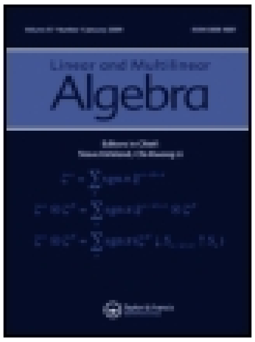

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Some extensions of the operator entropy type inequalities

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ABSTRACT

In this paper, we define the generalised relative operator entropy and investigate some of its properties such as subadditivity and homogeneity. As application of our result, we obtain the information inequality. In continuation, we establish some reverses of the operator entropy inequalities under certain conditions by using the Mond–Pečarić method.

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1. Introduction and preliminaries

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with the identity $I_{\mathscr{H}}$. In the case when $\dim\mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the full matrix algebra $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field. An operator $A \in \mathbb{B}(\mathscr{H})$ is called *positive* if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$ and in this case we write $A \geq 0$. We write $A > 0$ if A is a positive invertible operator. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $A \leq B$ if $B - A \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $\mathcal{C}(\mathsf{sp}(A))$ of continuous functions on the spectrum $\mathsf{sp}(A)$ of a self-adjoint operator A and the C^*-algebra generated by A and $I_{\mathscr{H}}$. If $f, g \in \mathcal{C}(\mathsf{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathsf{sp}(A))$ implies that $f(A) \geq g(A)$.

Let f be a continuous real-valued function defined on an interval J. It is called *operator monotone* if $A \leq B$ implies $f(A) \leq f(B)$ for all self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ with spectra in J; see [1] and references therein for some recent results. It said to be *operator concave* if $\lambda f(A) + (1-\lambda)f(B) \leq f(\lambda A + (1-\lambda)B)$ for all self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ with spectra in J and all $\lambda \in [0, 1]$. Every nonnegative continuous function f is operator monotone on $[0, +\infty)$ if and only if f is operator concave on $[0, +\infty)$; see [2, Theorem 8.1].

A linear map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$, where $\mathscr{H}$ and $\mathscr{K}$ are complex Hilbert spaces, is called *positive* if $\Phi(A) \geq 0$ whenever $A \geq 0$ and is said to be *normalized* if $\Phi(I_{\mathscr{H}}) =$

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$I_{\mathscr{K}}$. We denote by $\mathbf{P}_N[\mathbb{B}(\mathscr{H}),\mathbb{B}(\mathscr{K})]$ the set of all normalised positive linear maps Φ : $\mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$. If $\Phi \in \mathbf{P}_N[\mathbb{B}(\mathscr{H}),\mathbb{B}(\mathscr{K})]$ and f is an operator concave function on an interval J, then

$$f(\Phi(A)) \geq \Phi(f(A)) \qquad \text{(Davis-Choi-Jensen's inequality)} \tag{1.1}$$

for every selfadjoint operator A on $\mathscr{H}$, whose spectrum is contained in J, see also [2–4].

Let $\mathscr{A}$ be a C^*-algebra of operators acting on a Hilbert space, let T be a locally compact Hausdorff space and $\mu(t)$ be a Radon measure on T. A field $(A_t)_{t\in T}$ of operators in $\mathscr{A}$ is called a *continuous field of operators* if the function $t \mapsto A_t$ is norm continuous on T and the function $t \mapsto \|A_t\|$ is integrable, one can form the Bochner integral $\int_T A_t \mathrm{d}\mu(t)$, which is the unique element in $\mathscr{A}$ such that

$$\varphi\left(\int_T A_t \mathrm{d}\mu(t)\right) = \int_T \varphi(A_t)\mathrm{d}\mu(t) \tag{1.2}$$

for every linear functional φ in the norm dual $\mathscr{A}^*$ of $\mathscr{A}$; see [5].

In 1850 Clausius [6] introduced the notion of entropy in thermodynamics. Since then several extensions and reformulations have been developed in various disciplines; cf. [7–10]. There have been investigations of the so-called entropy inequalities by some mathematicians; see [11–13] and references therein. A relative operator entropy of strictly positive operators A, B was introduced in the noncommutative information theory by Fujii and Kamei [14] by

$$S(A|B) = A^{\frac{1}{2}} \log (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

In the same paper, it is shown that $S(A|B) \leq 0$ if $A \geq B$.

Next, recall that $X\natural_q Y$ is defined by $X^{\frac{1}{2}}\left(X^{-\frac{1}{2}}YX^{-\frac{1}{2}}\right)^q X^{\frac{1}{2}}$ for any real number q and any strictly positive operators X and Y. For $p \in [0,1]$, the operator $X\natural_p Y$ coincides with the well-known geometric mean of X, Y.

Furuta [15] defined the operator Shannon entropy by

$$S_p(A|B) = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$

where $p \in [0,1]$ and A, B are strictly positive operators on a Hilbert space $\mathscr{H}$. Suppose that $\mathbf{A} = (A_t)_{t\in T}, \mathbf{B} = (B_t)_{t\in T}$ are (continuous) fields of strictly positive operators, $q \in \mathbb{R}$ and f is a nonnegative operator monotone function on $(0,\infty)$. Then we have the definition of the generalised relative operator entropy

$$\widetilde{S}_q^f(\mathbf{A}|\mathbf{B}) := \int_T S_q^f(A_s|B_s)\mathrm{d}\mu(s), \tag{1.3}$$

where $S_q^f(A_s|B_s) = A_s^{1/2}\left(A_s^{-1/2}B_sA_s^{-1/2}\right)^q f\left(A_s^{-1/2}B_sA_s^{-1/2}\right)A_s^{1/2}$. In the discrete case $T = \{1,2,\ldots,n\}$, we get

$$S_q^f(\mathbf{A}|\mathbf{B}) := \sum_{j=1}^n S_q^f(A_j|B_j). \tag{1.4}$$

For $q = 0$, $f(t) = \log t$ and $A, B > 0$, we get the relative operator entropy $S_0^f(A|B) = A^{\frac{1}{2}} \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}} = S(A|B)$.

Moslehian et al. [16] showed the following operator entropy

$$f\left[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + t_0\left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j \natural_p B_j\right)\right] - f(t_0)\left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j \natural_p B_j\right)$$
$$\geq S_p^f(\mathbf{A}|\mathbf{B}) \qquad (p \in [0,1]), \tag{1.5}$$

where $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ are finite sequences of strictly positive operators such that $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I_{\mathscr{H}}$, f is a nonnegative operator monotone function on $(0, \infty)$ and t_0 is a positive fixed real number.

We present some extensions of the operator entropy inequality. Also, we show some reverses of the operator entropy inequalities under certain conditions using the Mond–Pečarić method. In this direction, we show a reverse of (1.5).

2. Some extensions of the operator entropy inequality

First, we present a variational form of $S_q^f(A|B)$ where A and B are two strictly positive operator in $\mathbb{B}(\mathscr{H})$ and q is an arbitrary real number.

Lemma 2.1: *If A and B are strictly positive, then*

$$S_q^f(A|B) = BS_{q-1}^f\left(B^{-1}|A^{-1}\right)B\,. \tag{2.1}$$

In particular, $S_0^f(A|B) = AS_0^f\left(B^{-1}|A^{-1}\right)B$.

***Proof*:** Since $Xg(X^*X) = g(XX^*)X$ for every $X \in \mathbb{B}(\mathscr{H})$ and every continuous function g on $[0, \|X\|^2]$ [2, Lemma 1.7], considering $X = B^{1/2}A^{-1/2}$ we have

$$\begin{aligned}
S_q^f(A|B) &= A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^q f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}\\
&= A^{\frac{1}{2}}\left(X^*X\right)^q f\left(X^*X\right)A^{\frac{1}{2}}\\
&= B^{\frac{1}{2}}B^{-\frac{1}{2}}A^{\frac{1}{2}}\left(X^*X\right)^q f\left(X^*X\right)A^{\frac{1}{2}}B^{-\frac{1}{2}}B^{\frac{1}{2}}\\
&= B^{\frac{1}{2}}X^{*-1}\left(X^*X\right)^q f\left(X^*X\right)X^{-1}B^{\frac{1}{2}}\\
&= B^{\frac{1}{2}}X(X^*X)^{-1}\left(X^*X\right)^q f\left(X^*X\right)(X^*X)^{-1}X^*B^{\frac{1}{2}}\\
&= B^{\frac{1}{2}}X(X^*X)^{q-2} f\left(X^*X\right)X^*B^{\frac{1}{2}}\\
&= B^{\frac{1}{2}}(XX^*)^{q-2}Xf\left(X^*X\right)X^*B^{\frac{1}{2}}\\
&= B^{\frac{1}{2}}(XX^*)^{q-2}f\left(XX^*\right)XX^*B^{\frac{1}{2}}\\
&= B^{\frac{1}{2}}(XX^*)^{q-1}f\left(XX^*\right)B^{\frac{1}{2}}\\
&= B^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{q-1} f\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)B^{\frac{1}{2}}\\
&= BS_{q-1}^f(B^{-1}|A^{-1})B\,,
\end{aligned}$$

as desired.

Also,

$$\begin{aligned} S_0^f(A|B) &= BS_{-1}^f(B^{-1}|A^{-1})B \\ &= B\left[B^{-\frac{1}{2}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{-1}f\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)B^{-\frac{1}{2}}\right]B \\ &= A\left[B^{-\frac{1}{2}}f\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)B^{-\frac{1}{2}}\right]B \\ &= AS_0^f\left(B^{-1}|A^{-1}\right)B\,. \end{aligned}$$

□

The above lemma says that if B is also invertible, then we can define $S_q^f(A|B)$ by (2.1). Furthermore, if A and B commute, then

$$S_q^f(A|B) = A\left(A^{-1}B\right)^q f\left(A^{-1}B\right)\,.$$

Note that, in the general case, the generalised relative operator entropy for noninvertible positive operators does not always exist. For instance, give $f(t) = \log t$ and $q = 0$, consequently $S_0^f(I_{\mathscr{H}}|\epsilon I_{\mathscr{H}}) = (\log\epsilon)I_{\mathscr{H}}$ is not bounded below and hence $S_0^f(I_{\mathscr{H}}|0)$ does not make sense. For more information, see [2].

Now, we have the following lemmas.

Lemma 2.2: *Let $\mathbf{A} = (A_t)_{t\in T}$ and $\mathbf{B} = (B_t)_{t\in T}$ be continuous fields of strictly positive operators. Then*

$$\int_T (A_s\natural_p B_s)\mathrm{d}\mu(s) \le \left(\int_T A_s\mathrm{d}\mu(s)\right)\natural_p\left(\int_T B_s\mathrm{d}\mu(s)\right), \tag{2.2}$$

where $p \in [0,1]$.

***Proof*:** For continuous fields of strictly positive operators $\mathbf{A} = (A_t)_{t\in T}$ and $\mathbf{B} = (B_t)_{t\in T}$, we take the positive unital linear map $\Phi(X) = \int_T C^*XC\mathrm{d}\mu(t)$ $(X \in \mathscr{A})$, where $C = B_t^{\frac{1}{2}}\left(\int_T B_s\mathrm{d}\mu(s)\right)^{-\frac{1}{2}}$. Thus for $p \in [0,1]$ we have

$$\begin{aligned} &\left(\int_T A_t\mathrm{d}\mu(t)\right)\natural_p\left(\int_T B_s\mathrm{d}\mu(s)\right) \\ &= \left(\int_T B_s\mathrm{d}\mu(s)\right)^{\frac{1}{2}}\left(\left(\int_T B_s\mathrm{d}\mu(s)\right)^{-\frac{1}{2}}\int_T A_t\mathrm{d}\mu(t)\left(\int_T B_s\mathrm{d}\mu(s)\right)^{-\frac{1}{2}}\right)^p\left(\int_T B_s\mathrm{d}\mu(s)\right)^{\frac{1}{2}} \\ &= \left(\int_T B_s\mathrm{d}\mu(s)\right)^{\frac{1}{2}}\left(\int_T\left(\int_T B_s\mathrm{d}\mu(s)\right)^{-\frac{1}{2}}B_t^{\frac{1}{2}}(B_t^{-\frac{1}{2}}A_tB_t^{-\frac{1}{2}})B_t^{\frac{1}{2}}\left(\int_T B_s\mathrm{d}\mu(s)\right)^{-\frac{1}{2}}\mathrm{d}\mu(t)\right)^p \\ &\qquad\times\left(\int_T B_s\mathrm{d}\mu(t)\right)^{\frac{1}{2}} \\ &= \left(\int_T B_s\mathrm{d}\mu(s)\right)^{\frac{1}{2}}\left(\int_T C^*B_t^{-\frac{1}{2}}A_tB_t^{-\frac{1}{2}}C\mathrm{d}\mu(t)\right)^p\left(\int_T B_s\mathrm{d}\mu(t)\right)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&= \left(\int_T B_s \mathrm{d}\mu(s)\right)^{\frac{1}{2}} \left(\Phi\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right)\right)^p \left(\int_T B_s \mathrm{d}\mu(s)\right)^{\frac{1}{2}} \\
&\geq \left(\int_T B_s \mathrm{d}\mu(s)\right)^{\frac{1}{2}} \Phi\left(\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right)^p\right) \left(\int_T B_s \mathrm{d}\mu(s)\right)^{\frac{1}{2}} \qquad \text{(by (1.1))} \\
&= \left(\int_T B_s \mathrm{d}\mu(s)\right)^{\frac{1}{2}} \left(\int_T C^* \left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right)^p C \mathrm{d}\mu(t)\right) \left(\int_T B_s \mathrm{d}\mu(t)\right)^{\frac{1}{2}} \\
&= \int_T B_t^{\frac{1}{2}} \left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right)^p B_t^{\frac{1}{2}} \mathrm{d}\mu(t) \\
&= \int_T (A_t \natural_p B_t) \mathrm{d}\mu(t)\,.
\end{aligned}
$$

□

Lemma 2.3: *If $X, C_s \in \mathscr{A}$ $(s \in T)$ such that $0 < m \leq X \leq M$, $f : (0, \infty) \to [0, \infty)$ is an operator monotone function and $t_0 \in [m, M]$, then*

$$
\begin{aligned}
& f\left(\int_T C_s^* X C_s \mathrm{d}\mu(s) + t_0\Big(I_{\mathscr{H}} - \int_T C_s^* C_s \mathrm{d}\mu(s)\Big)\right) \\
& \quad \geq \int_T C_s^* f(X) C_s \mathrm{d}\mu(s) + f(t_0)\Big(I_{\mathscr{H}} - \int_T C_s^* C_s \mathrm{d}\mu(s)\Big),
\end{aligned}
$$

where $\int_T C_s^* C_s \mathrm{d}\mu(s) \leq I_{\mathscr{H}}$.

***Proof*:** We put $D = (I_{\mathscr{H}} - \int_T C_s^* C_s \mathrm{d}\mu(s))^{\frac{1}{2}}$. Assume that the positive unital linear map $\Phi\big(\mathrm{diag}(X, Y)\big) = \int_T C_s^* X C_s \mathrm{d}\mu(s) + D^* Y D$ $(X, Y \in \mathscr{A})$, where

$$
\mathrm{diag}(X, Y) = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}.
$$

Using inequality (1.1) and the operator monotonicity of f, we have

$$
\begin{aligned}
& f\left(\int_T C_s^* X C_s \mathrm{d}\mu(s) + t_0 \left(I_{\mathscr{H}} - \int_T C_s^* C_s \mathrm{d}\mu(s)\right)\right) \\
& = f\left(\Phi\left(\begin{bmatrix} X & 0 \\ 0 & t_0 \end{bmatrix}\right)\right) \\
& \geq \left(\Phi\left(\begin{bmatrix} f(X) & 0 \\ 0 & f(t_0) \end{bmatrix}\right)\right) \qquad \text{(by (1.1))} \\
& = \int_T C_s^* f(X) C_s \mathrm{d}\mu(s) + D^* f(t_0) D,
\end{aligned}
$$

whenever $0 < m \leq X \leq M$ and $t_0 \in [m, M]$. Therefore we get the desired inequality. □

In the next theorem, we have an extension of (1.5).

Theorem 2.4: *Let $\mathbf{A} = (A_t)_{t \in T}, \mathbf{B} = (B_t)_{t \in T}$ be continuous fields of strictly positive operators such that $0 < mA_s \leq B_s \leq MA_s$ $(s \in T)$ for some positive real numbers m, M, where $m < 1 < M$, and $\int_T A_s \mathrm{d}\mu(s) = \int_T B_s \mathrm{d}\mu(s) = I_{\mathscr{H}}$, $f : (0, \infty) \to [0, \infty)$ is operator* *concave and* $p \in [0,1]$. *Then*

$$f\left[\int_T (A_s \natural_{p+1} B_s)\mathrm{d}\mu(s) + t_0\left(I_{\mathscr{H}} - \int_T A_s\natural_p B_s \mathrm{d}\mu(s)\right)\right] - f(t_0)\left(I_{\mathscr{H}} - \int_T A_s\natural_p B_s\mathrm{d}\mu(s)\right)$$
$$\geq \widetilde{S}_p^f(\mathbf{A}|\mathbf{B})\,. \tag{2.3}$$

***Proof*:** It follows from (2.2) and $\int_T A_s \mathrm{d}\mu(s) = \int_T B_s\mathrm{d}\mu(s) = I_{\mathscr{H}}$ that $\int_T A_s\natural_p B_s\mathrm{d}\mu(s) \leq I_{\mathscr{H}}$ $(p\in[0,1])$.

$$\begin{aligned}
& f\left[\int_T (A_s \natural_{p+1} B_s)\mathrm{d}\mu(s) + t_0\left(I_{\mathscr{H}} - \int_T A_s\natural_p B_s \mathrm{d}\mu(s)\right)\right] \\
&= f\left[\int_T \left(\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right)^{\frac{p}{2}} A_s^{\frac{1}{2}}\right)^* \left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right)\left(\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right)^{\frac{p}{2}} A_s^{\frac{1}{2}}\right)\mathrm{d}\mu(s)\right. \\
&\quad \left. + t_0\left(I_{\mathscr{H}} - \int_T A_s\natural_p B_s\mathrm{d}\mu(s)\right)\right] \\
&\geq \int_T A_s^{\frac{1}{2}}\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right)^{\frac{p}{2}} f\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right)\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right)^{\frac{p}{2}} A_s^{\frac{1}{2}}\mathrm{d}\mu(s) \\
&\quad + f(t_0)\left(I_{\mathscr{H}} - \int_T A_s\natural_p B_s\mathrm{d}\mu(s)\right) \qquad \text{(by Lemma 2.3)} \\
&= \int_T A_s^{\frac{1}{2}}\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right)^{p} f\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right) A_s^{\frac{1}{2}}\mathrm{d}\mu(s) + f(t_0)\left(I_{\mathscr{H}} - \int_T A_s\natural_p B_s\mathrm{d}\mu(s)\right) \\
&= \int_T S_p^f(A_s|B_s)\mathrm{d}\mu(s) + f(t_0)\left(I_{\mathscr{H}} - \int_T A_s\natural_p B_s\mathrm{d}\mu(s)\right)\,.
\end{aligned}$$

□

In the next theorem, we present the lower and upper bounds of the generalised relative operator entropy.

Theorem 2.5: *With above notations, the following statements hold:*

(i) $\widetilde{S}_q^f(\mathbf{A}|\mathbf{B}) \geq 0$.

(ii) *If* $f(t) \leq t-1$, *then* $\widetilde{S}_q^f(\mathbf{A}|\mathbf{B}) \leq \int_T (A_s\natural_{q+1}B_s - A_s\natural_q B_s)\mathrm{d}\mu(s)$. *In particular,* $\widetilde{S}_0^f(\mathbf{A}|\mathbf{B}) \leq \int_T (B_s - A_s)\mathrm{d}\mu(s)$ *and* $\widetilde{S}_1^f(\mathbf{A}|\mathbf{B}) \leq \int_T (B_sA_s^{-1}B_s - B_s)\mathrm{d}\mu(s)$.

***Proof*:** (i) Since f is a continuous nonnegative function, $X^q f(X) \geq 0$ for every $X \geq 0$ and $q\in\mathbb{R}$. Hence

$$\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right)^q f\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right) \geq 0\,.$$

Consequently, $\widetilde{S}_q^f(\mathbf{A}|\mathbf{B}) \geq 0$.

(ii) Since $f(t) \leq t - 1$, we have

$$\begin{aligned}
\widetilde{\mathbb{S}}_q^f(\mathbf{A}|\mathbf{B}) &= \int_T A_s^{\frac{1}{2}} \left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right)^q f\left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right) A_s^{\frac{1}{2}} \mathrm{d}\mu(s) \\
&\leq \int_T A_s^{\frac{1}{2}} \left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right)^q \left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}} - I_{\mathscr{H}}\right) A_s^{\frac{1}{2}} \mathrm{d}\mu(s) \\
&= \int_T \left(A_s \natural_{q+1} B_s - A_s \natural_q B_s\right) \mathrm{d}\mu(s) .
\end{aligned}$$

Hence

$$\widetilde{\mathbb{S}}_0^f(\mathbf{A}|\mathbf{B}) \leq \int_T \left(A_s \natural_1 B_s - A_s \natural_0 B_s\right) \mathrm{d}\mu(s) = \int_T \left(B_s - A_s\right) \mathrm{d}\mu(s) ,$$

and

$$\begin{aligned}
\widetilde{\mathbb{S}}_1^f(\mathbf{A}|\mathbf{B}) &\leq \int_T \left(A_s \natural_2 B_s - A_s \natural_1 B_s\right) \mathrm{d}\mu(s) \\
&= \int_T \left[A_s^{\frac{1}{2}} \left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right)^2 A_s^{\frac{1}{2}} - A_s^{\frac{1}{2}} \left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right) A_s^{\frac{1}{2}}\right] \mathrm{d}\mu(s) \\
&= \int_T \left(B_s A_s^{-1} B_s - B_s\right) \mathrm{d}\mu(s) .
\end{aligned}$$

□

Corollary 2.6 [2, Theorem 5.12]: *Assume that A and B are two strictly positive operators in $\mathbb{B}(\mathscr{H})$. Then the relative operator entropy is upper bounded; i.e.*

$$S(A|B) \leq B - A . \tag{2.4}$$

Equality holds if and only if $A = B$.

***Proof*:** By taking $T = \{1\}$, $f(t) = \log t$ and $q = 0$ in Theorem 2.5(ii), (2.4) follows from the Klein inequality $\log t \leq t - 1$. Also, in the Klein inequality, equality holds if and only if $t = 1$ [17, Lemma 3.8]. So equality holds in (2.4) if and only if $A = B$. □

Corollary 2.7 (Information inequality, [18, Lemma 3.1]): *Given two probability mass functions $\{a_j\}$ and $\{b_j\}$, that is, two countable or finite sequences of positive numbers that sum to one, then*

$$\sum_j a_j \log \frac{a_j}{b_j} \geq 0 , \tag{2.5}$$

with equality if and only if $a_j = b_j$, for all j.

***Proof*:** If we take A and B in Corollary 2.6 as follows

$$A = \begin{pmatrix} a_1 & 0 & 0 & \cdots \\ 0 & a_2 & 0 & \cdots \\ 0 & 0 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & 0 & 0 & \cdots \\ 0 & b_2 & 0 & \cdots \\ 0 & 0 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then, we have

$$S(A|B) = \begin{pmatrix} a_1 & 0 & 0 & \cdots \\ 0 & a_2 & 0 & \cdots \\ 0 & 0 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \log \begin{pmatrix} \frac{b_1}{a_1} & 0 & 0 & \cdots \\ 0 & \frac{b_2}{a_2} & 0 & \cdots \\ 0 & 0 & \frac{b_3}{a_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \log\left(\frac{b_1}{a_1}\right) & 0 & 0 & \cdots \\ 0 & a_2 \log\left(\frac{b_2}{a_2}\right) & 0 & \cdots \\ 0 & 0 & a_3 \log\left(\frac{b_3}{a_3}\right) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Consequently,

$$\left\langle S(A|B) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} a_1 \log\left(\frac{b_1}{a_1}\right) & 0 & 0 & \cdots \\ 0 & a_2 \log\left(\frac{b_2}{a_2}\right) & 0 & \cdots \\ 0 & 0 & a_3 \log\left(\frac{b_3}{a_3}\right) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \right\rangle$$

$$= \sum_j a_j \log\left(\frac{b_j}{a_j}\right).$$

On the other hand,

$$\left\langle (B-A) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} b_1 - a_1 & 0 & 0 & \cdots \\ 0 & b_2 - a_2 & 0 & \cdots \\ 0 & 0 & b_3 - a_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \right\rangle$$

$$= \sum_j (b_j - a_j) = 0.$$

We, therefore, deduce the desired inequality (2.5). Using Corollary 2.6, equality holds if and only if $A = B$, or equivalently $a_j = b_j$ for all j. □

In the next theorem, we show that the generalised relative operator entropy is subadditive.

Theorem 2.8: *For $q = 0$, the generalised relative operator entropy is subadditive,*

$$\widetilde{S}_0^f(\mathbf{A}+\mathbf{B}|\mathbf{C}+\mathbf{D}) \geq \widetilde{S}_0^f(\mathbf{A}|\mathbf{C}) + \widetilde{S}_0^f(\mathbf{B}|\mathbf{D}).$$

Proof: Without loss of generality, we assume that A_s and B_s $(s \in T)$ are invertible. Put $X_s = A_s^{1/2}(A_s + B_s)^{-1/2}$ and $Y_s = B_s^{1/2}(A_s + B_s)^{-1/2}$. Therefore,

$$\begin{aligned}X_s^* X_s + Y_s^* Y_s &= \left[(A_s + B_s)^{-\frac{1}{2}} A_s^{\frac{1}{2}}\right]\left[A_s^{\frac{1}{2}}(A_s + B_s)^{-\frac{1}{2}}\right] \\ &\quad + \left[(A_s + B_s)^{-\frac{1}{2}} B_s^{\frac{1}{2}}\right]\left[B_s^{\frac{1}{2}}(A_s + B_s)^{-\frac{1}{2}}\right] \\ &= (A_s + B_s)^{-\frac{1}{2}} A_s (A_s + B_s)^{-\frac{1}{2}} + (A_s + B_s)^{-\frac{1}{2}} B_s (A_s + B_s)^{-\frac{1}{2}} \\ &= (A_s + B_s)^{-\frac{1}{2}}(A_s + B_s)(A_s + B_s)^{-\frac{1}{2}} \\ &= I_{\mathscr{H}},\end{aligned}$$

and this implies that

$$\begin{aligned}&\widetilde{S}_0^f(\mathbf{A} + \mathbf{B}|\mathbf{C} + \mathbf{D}) \\ &\quad = \int_T S_0^f(A_s + B_s|C_s + D_s)\mathrm{d}\mu(s) \\ &\quad = \int_T (A_s + B_s)^{\frac{1}{2}} f\left[(A_s + B_s)^{-\frac{1}{2}}(C_s + D_s)(A_s + B_s)^{-\frac{1}{2}}\right](A_s + B_s)^{\frac{1}{2}}\mathrm{d}\mu(s) \\ &\quad = \int_T (A_s + B_s)^{\frac{1}{2}} f\left[(A_s + B_s)^{-\frac{1}{2}} C_s (A_s + B_s)^{-\frac{1}{2}} + (A_s + B_s)^{-\frac{1}{2}} D_s (A_s + B_s)^{-\frac{1}{2}}\right] \\ &\quad\quad \times (A_s + B_s)^{\frac{1}{2}}\mathrm{d}\mu(s) \\ &\quad = \int_T (A_s + B_s)^{\frac{1}{2}} f\left[X_s^*\left(A_s^{-\frac{1}{2}} C_s A_s^{-\frac{1}{2}}\right)X_s + Y_s^*\left(B_s^{-\frac{1}{2}} D_s B_s^{-\frac{1}{2}}\right)Y_s\right](A_s + B_s)^{\frac{1}{2}}\mathrm{d}\mu(s) \\ &\quad \geq \int_T (A_s + B_s)^{\frac{1}{2}}\left[X_s^* f\left(A_s^{-\frac{1}{2}} C_s A_s^{-\frac{1}{2}}\right)X_s + Y_s^* f\left(B_s^{-\frac{1}{2}} D_s B_s^{-\frac{1}{2}}\right)Y_s\right](A_s + B_s)^{\frac{1}{2}}\mathrm{d}\mu(s) \\ &\qquad\qquad\qquad\qquad \text{(by operator concavity of } f \text{ and Theorem 1.9 in [2])} \\ &\quad = \int_T \left[A_s^{\frac{1}{2}} f\left(A_s^{-\frac{1}{2}} C_s A_s^{-\frac{1}{2}}\right)A_s^{\frac{1}{2}} + B_s^{\frac{1}{2}} f\left(B_s^{-\frac{1}{2}} D_s B_s^{-\frac{1}{2}}\right)B_s^{\frac{1}{2}}\right]\mathrm{d}\mu(s) \\ &\quad = \int_T S_0^f(A_s|C_s)\mathrm{d}\mu(s) + \int_T S_0^f(B_s|D_s)\mathrm{d}\mu(s) \\ &\quad = \widetilde{S}_0^f(\mathbf{A}|\mathbf{C}) + \widetilde{S}_0^f(\mathbf{B}|\mathbf{D}).\end{aligned}$$

□

Lemma 2.9: *The generalised relative operator entropy is homogenous; i.e. for any real number* $\alpha > 0$

$$\widetilde{S}_q^f(\alpha\mathbf{A}|\alpha\mathbf{B}) = \alpha\widetilde{S}_q^f(\mathbf{A}|\mathbf{B}),$$

where $\alpha\mathbf{A} = (\alpha A_s)_{s\in T}$.

Theorem 2.10: *For* $q = 0$, *the generalised relative operator entropy is jointly concave; i.e. if* $\mathbf{A} = \alpha\mathbf{A}_1 + \beta\mathbf{A}_2$ *and* $\mathbf{B} = \alpha\mathbf{B}_1 + \beta\mathbf{B}_2$ *for* $\alpha, \beta > 0$ *with* $\alpha + \beta = 1$, *then*

$$\widetilde{S}_0^f(\mathbf{A}|\mathbf{B}) \geq \alpha\widetilde{S}_0^f(\mathbf{A}_1|\mathbf{B}_1) + \beta S_0^f(\mathbf{A}_2|\mathbf{B}_2).$$

***Proof*:** Assume that $\mathbf{A}_k = (A_{ks})$ and $\mathbf{B}_k = (B_{ks})$ $(k = 1, 2,\ s \in T)$. By using of subadditivity and homogeneity of the generalised relative operator entropy, we get

$$\begin{aligned}
\widetilde{S}_0^f(\mathbf{A}|\mathbf{B}) &= \widetilde{S}_0^f(\alpha\mathbf{A}_1 + \beta\mathbf{A}_2|\alpha\mathbf{B}_1 + \beta\mathbf{B}_2) \\
&= \int_T S_0^f(\alpha A_{1s} + \beta A_{2s}|\alpha B_{1s} + \beta B_{2s}) \mathrm{d}\mu(s) \\
&\geq \int_T \left[S_0^f(\alpha A_{1s}|\alpha B_{1s}) + S_0^f(\beta A_{2s}|\beta B_{2s}) \right] \mathrm{d}\mu(s) \\
&= \int_T \left[\alpha S_0^f(A_{1s}|B_{1s}) + \beta S_0^f(A_{2s}|B_{2s}) \right] \mathrm{d}\mu(s) \\
&= \alpha\widetilde{S}_0^f(\mathbf{A}_1|\mathbf{B}_1) + \beta\widetilde{S}_0^f(\mathbf{A}_2|\mathbf{B}_2) .
\end{aligned}$$

□

We say that $\mathbf{A} = (A_s)_{s\in T}$ is *invertible*, if A_s is invertible for every $s \in T$.

In the following theorem, we show that the generalised relative operator entropy has informational monotonicity.

Theorem 2.11: *For $q = 0$ and $\Phi \in \mathbf{P}_N[\mathbb{B}(\mathscr{H}), \mathbb{B}(\mathscr{K})]$,*

$$\Phi\left(\widetilde{S}_0^f(\mathbf{A}|\mathbf{B})\right) \leq \widetilde{S}_0^f\left(\Phi(\mathbf{A})|\Phi(\mathbf{B})\right) .$$

***Proof*:** Assume that $\mathbf{A}$ is invertible. Then so does $\Phi(\mathbf{A}) = \left(\Phi(A_s)\right)_{s\in T}$. Define

$$\Psi(X) = \Phi(A_s)^{-\frac{1}{2}}\Phi\left(A_s^{\frac{1}{2}}XA_s^{\frac{1}{2}}\right)\Phi(A_s)^{-\frac{1}{2}} .$$

So Ψ is a normalised positive linear map. Consequently,

$$\begin{aligned}
\Phi\left(\widetilde{S}_0^f(\mathbf{A}|\mathbf{B})\right) &= \int_T \Phi\left(S_0^f(A_s|B_s)\right) \mathrm{d}\mu(s) \qquad \text{(by (1.2))} \\
&= \int_T \Phi\left(A_s^{\frac{1}{2}} f\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right)A_s^{\frac{1}{2}}\right) \mathrm{d}\mu(s) \\
&= \int_T \Phi(A_s)^{\frac{1}{2}}\left[\Phi(A_s)^{-\frac{1}{2}}\Phi\left(A_s^{\frac{1}{2}} f\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right)A_s^{\frac{1}{2}}\right)\Phi(A_s)^{-\frac{1}{2}}\right] \\
&\qquad \times \Phi(A_s)^{\frac{1}{2}}\mathrm{d}\mu(s) \\
&= \int_T \Phi(A_s)^{\frac{1}{2}}\Psi\left(f\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right)\right)\Phi(A_s)^{\frac{1}{2}}\mathrm{d}\mu(s) \\
&\leq \int_T \Phi(A_s)^{\frac{1}{2}} f\left(\Psi\left(A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}\right)\right)\Phi(A_s)^{\frac{1}{2}}\mathrm{d}\mu(s) \qquad \text{(by (1.1))} \\
&= \int_T \Phi(A_s)^{\frac{1}{2}} f\left[\Phi(A_s)^{-\frac{1}{2}}\Phi\left(A_s^{\frac{1}{2}}A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}A_s^{\frac{1}{2}}\right)\Phi(A_s)^{-\frac{1}{2}}\right]\Phi(A_s)^{\frac{1}{2}}\mathrm{d}\mu(s) \\
&= \int_T \Phi(A_s)^{\frac{1}{2}} f\left[\Phi(A_s)^{-\frac{1}{2}}\Phi(B_s)\Phi(A_s)^{-\frac{1}{2}}\right]\Phi(A_s)^{\frac{1}{2}}\mathrm{d}\mu(s)
\end{aligned}$$

$$
\begin{aligned}
&= \int_T S_0^f \left(\Phi(A_s) | \Phi(B_s) \right) \mathrm{d}\mu(s) \\
&= \widetilde{S}_0^f \left(\Phi(\mathbf{A}) | \Phi(\mathbf{B}) \right) .
\end{aligned}
$$

□

3. Some operator entropy inequalities

There is an impressive method for finding inverses of some operator inequalities. It was introduced for investigation of converses of the Jensen inequality associated with convex functions, see [2] and [19] and references therein. We need the essence of the Mond–Pečarić method where appeared in [2, Chapter 2] in some general forms.

If f is a strictly concave differentiable function on an interval $[m, M]$ with $m < M$ and $\Phi : \mathbb{B}(\mathscr{H}) \longrightarrow \mathbb{B}(\mathscr{K})$ is a positive unital linear map,

$$
\begin{aligned}
\mu_f &= \frac{f(M) - f(m)}{M - m}, \quad \nu_f = \frac{Mf(m) - mf(M)}{M - m} \quad \text{and} \\
\gamma_f &= \max \left\{ \frac{f(t)}{\mu_f t + \nu_f} : m \le t \le M \right\},
\end{aligned} \tag{3.1}
$$

then

$$
f(\Phi(A)) \le \gamma_f \Phi(f(A)). \tag{3.2}
$$

Lemma 3.1: *Assume that $X, C_s \in \mathscr{A}$ $(s \in T)$ such that $0 < m \le X \le M$, $f : (0, \infty) \to [0, \infty)$ is an operator monotone function, $t_0 \in [m, M]$ and γ_f is given by (3.1). Then*

$$
\begin{aligned}
&f\left(\int_T C_s^* X C_s \mathrm{d}\mu(s) + t_0 \left(I_{\mathscr{H}} - \int_T C_s^* C_s \mathrm{d}\mu(s) \right) \right) \\
&\quad \le \gamma_f \left[\int_T C_s^* f(X) C_s \mathrm{d}\mu(s) + f(t_0) \left(I_{\mathscr{H}} - \int_T C_s^* C_s \mathrm{d}\mu(s) \right) \right],
\end{aligned}
$$

where $\int_T C_s^* C_s \mathrm{d}\mu(s) \le I_{\mathscr{H}}$.

***Proof*:** Using (3.2), the proof is similar to Lemma 2.3. □

Theorem 3.2: *Let $\mathbf{A} = (A_t)_{t \in T}, \mathbf{B} = (B_t)_{t \in T}$ be continuous fields of strictly positive operators such that $0 < mA_s \le B_s \le MA_s$ $(s \in T)$ for some positive real numbers m, M, where $m < 1 < M$, $\int_T A_s \mathrm{d}\mu(s) = \int_T B_s \mathrm{d}\mu(s) = I_{\mathscr{H}}$, $f : (0, \infty) \to [0, \infty)$ is operator concave and $p \in [0, 1]$. Then*

$$
\begin{aligned}
&f\left[\int_T (A_s \natural_{p+1} B_s) \mathrm{d}\mu(s) + t_0 \left(I_{\mathscr{H}} - \int_T A_s \natural_p B_s \mathrm{d}\mu(s) \right) \right] \\
&\qquad - \gamma_f f(t_0) \left(I_{\mathscr{H}} - \int_T A_s \natural_p B_s \mathrm{d}\mu(s) \right) \\
&\quad \le \gamma_f \widetilde{S}_p^f (\mathbf{A} | \mathbf{B}),
\end{aligned} \tag{3.3}
$$

where $t_0 \in [m, M]$ and γ_f is given by (3.1).

***Proof*:** It follows from (2.2) and $\int_T A_s \mathrm{d}\mu(s) = \int_T B_s \mathrm{d}\mu(s) = I_{\mathscr{H}}$ that $\int_T A_s \natural_p B_s \mathrm{d}\mu(s) \leq I_{\mathscr{H}}$ $(p \in [0,1])$.

$$\begin{aligned}
& f\left[\int_T (A_s \natural_{p+1} B_s)\mathrm{d}\mu(s) + t_0\left(I_{\mathscr{H}} - \int_T A_s \natural_p B_s \mathrm{d}\mu(s)\right)\right] \\
=& f\left[\int_T \left(\left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right)^{\frac{p}{2}} A_s^{\frac{1}{2}}\right)^* \left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right)\left(\left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right)^{\frac{p}{2}} A_s^{\frac{1}{2}}\right) \mathrm{d}\mu(s)\right. \\
& \left. + t_0\left(I_{\mathscr{H}} - \int_T A_s \natural_p B_s \mathrm{d}\mu(s)\right)\right] \\
\leq& \gamma_f\left[\int_T A_s^{\frac{1}{2}}\left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right)^{\frac{p}{2}} f\left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right)\left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right)^{\frac{p}{2}} A_s^{\frac{1}{2}} \mathrm{d}\mu(s)\right. \\
& \left. + f(t_0)\left(I_{\mathscr{H}} - \int_T A_s \natural_p B_s \mathrm{d}\mu(s)\right)\right] \qquad \text{(by Lemma 3.1)} \\
=& \gamma_f\left[\int_T A_s^{\frac{1}{2}}\left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right)^{p} f\left(A_s^{-\frac{1}{2}} B_s A_s^{-\frac{1}{2}}\right) A_s^{\frac{1}{2}} \mathrm{d}\mu(s)\right. \\
& \left. + f(t_0)\left(I_{\mathscr{H}} - \int_T A_s \natural_p B_s \mathrm{d}\mu(s)\right)\right] \\
=& \gamma_f\left[\int_T S_p^f(A_s|B_s)\mathrm{d}\mu(s) + f(t_0)\left(I_{\mathscr{H}} - \int_T A_s \natural_p B_s \mathrm{d}\mu(s)\right)\right].
\end{aligned}$$

□

For the discrete case $T = \{1, 2, \ldots, n\}$, we give a reverse of (2.3).

Corollary 3.3: *Let $0 < mA_j \leq B_j \leq MA_j$ $(1 \leq j \leq n)$ for some positive real numbers m, M such that $m < 1 < M$, $\sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I_{\mathscr{H}}$ and $f : (0,\infty) \to [0,\infty)$ be operator concave. Then*

$$\begin{aligned}
& f\left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + t_0\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j \natural_p B_j\right)\right] - \gamma_f f(t_0)\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j \natural_p B_j\right) \\
& \leq \gamma_f S_p^f(\mathbf{A}|\mathbf{B}),
\end{aligned} \tag{3.4}$$

where $t_0 \in [m, M]$, $p \in [0,1]$ and γ_f is given by (3.1).

There is another method to find a reverse of the Choi–Davis–Jensen inequality. If f is a strictly concave differentiable function on an interval $[m, M]$ with $m < M$ and Φ is a unital positive linear map, then

$$\zeta_f I_{\mathscr{H}} + \Phi(f(A)) \geq f(\Phi(A)), \tag{3.5}$$

Where

$$\mu_f = \frac{f(M) - f(m)}{M - m}, \ \nu_f = \frac{Mf(m) - mf(M)}{M - m} \ \text{ and } \ \zeta_f = \max_{m \leq t \leq M}\left\{f(t) - \mu_f t - \nu_f\right\} \tag{3.6}$$

$A \in \mathbb{B}(\mathscr{H})$ is a self-adjoint operator with spectrum in $[m, M]$; see [2, p.101].

Lemma 3.4: *Let $X, C_s \in \mathscr{A}$ $(s \in T)$ such that $\int_T C_s^* C_s \mathrm{d}\mu(s) \le I_{\mathscr{H}}$ and $0 < m \le X \le M$, let $f : (0,\infty) \to [0,\infty)$ be an operator monotone function and $t_0 \in [m, M]$. Then*

$$f\left(\int_T C_s^* X C_s \mathrm{d}\mu(s) + t_0\left(I_{\mathscr{H}} - \int_T C_s^* C_s \mathrm{d}\mu(s)\right)\right) \\ \le \int_T C_s^* f(X) C_s \mathrm{d}\mu(s) + f(t_0)\left(I_{\mathscr{H}} - \int_T C_s^* C_s \mathrm{d}\mu(s)\right) + \zeta_f,$$

where ζ_f is given by (3.6).

Proof: We put $D = (I_{\mathscr{H}} - \int_T C_s^* C_s \mathrm{d}\mu(s))^{\frac{1}{2}}$. Applying inequality (3.5) for the positive unital linear map $\Phi\left(\mathrm{diag}(X, Y)\right) = \int_T C_s^* X C_s \mathrm{d}\mu(s) + D^* Y D$ $(X, Y \in \mathscr{A})$ we get the desired inequality. □

Using Lemma 3.4, by the same argument in the proof of Theorem 3.3 we give the next result.

Theorem 3.5: *Let $\mathbf{A} = (A_t)_{t\in T}, \mathbf{B} = (B_t)_{t\in T}$ be continuous fields of strictly positive operators such that $0 < mA_s \le B_s \le MA_s$ $(s \in T)$ for some positive real numbers m, M, where $m < 1 < M$, $\int_T A_s \mathrm{d}\mu(s) = \int_T B_s \mathrm{d}\mu(s) = I_{\mathscr{H}}$, $f : (0,\infty) \to [0,\infty)$ is operator concave, $t_0 \in [m, M]$ and $p \in [0, 1]$. Then*

$$f\left[\int_T (A_s \natural_{p+1} B_s)\mathrm{d}\mu(s) + t_0\left(I_{\mathscr{H}} - \int_T A_s \natural_p B_s \mathrm{d}\mu(s)\right)\right] - f(t_0)\left(I_{\mathscr{H}} - \int_T A_s \natural_p B_s \mathrm{d}\mu(s)\right) \\ \le \widetilde{S}_p^f(\mathbf{A}|\mathbf{B}) + \zeta_f, \tag{3.7}$$

where ζ_f is given by (3.6).

In the discrete case, we get a reverse of inequality (2.3).

Corollary 3.6: *Let $0 < mA_j \le B_j \le MA_j$ $(1 \le j \le n)$ for some positive real numbers m, M such that $m < 1 < M$, $\sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I_{\mathscr{H}}$ and $f : (0,\infty) \to [0,\infty)$ be operator concave, $p \in [0, 1]$ and $t_0 \in [m, M]$. Then*

$$f\left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + t_0\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j \natural_p B_j\right)\right] - f(t_0)\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j \natural_p B_j\right) \\ \le S_p^f(\mathbf{A}|\mathbf{B}) + \zeta_f, \tag{3.8}$$

where ζ_f is given by (3.6).

Using Corollary 3.6 for the operator monotone functions $f(t) = -t\log t$ and $g(t) = \log t$, respectively, we get the following example.

Example 3.7: Let $0 < mA_j \le B_j \le MA_j$ $(1 \le j \le n)$ for some positive real numbers m, M such that $m < 1 < M$, $\sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I_{\mathscr{H}}$ and $t_0 \in [m, M]$. Then

$$\left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + t_0\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j \natural_p B_j\right)\right]$$

$$
\begin{aligned}
&\times \log\left[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j)+t_0\left(I_{\mathscr{H}}-\sum_{j=1}^{n}A_j\natural_pB_j\right)\right]\\
&-t_0\log(t_0)\left(I_{\mathscr{H}}-\sum_{j=1}^{n}A_j\natural_pB_j\right)\\
\geq\ & S_1(\mathbf{A}|\mathbf{B})+L(1/m,1/M)^{-1}-I(m,M)
\end{aligned}
$$

and

$$
\begin{aligned}
&\log\left[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j)+t_0\left(I_{\mathscr{H}}-\sum_{j=1}^{n}A_j\natural_pB_j\right)\right]-\log(t_0)\left(I_{\mathscr{H}}-\sum_{j=1}^{n}A_j\natural_pB_j\right)\\
&\leq S(\mathbf{A}|\mathbf{B})+\log\left[\frac{1}{e}\left(\frac{M^m}{m^M}\right)^{\frac{1}{M-m}}L(m,M)\right],
\end{aligned}
$$

where

$$
L(a,b)=\begin{cases}\frac{b-a}{\log b-\log a} & ;\, a\neq b\\ a & ;\, a=b\end{cases}\quad \text{and}\quad I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & ;\, a\neq b\\ a & ;\, a=b\end{cases}
$$

are the logarithmic mean and the identric mean of positive real numbers a and b, respectively.

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