)X,X\right\rangle^{\frac{m}{2}}\right)^2.$$

Proof. Let $X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}\in\mathscr{H}_1\oplus\mathscr{H}_2$ be a unit vector. Applying Lemmas 2.5, 2.3 and inequality (1.3), respectively, we have

$$\begin{aligned} &\mid\langle TX,X\rangle\mid^r\leqslant\| f(\mid T\mid)X\|^r\| g(\mid T^*\mid)X\|^r \\ &\quad=\left(\langle f^2(\mid T\mid)X,X\rangle^{\frac{r}{2m}}\langle g^2(\mid T^*\mid)X,X\rangle^{\frac{r}{2m}}\right)^m \\ &\quad\leqslant\left(\langle f^{\frac{2r}{m}}(\mid T\mid)X,X\rangle^{\frac{1}{2}}\langle g^{\frac{2r}{m}}(\mid T^*\mid)X,X\rangle^{\frac{1}{2}}\right)^m \\ &\qquad-2^{-m}\left(\langle f^{\frac{2r}{m}}(\mid T\mid)X,X\rangle^{\frac{m}{2}}-\langle g^{\frac{2r}{m}}(\mid T^*\mid)X,X\rangle^{\frac{m}{2}}\right)^2 \\ &\quad\leqslant\left(\frac{1}{2}\left\langle f^{\frac{2r}{m}}\left(\begin{bmatrix}\mid C\mid & 0\\ 0 & \mid B\mid\end{bmatrix}\right)X,X\right\rangle+\frac{1}{2}\left\langle g^{\frac{2r}{m}}\left(\begin{bmatrix}\mid B^*\mid & 0\\ 0 & \mid C^*\mid\end{bmatrix}\right)X,X\right\rangle\right)^m \end{aligned}$$

$$-2^{-m}\left(\left\langle f^{\frac{2r}{m}}\left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix}\right)X,X\right\rangle^{\frac{m}{2}} - \left\langle g^{\frac{2r}{m}}\left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix}\right)X,X\right\rangle^{\frac{m}{2}}\right)^2$$

$$= \left(\frac{1}{2}\left\langle\begin{bmatrix} f^{\frac{2r}{m}}(|C|)+g^{\frac{2r}{m}}(|B^*|) & 0 \\ 0 & f^{\frac{2r}{m}}(|B|)+g^{\frac{2r}{m}}(|C^*|) \end{bmatrix}X,X\right\rangle\right)^m$$

$$-2^{-m}\left(\left\langle f^{\frac{2r}{m}}\left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix}\right)X,X\right\rangle^{\frac{m}{2}} - \left\langle g^{\frac{2r}{m}}\left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix}\right)X,X\right\rangle^{\frac{m}{2}}\right)^2.$$

Therefore

$$w^r(T) \leqslant \left(\frac{1}{2}\right)^m \max\{\| f^{\frac{2r}{m}}(|C|)+g^{\frac{2r}{m}}(|B^*|)\|^m, \| f^{\frac{2r}{m}}(|B|)+g^{\frac{2r}{m}}(|C^*|)\|^m\} - \inf_{\|X\|=1}\zeta(X).$$

Hence we get the desired inequality. □

REMARK 2.14. In inequality (2.5) if $m = 1$, then we get a refinement of inequality (2.1).

3. Numerical radius of the operator matrix 2×2

In this section, we estimate numerical radius of matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

LEMMA 3.1. *Let* $T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathbb{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$. *Then*

$$w^r(T) \leqslant \frac{1}{2}\max\{\||A|^r+|A^*|^r\|, \||D|^r+|D^*|^r\|\} \tag{3.1}$$

for $r \geqslant 1$.

Proof. Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathscr{H}_1 \oplus \mathscr{H}_2$ be any unit vector. Then

$$\begin{aligned} |\langle TX,X\rangle| &\leqslant \langle |T|X,X\rangle^{\frac{1}{2}}\langle |T^*|X,X\rangle^{\frac{1}{2}} \\ &\leqslant \frac{1}{2}\left\langle\begin{bmatrix} |A| & 0 \\ 0 & |D| \end{bmatrix}X,X\right\rangle + \frac{1}{2}\left\langle\begin{bmatrix} |A^*| & 0 \\ 0 & |D^*| \end{bmatrix}X,X\right\rangle \\ &\leqslant \left(\frac{1}{2}\left\langle\begin{bmatrix} |A| & 0 \\ 0 & |D| \end{bmatrix}X,X\right\rangle^r + \frac{1}{2}\left\langle\begin{bmatrix} |A^*| & 0 \\ 0 & |D^*| \end{bmatrix}X,X\right\rangle^r\right)^{\frac{1}{r}} \\ &\leqslant \left(\frac{1}{2}\left\langle\begin{bmatrix} |A|^r & 0 \\ 0 & |D|^r \end{bmatrix}X,X\right\rangle + \frac{1}{2}\left\langle\begin{bmatrix} |A^*|^r & 0 \\ 0 & |D^*|^r \end{bmatrix}X,X\right\rangle\right)^{\frac{1}{r}} \end{aligned}$$

$$= \left(\left\langle \begin{bmatrix} \frac{1}{2}(|A|^r + |A^*|^r) & 0 \\ 0 & \frac{1}{2}(|D|^r + |D^*|^r) \end{bmatrix} X, X \right\rangle \right)^{\frac{1}{r}},$$

and so

$$|\langle TX, X\rangle|^r \leqslant \left\langle \begin{bmatrix} \frac{1}{2}(|A|^r + |A^*|^r) & 0 \\ 0 & \frac{1}{2}(|D|^r + |D^*|^r) \end{bmatrix} X, X \right\rangle.$$

Therefore

$$w^r(T) \leqslant \frac{1}{2} \max\{\||A|^r + |A^*|^r\|, \||D|^r + |D^*|^r\|\}. \quad \square$$

REMARK 3.2. By letting $r = 1$ and $A = D$ in inequality (3.1), we obtain inequality (1.1), that is

$$w(A) \leqslant \frac{1}{2} \||A| + |A^*|\|.$$

The following proposition follows from inequalities (2.1) and (3.1).

PROPOSITION 3.3. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *with* $A, B, C, D \in \mathbb{B}(\mathscr{H})$. *Then*

$$w(T) \leqslant \frac{1}{2} \max\{\||C| + |B^*|\|, \||B| + |C^*|\|\} + \frac{1}{2} \max\{\||A| + |A^*|\|, \||D| + |D^*|\|\}.$$

In particular,

$$w\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) \leqslant \frac{1}{2} (\||A| + |A^*|\| + \||B| + |B^*|\|).$$

THEOREM 3.4. *Let* $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(\mathscr{H}_2 \oplus \mathscr{H}_1)$ *and* $r \geqslant 2$. *Then*

$$\frac{1}{2^{\frac{3}{2}(r-1)}} \max\{\|\mu\|, \|\eta\|\} \leqslant w^r(T) \leqslant \left(\frac{1}{2}\right)^{r+1} \max\{\|\mu\|, \|\eta\|\}, \tag{3.2}$$

where

$$\mu = |(C - B^*) + i(C + B^*)|^r + |(B^* - C) + i(C + B^*)|^r,$$

and

$$\eta = |(B - C^*) + i(B + C^*)|^r + |(C^* - B) + i(B + C^*)|^r.$$

Proof. Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathscr{H}_1 \oplus \mathscr{H}_2$ be a unit vector. Let $T = S + iW$ be the Cartesian decomposition of T. Then applying [9, Theorem 1], we have

$$w^2(T) \geqslant \frac{1}{2} \|(S \pm W)^2\|.$$

Therefore

$$w^r(T) \geqslant 2^{-\frac{r}{2}}\|(S \pm W)^2\|^{\frac{r}{2}} = 2^{-\frac{r}{2}}\||S \pm W|^r\|,$$

and so

$$\begin{aligned} 2w^r(T) &\geqslant 2^{-\frac{r}{2}}(\||S+W|^r\| + \||S-W|^r\|) \\ &\geqslant 2^{-\frac{r}{2}}\||S+W|^r + |S-W|^r\| \\ &\geqslant 2^{-\frac{r}{2}-1}|\langle(|S+W|^r + |S-W|^r)X, X\rangle| \\ &= 2^{-\frac{r}{2}-1}\left|\left\langle \begin{bmatrix} (\frac{1}{2})^r\mu & 0 \\ 0 & (\frac{1}{2})^r\eta \end{bmatrix} X, X \right\rangle\right|, \end{aligned}$$

where

$$\mu = |(C - B^*) + i(C + B^*)|^r + |(B^* - C) + i(C + B^*)|^r,$$

and

$$\eta = |(B - C^*) + i(B + C^*)|^r + |(C^* - B) + i(B + C^*)|^r.$$

Taking the supremum over $X \in \mathbb{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ with $\|X\| = 1$ in the above inequality and applying the numerical radius of diagonal matrices, we deduce the first inequality.

For the second inequality, we have

$$\begin{aligned} |\langle TX, X\rangle|^r &= (\langle SX, X\rangle^2 + \langle WX, X\rangle^2)^{\frac{r}{2}} \\ &= 2^{-\frac{r}{2}}(\langle(S+W)X, X\rangle^2 + \langle(S-W)X, X\rangle^2)^{\frac{r}{2}} \\ &\leqslant 2^{-\frac{r}{2}}2^{\frac{r}{2}-1}(|\langle(S+W)X, X\rangle|^r + |\langle(S-W)X, X\rangle|^r) \\ &\qquad\qquad (\text{since } f(t) = t^{\frac{r}{2}} \text{ is convex}) \\ &\leqslant \frac{1}{2}(\langle|S+W|X, X\rangle^r + \langle|S-W|X, X\rangle^r) \\ &\leqslant \frac{1}{2}(\langle|S+W|^r X, X\rangle + \langle|S-W|^r X, X\rangle) \\ &= \frac{1}{2}\langle(|S+W|^r + |S-W|^r)X, X\rangle \\ &= \frac{1}{2}\left\langle \begin{bmatrix} (\frac{1}{2})^r\mu & 0 \\ 0 & (\frac{1}{2})^r\eta \end{bmatrix} X, X \right\rangle. \end{aligned}$$

Now, applying the definition of numerical radius and Lemma 2.1, we get the desired inequality. □

REMARK 3.5. If $T^2 = 0$, then $w(T) = \frac{1}{2}\|T\|$, $\|T^*T + TT^*\| = \|T\|^2$, and

$$\||S+W|^r + |S-W|^r\| = 2^{-\frac{r}{2}+1}\|T^*T + TT^*\|^{\frac{r}{2}} = 2^{-\frac{r}{2}+1}\|T\|^r.$$

On the other hand, from $\||S+W|^r + |S-W|^r\| = \sup_{\|X\|=1}|\langle|S+W|^r + |S-W|^r X, X\rangle|$, we conclude that $(\frac{1}{2})^r \max\{\|\mu\|, \|\eta\|\} = 2^{-\frac{r}{2}+1}\|T\|^r$. Therefore $2^{\frac{-3}{2}r-1}\max\{\|\mu\|, \|\eta\|\} = 2^{-r}\|T\|^r = w^r(T)$, where μ and η are defined above.

REFERENCES

[1] A. ABU-OMAR AND F. KITTANEH, *Estimates for the numerical radius and the spectral radius of the Frobenius companion matrix and bounds for the zeros of polynomials*, Ann. Func. Anal. **5** (2014), no. 1, 56–62.
[2] A. ABU-OMAR AND F. KITTANEH, *Numerical radius inequalities for $n \times n$ operator matrices*, Linear Algebra Appl. **468** (2015), 18–26.
[3] Y. AL-MANASRAH AND F. KITTANEH, *A generalization of two refined Young inequalities*, Positivity **19** (2015), no. 4, 757–768.
[4] P. R. HALMOS, *A Hilbert Space Problem Book*, 2nd ed., Springer, New York, 1982.
[5] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1988.
[6] O. HIRZALLAH, F. KITTANEH AND K. SHEBRAWI, *Numerical radius inequalities for certain 2×2 operator matrices*, Integral equations Operator Theory **71** (2011), 129–149.
[7] F. KITTANEH, *Notes on some inequalitis for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24** (2) (1988), 283–293.
[8] F. KITTANEH, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. **158** (2003), 11–17.
[9] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Studia Math. **168** (2005), no. 1, 73–80.
[10] F. KITTANEH, M. S. MOSLEHIAN AND T. YAMAZAKI, *Cartesian decomposition and numerical radius inequalities*, Linear Algebra Appl. **471** (2015), 46–53.
[11] K. E. GUSTAFSON AND D. K. M. RAO, *Numerical Range, The Field of Values of Linear Operators and Matrices*, Springer, New York, 1997.
[12] M. SATTARI, M. S. MOSLEHIAN AND T. YAMAZAKI, *Some genaralized numerical radius inequalities for Hilbert space operators*, Linear Algebra Appl, **470** (2014), 1–12.
[13] M. SATTARI, M. S. MOSLEHIAN AND K. SHEBRAWI, *Extension of Euclidean operator radius inequalities*, Math. Scand. (2016) in press, arXiv 1502.00083.
[14] A. SHEIKHHOSSEINI, M. S. MOSLEHIAN AND K. SHEBRAWI, *Inequalities for generalized Euclidean operator radius via Young's inequality*, J. Math. Anal. Appl. **445** (2017), no. 2, 1516–1529.
[15] T. YAMAZAKI, *On upper and lower bounds of the numerical radius and an equality condition*, Studia Math. **178** (2007), 83–89.

(Received August 3, 2016)

Monire Hajmohamadi
Department of Mathematics, Faculty of Mathematics
University of Sistan and Baluchestan
Zahedan, I. R. Iran
e-mail: monire.hajmohamadi@yahoo.com

Rahmatollah Lashkaripour
Department of Mathematics, Faculty of Mathematics
University of Sistan and Baluchestan
Zahedan, I. R. Iran
e-mail: lashkari@hamoon.usb.ac.ir

Mojtaba Bakherad
Department of Mathematics, Faculty of Mathematics
University of Sistan and Baluchestan
Zahedan, I. R. Iran
e-mail: mojtaba.bakherad@yahoo.com;
bakherad@member.ams.org