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Some Reversed and Refined Callebaut Inequalities Via Kontorovich Constant

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Abstract In this paper, we employ some operator techniques to establish some refinements and reverses of the Callebaut inequality involving the geometric mean and Hadamard product under some mild conditions. In particular, we show

$$\begin{aligned} &K\left(\frac{M^{2t-1}}{m^{2t-1}},2\right)^{r'}\sum_{j=1}^{n}(A_j\sharp_s B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-s}B_j)\\ &\quad+\left(\frac{t-s}{t-1/2}\right)\left(\sum_{j=1}^{n}(A_j\sharp_t B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-t}B_j)-\sum_{j=1}^{n}(A_j\sharp B_j)\circ\sum_{j=1}^{n}(A_j\sharp B_j)\right)\\ &\le\sum_{j=1}^{n}(A_j\sharp_t B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-t}B_j), \end{aligned}$$

where $A_j, B_j \in \mathbb{B}(\mathscr{H})$ $(1 \le j \le n)$ are positive operators such that $0 < m' \le B_j \le m < M \le A_j \le M'$ $(1 \le j \le n)$, either $1 \ge t \ge s > \frac{1}{2}$ or $0 \le t \le s < \frac{1}{2}$, $r' = \min\left\{\frac{t-s}{t-1/2}, \frac{s-1/2}{t-1/2}\right\}$ and $K(t,2) = \frac{(t+1)^2}{4t}$ $(t > 0)$.

Keywords Callebaut inequality · Cauchy–Schwarz inequality · Hadamard product · Operator geometric mean · Kontorovich constant

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1 Introduction and Preliminaries

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with the identity I. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$, and we then write $A \geq 0$. We write $A > 0$ if A is a positive invertible operator. The set of all positive invertible operators is denoted by $\mathbb{B}(\mathscr{H})_+$. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $B \geq A$ if $B - A \geq 0$. The Gelfand map $ftof(A)$ is an isometric $*$-isomorphism between the C^*-algebra $C(\mathrm{sp(A)})$ of a complex-valued continuous functions on the spectrum sp(A) of a self-adjoint operator A and the C^*-algebra generated by I and A. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \geq g(A)$.

It is known that the Hadamard product can be presented by filtering the tensor product $A \otimes B$ through a positive linear map. In fact,$A \circ B = U^*(A \otimes B)U$, where $U : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is the isometry defined by $Ue_j = e_j \otimes e_j$, where (e_j) is an orthonormal basis of the Hilbert space $\mathscr{H}$; see [7].

For $A, B \in \mathbb{B}(\mathscr{H})_+$, the operator geometric mean $A\sharp B$ is defined by $A\sharp B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}$. For $\alpha \in (0, 1)$, the operator-weighted geometric mean is defined by

$$A\sharp_\alpha B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\alpha}A^{\frac{1}{2}}.$$

Callebaut [4] showed the following refinement of the Cauchy–Schwarz inequality

$$\begin{aligned}\left(\sum_{j=1}^{n} x_j^{\frac{1}{2}} y_j^{\frac{1}{2}}\right)^2 &\leq \sum_{j=1}^{n} x_j^{\frac{1+s}{2}} y_j^{\frac{1-s}{2}} \sum_{j=1}^{n} x_j^{\frac{1-s}{2}} y_j^{\frac{1+s}{2}} \\ &\leq \sum_{j=1}^{n} x_j^{\frac{1+t}{2}} y_j^{\frac{1-t}{2}} \sum_{j=1}^{n} x_j^{\frac{1-t}{2}} y_j^{\frac{1+t}{2}} \\ &\leq \left(\sum_{j=1}^{n} x_j\right)\left(\sum_{j=1}^{n} y_j\right), \end{aligned} \tag{1.1}$$

where x_j, y_j $(1 \leq j \leq n)$ are positive real numbers and either $1 \geq t \geq s > \frac{1}{2}$ or $0 \leq t \leq s < \frac{1}{2}$. This is indeed an extension of the Cauchy–Schwarz inequality.
Wada [10] gave an operator version of the Callebaut inequality by showing that if $A, B \in \mathbb{B}(\mathscr{H})_+$, then

$$(A\sharp B)\otimes(A\sharp B)\le\frac{1}{2}\left\{(A\sharp_{\alpha}B)\otimes(A\sharp_{1-\alpha}B)+(A\sharp_{1-\alpha}B)\otimes(A\sharp_{\alpha}B)\right\}$$
$$\le\frac{1}{2}\left\{(A\otimes B)+(B\otimes A)\right\},$$

where $\alpha\in[0,1]$. In [6], the authors showed another operator version of the Callebaut inequality:

$$\begin{aligned}\sum_{j=1}^{n}(A_j\sharp B_j)\circ\sum_{j=1}^{n}(A_j\sharp B_j)&\le\sum_{j=1}^{n}(A_j\sharp_s B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-s}B_j)\\&\le\sum_{j=1}^{n}(A_j\sharp_t B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-t}B_j)\\&\le\left(\sum_{j=1}^{n}A_j\right)\circ\left(\sum_{j=1}^{n}B_j\right),\end{aligned}\tag{1.2}$$

where $A_j, B_j\in\mathbb{B}(\mathscr{H})_+$ $(1\le j\le n)$ and either $1\ge t\ge s>\frac{1}{2}$ or $0\le t\le s<\frac{1}{2}$. In [3], the authors presented the following refinement of inequality (1.2) as follows:

$$\begin{aligned}&\sum_{j=1}^{n}(A_j\sharp_s B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-s}B_j)\\&\quad\le\sum_{j=1}^{n}(A_j\sharp_s B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-s}B_j)\\&\qquad+\left(\frac{t-s}{s-1/2}\right)\left(\sum_{j=1}^{n}(A_j\sharp_s B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-s}B_j)-\sum_{j=1}^{n}(A_j\sharp B_j)\circ\sum_{j=1}^{n}(A_j\sharp B_j)\right)\\&\quad\le\sum_{j=1}^{n}(A_j\sharp_t B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-t}B_j),\end{aligned}\tag{1.3}$$

in which $A_j, B_j\in\mathbb{B}(\mathscr{H})_+$ $(1\le j\le n)$ and either $1\ge t\ge s>\frac{1}{2}$ or $0\le t\le s<\frac{1}{2}$. There have been obtained several Cauchy–Schwarz type inequalities for Hilbert space operators and matrices; see [1,2,5,8] and references therein.

In this paper, we present some refinements and reverses of the Callebaut inequality involving the weighted geometric mean and Hadamard product of Hilbert space operators.

2 Further Refienements of the Callebaut Inequality Involving Hadamard Product

The Kontorovich constant is

$$K(t,2)=\frac{(t+1)^2}{4t}\qquad(t>0).$$

The classical Young inequality states that

$$a^{\nu}b^{1-\nu} \le \nu a + (1-\nu)b,$$

where $a, b \ge 0$ and $\nu \in [0, 1]$. Recently, Zuo et. al. [11] showed an improvement of the Young inequality as follows:

$$K\left(\sqrt{\frac{a}{b}}, 2\right)^{r} a^{\nu}b^{1-\nu} \le \nu a + (1-\nu)b,$$

where $a, b > 0$, $\nu \in [0, 1]$, $r = \min\{\nu, 1-\nu\}$. Applying this inequality, J. Wu and J. Zhao [9] showed the following refinement of the Young inequality:

$$K\left(\sqrt{\frac{a}{b}}, 2\right)^{r'} a^{\nu}b^{1-\nu} + r\left(\sqrt{a}-\sqrt{b}\right)^{2} \le \nu a + (1-\nu)b, \tag{2.1}$$

where $a, b > 0$, $\nu \in [0, 1] - \left\{\frac{1}{2}\right\}$, $r = \min\{\nu, 1-\nu\}$ and $r' = \min\{2r, 1-2r\}$. Using (2.1), we get the following lemmas.

Lemma 2.1 *Let* $a, b > 0$ *and* $\nu \in [0, 1] - \left\{\frac{1}{2}\right\}$. *Then*

$$K\left(\sqrt{\frac{a}{b}}, 2\right)^{r'} \left(a^{\nu}b^{1-\nu} + a^{1-\nu}b^{\nu}\right) + 2r\left(\sqrt{a}-\sqrt{b}\right)^{2} \le a + b, \tag{2.2}$$

where $r = \min\{\nu, 1-\nu\}$ *and* $r' = \min\{2r, 1-2r\}$.

Lemma 2.2 *Let* $0 < m' \le B \le m < M \le A \le M'$ *and either* $1 \ge t \ge s > \frac{1}{2}$ *or* $0 \le t \le s < \frac{1}{2}$. *Then*

$$\begin{aligned} K\left(\frac{M^{2t-1}}{m^{2t-1}}, 2\right)^{r'} & (A^{s} \otimes B^{1-s} + A^{1-s} \otimes B^{s}) \\ & + \left(\frac{t-s}{t-1/2}\right)\left(A^{t} \otimes B^{1-t} + A^{1-t} \otimes B^{t} - 2(A^{\frac{1}{2}} \otimes B^{\frac{1}{2}})\right) \\ & \le A^{t} \otimes B^{1-t} + A^{1-t} \otimes B^{t}, \end{aligned} \tag{2.3}$$

where $r' = \min\left\{\frac{t-s}{t-1/2}, \frac{s-1/2}{t-1/2}\right\}$.

Proof Let $a > 0$. If we replace b by a^{-1} and take $\nu = \frac{1-\mu}{2}$ (2.2), then we get

$$K(a, 2)^{r'}\left(a^{\mu} + a^{-\mu}\right) + (1-\mu)\left(a + a^{-1} - 2\right) \le a + a^{-1}, \tag{2.4}$$

in which $\mu \in (0, 1]$ and $r' = \min\{1-\mu, \mu\}$. Let us fix positive real numbers α, β such that $\beta < \alpha$. It follows from $0 < m' \le B \le m < M \le A \le M'$ that

$I \leq h = \left(\frac{M}{m}\right)^{\alpha} \leq A^{\alpha} \otimes B^{-\alpha} \leq h' = \left(\frac{M'}{m'}\right)^{\alpha}$ and $\mathrm{sp}(A^{\alpha} \otimes B^{-\alpha}) \subseteq [h, h'] \subseteq (1, +\infty)$. Since the Kontorovich constant $\frac{(a+1)^2}{4a}$ is an increasing function on $(1, +\infty)$, by (2.4), we have

$$K\left(\frac{M^{\alpha}}{m^{\alpha}}, 2\right)^{r'} \left(a^{\mu} + a^{-\mu}\right) + (1-\mu)\left(a + a^{-1} - 2\right) \leq a + a^{-1},$$

where $\mu \in (0, 1]$ and $r' = \min\{1-\mu, \mu\}$. Using the functional calculus, if we replace a by the operator $A^{\alpha} \otimes B^{-\alpha}$ and μ by $\frac{\beta}{\alpha}$ we have

$$\begin{aligned} K\left(\frac{M^{\alpha}}{m^{\alpha}}, 2\right)^{r'} & \left(A^{\beta} \otimes B^{-\beta} + A^{-\beta} \otimes B^{\beta}\right) \\ & + \left(1 - \frac{\beta}{\alpha}\right)\left(A^{\alpha} \otimes B^{-\alpha} + A^{-\alpha} \otimes B^{\alpha} - 2I\right) \\ & \leq A^{\alpha} \otimes B^{-\alpha} + A^{-\alpha} \otimes B^{\alpha}, \end{aligned} \tag{2.5}$$

where $r' = \min\left\{1 - \frac{\beta}{\alpha}, \frac{\beta}{\alpha}\right\}$. Multiplying, we both sides of (2.5) by $A^{\frac{1}{2}} \otimes B^{\frac{1}{2}}$, we obtain

$$\begin{aligned} K\left(\frac{M^{\alpha}}{m^{\alpha}}, 2\right)^{r'} & \left(A^{1+\beta} \otimes B^{1-\beta} + A^{1-\beta} \otimes B^{1+\beta}\right) \\ & + \left(1 - \frac{\beta}{\alpha}\right)\left(A^{1+\alpha} \otimes B^{1-\alpha} + A^{1-\alpha} \otimes B^{1+\alpha} - 2(A \otimes B)\right) \\ & \leq A^{1+\alpha} \otimes B^{1-\alpha} + A^{1-\alpha} \otimes B^{1+\alpha}. \end{aligned} \tag{2.6}$$

Now, if we replace α, β, A, B by $2t-1, 2s-1, A^{\frac{1}{2}}, B^{\frac{1}{2}}$, respectively, in (2.6), we obtain

$$\begin{aligned} K\left(\frac{M^{2t-1}}{m^{2t-1}}, 2\right)^{r'} & (A^{s} \otimes B^{1-s} + A^{1-s} \otimes B^{s}) \\ & + \left(\frac{t-s}{t-1/2}\right)\left(A^{t} \otimes B^{1-t} + A^{1-t} \otimes B^{t} - 2(A^{\frac{1}{2}} \otimes B^{\frac{1}{2}})\right) \\ & \leq A^{t} \otimes B^{1-t} + A^{1-t} \otimes B^{t} \end{aligned}$$

for either $1 \geq t \geq s > \frac{1}{2}$ or $0 \leq t \leq s < \frac{1}{2}$ and $r' = \min\left\{\frac{t-s}{t-1/2}, \frac{s-1/2}{t-1/2}\right\}$. □

We are ready to prove the first result of this section.

Theorem 2.3 *Let* $0 < m' \leq B_j \leq m < M \leq A_j \leq M'$ $(1 \leq j \leq n)$ *and either* $1 \geq t \geq s > \frac{1}{2}$ *or* $0 \leq t \leq s < \frac{1}{2}$. *Then*

$$K\left(\frac{M^{2t-1}}{m^{2t-1}},2\right)^{r'}\sum_{j=1}^{n}(A_j\sharp_s B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-s}B_j)$$
$$+\left(\frac{t-s}{t-1/2}\right)\left(\sum_{j=1}^{n}(A_j\sharp_t B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-t}B_j)-\sum_{j=1}^{n}(A_j\sharp B_j)\circ\sum_{j=1}^{n}(A_j\sharp B_j)\right)$$
$$\leq\sum_{j=1}^{n}(A_j\sharp_t B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-t}B_j), \tag{2.7}$$

where $r'=\min\left\{\frac{t-s}{t-1/2},\frac{s-1/2}{t-1/2}\right\}$.

Proof Put $C_j=A_j^{-\frac{1}{2}}B_jA_j^{-\frac{1}{2}}$ $(1\leq j\leq n)$. By inequality (2.3), we get

$$K\left(\frac{M^{2t-1}}{m^{2t-1}},2\right)^{r'}\left(C_j^s\otimes C_i^{1-s}+C_j^{1-s}\otimes C_i^s\right)$$
$$+\left(\frac{t-s}{t-1/2}\right)\left(C_j^t\otimes C_i^{1-t}+C_j^{1-t}\otimes C_i^t-2\big(C_j^{\frac{1}{2}}\otimes C_i^{\frac{1}{2}}\big)\right)$$
$$\leq C_j^t\otimes C_i^{1-t}+C_j^{1-t}\otimes C_i^t\qquad(1\leq i,j\leq n). \tag{2.8}$$

Multiplying both sides of (2.8) by $A_j^{\frac{1}{2}}\otimes A_i^{\frac{1}{2}}$, we get

$$K\left(\frac{M^{2t-1}}{m^{2t-1}},2\right)^{r'}\big((A_j\sharp_s B_j)\otimes(A_i\sharp_{1-s}B_i)+(A_j\sharp_{1-s}B_j)\otimes(A_i\sharp_s B_i)\big)$$
$$+\left(\frac{t-s}{t-1/2}\right)\Big((A_j\sharp_t B_j)\otimes(A_i\sharp_{1-t}B_i)+(A_j\sharp_{1-t}B_j)$$
$$\otimes(A_i\sharp_t B_i)-2(A_j\sharp B_j)\otimes(A_i\sharp B_i)\Big)$$
$$\leq(A_j\sharp_t B_j)\otimes(A_i\sharp_{1-t}B_i)+(A_j\sharp_{1-t}B_j)\otimes(A_i\sharp_t B_i) \tag{2.9}$$

for all $1\leq i,j\leq n$. Therefore

$$K\left(\frac{M^{2t-1}}{m^{2t-1}},2\right)^{r'}\left(\sum_{j=1}^{n}(A_j\sharp_s B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-s}B_j)\right)$$
$$+\left(\frac{t-s}{t-1/2}\right)\left(\sum_{j=1}^{n}(A_j\sharp_t B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-t}B_j)-\left(\sum_{j=1}^{n}A_j\sharp B_j\right)\circ\left(\sum_{j=1}^{n}A_j\sharp B_j\right)\right)$$
$$=\frac{1}{2}\sum_{i,j=1}^{n}\Big[K\left(\frac{M^{2t-1}}{m^{2t-1}},2\right)^{r'}\big((A_j\sharp_s B_j)\circ(A_i\sharp_{1-s}B_i)+(A_j\sharp_{1-s}B_j)\circ(A_i\sharp_s B_i)\big)$$

$$+\left(\frac{t-s}{t-1/2}\right)\Big((A_j\sharp_t B_j)\circ(A_i\sharp_{1-t}B_i)+(A_j\sharp_{1-t}B_j)$$
$$\circ(A_i\sharp_t B_i)-2(A_j\sharp B_j)\circ(A_i\sharp B_i)\Big)\Big]$$
$$\le\frac{1}{2}\sum_{i,j=1}^{n}\big((A_j\sharp_t B_j)\circ(A_i\sharp_{1-t}B_i)+(A_j\sharp_{1-t}B_j)\circ(A_i\sharp_t B_i)\big)\ \text{(by inequality (2.9))}$$
$$=\sum_{j=1}^{n}(A_j\sharp_t B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-t}B_j).$$

Remark 2.4 It follows from

$$\left(\frac{t-s}{t-1/2}\right)\left(\sum_{j=1}^{n}(A_j\sharp_t B_j)\circ\sum_{j=1}^{n}(A_j\sharp_{1-t}B_j)-\sum_{j=1}^{n}(A_j\sharp B_j)\circ\sum_{j=1}^{n}(A_j\sharp B_j)\right)\ge 0,$$

where $A_j, B_j\in\mathbb{B}(\mathscr{H})_+$ $(1\le j\le n)$, either $1\ge t\ge s>\frac{1}{2}$ or $0\le t\le s<\frac{1}{2}$ and $K(t,2)=\frac{(t+1)^2}{4t}\ge 1$ $(t>0)$ that inequality (2.7) is a refinement of the second inequality of inequalities (1.2) and (1.3).

We conclude an application of Theorem 2.3 for numerical cases which is a refinement of inequality (1.1).

Corollary 2.5 *Let* $0<m'\le y_j\le m<M\le x_j\le M'$ $(1\le j\le n)$ *and either* $-1\le t\le s<0$ *or* $1\ge t\ge s>0$. *Then*

$$\sum_{j=1}^{n}x_j^{\frac{1+s}{2}}y_j^{\frac{1-s}{2}}\sum_{j=1}^{n}x_j^{\frac{1-s}{2}}y_j^{\frac{1+s}{2}}$$
$$\le K\left(\frac{M^t}{m^t},2\right)^{r'}\sum_{j=1}^{n}x_j^{\frac{1+s}{2}}y_j^{\frac{1-s}{2}}\sum_{j=1}^{n}x_j^{\frac{1-s}{2}}y_j^{\frac{1+s}{2}}$$
$$+\left(\frac{t-s}{t}\right)\left(\sum_{j=1}^{n}x_j^{\frac{1+t}{2}}y_j^{\frac{1-t}{2}}\sum_{j=1}^{n}x_j^{\frac{1-t}{2}}y_j^{\frac{1+t}{2}}-\left(\sum_{j=1}^{n}x_j^{\frac{1}{2}}y_j^{\frac{1}{2}}\right)^2\right)$$
$$\le\sum_{j=1}^{n}x_j^{\frac{1+t}{2}}y_j^{\frac{1-t}{2}}\sum_{j=1}^{n}x_j^{\frac{1-t}{2}}y_j^{\frac{1+t}{2}},$$

where $r'=\min\left\{\frac{t-s}{t},\frac{s}{t}\right\}$.

Proof If we put $A_j=x_j$, $B_j=y_j$ $(1\le j\le n)$ in Theorem 2.3 and inequality (1.2), then we get

$$\begin{aligned}
&\sum_{j=1}^{n} x_j^{1-s} y_j^{s} \sum_{j=1}^{n} x_j^{s} y_j^{1-s} \\
&\quad \le K\left(\frac{M^{2t-1}}{m^{2t-1}}, 2\right)^{r'} \sum_{j=1}^{n} x_j^{1-s} y_j^{s} \sum_{j=1}^{n} x_j^{s} y_j^{1-s} \\
&\qquad + \left(\frac{t-s}{t-1/2}\right)\left(\sum_{j=1}^{n} x_j^{1-t} y_j^{t} \sum_{j=1}^{n} x_j^{t} y_j^{1-t} - \left(\sum_{j=1}^{n} x_j^{\frac{1}{2}} y_j^{\frac{1}{2}}\right)^2\right) \\
&\quad \le \sum_{j=1}^{n} x_j^{1-t} y_j^{t} \sum_{j=1}^{n} x_j^{t} y_j^{1-t},
\end{aligned}$$

where $r' = \min\left\{\frac{t-s}{t-1/2}, \frac{s-1/2}{t-1/2}\right\}$ and either $0 \le t \le s < \frac{1}{2}$ or $1 \ge t \ge s > \frac{1}{2}$. Now if we replace s by $\frac{s+1}{2}$ and t by $\frac{t+1}{2}$, respectively, where either $-1 \le t \le s < 0$ or $1 \ge t \ge s > 0$, then we obtain the desired inequalities. □

Lemma 2.6 *Let $a, b > 0$ and $\nu \in (0, 1)$. Then*

$$\begin{aligned}
&a^{\nu} b^{1-\nu} + a^{1-\nu} b^{\nu} + 2r(\sqrt{a} - \sqrt{b})^2 \\
&\quad + r'\left(2\sqrt{ab} + a + b - 2a^{\frac{1}{4}} b^{\frac{3}{4}} - 2a^{\frac{3}{4}} b^{\frac{1}{4}}\right) \le a + b,
\end{aligned}$$

where $r = \min\{\nu, 1-\nu\}$ and $r' = \min\{2r, 1-2r\}$.

Proof Let $a, b > 0$, $\nu \in (0, 1)$, $r = \min\{\nu, 1-\nu\}$ and $r' = \min\{2r, 1-2r\}$. Applying [12, Lemma1], we have the inequalities

$$a^{1-\nu} b^{\nu} + \nu(\sqrt{a} - \sqrt{b})^2 + r'(\sqrt[4]{ab} - \sqrt{a})^2 \le (1-\nu)a + \nu b,$$

where $0 < \nu \le \frac{1}{2}$ and

$$a^{1-\nu} b^{\nu} + (1-\nu)(\sqrt{a} - \sqrt{b})^2 + r'(\sqrt[4]{ab} - \sqrt{b})^2 \le (1-\nu)a + \nu b,$$

where $\frac{1}{2} < \nu < 1$. Summing these inequalities, we get the desired result. □

Lemma 2.7 *Let $A_j, B_j \in \mathbb{B}(\mathscr{H})_+$ $(1 \le j \le n)$ and either $1 \ge t \ge s > \frac{1}{2}$ or $0 \le t \le s < \frac{1}{2}$. Then*

$$\begin{aligned}
&(A^s \otimes B^{1-s} + A^{1-s} \otimes B^s) \\
&\quad + \left(\frac{t-s}{t-1/2}\right)\left(A^s \otimes B^{1-s} + A^{1-s} \otimes B^s - 2(A^{\frac{1}{2}} \otimes B^{\frac{1}{2}})\right)
\end{aligned}$$

$$+ r'\Big(A^t \otimes B^{1-s} + A^{1-s} \otimes B^s + 2(A^{\frac{1}{2}} \otimes B^{\frac{1}{2}}) - 2A^{\frac{1+2s}{4}} \otimes B^{\frac{3-2s}{4}} - 2A^{\frac{3-2s}{4}} \otimes B^{\frac{1+2s}{4}}\Big)$$
$$\le A^t \otimes B^{1-t} + A^{1-t} \otimes B^t,$$

where $r' = \min\left\{\frac{t-s}{t-1/2}, \frac{s-1/2}{t-1/2}\right\}$.

Using Lemma 2.7 and the same argument in the proof of Lemma 2.2, we get another refinement of inequality (1.2).

Theorem 2.8 *Let* $A_j, B_j \in \mathbb{B}(\mathscr{H})_+$ $(1 \le j \le n)$ *and either* $1 \ge t \ge s > \frac{1}{2}$ *or* $0 \le t \le s < \frac{1}{2}$. *Then*

$$\begin{aligned}
&\sum_{j=1}^n (A_j \sharp_s B_j) \circ \sum_{j=1}^n (A_j \sharp_{1-s} B_j) \\
&\quad + \left(\frac{t-s}{t-1/2}\right)\left(\sum_{j=1}^n (A_j \sharp_s B_j) \circ \sum_{j=1}^n (A_j \sharp_{1-s} B_j) - \sum_{j=1}^n (A_j \sharp B_j) \circ \sum_{j=1}^n (A_j \sharp B_j)\right) \\
&\quad + r'\Big(\sum_{j=1}^n (A_j \sharp_s B_j) \circ \sum_{j=1}^n (A_j \sharp_{1-s} B_j) + \sum_{j=1}^n (A_j \sharp B_j) \circ \sum_{j=1}^n (A_j \sharp B_j) \\
&\quad - 2\sum_{j=1}^n (A_j \sharp_{\frac{3-2s}{4}} B_j) \circ \sum_{j=1}^n (A_j \sharp_{\frac{1+2s}{4}} B_j)\Big) \\
&\le \sum_{j=1}^n (A_j \sharp_t B_j) \circ \sum_{j=1}^n (A_j \sharp_{1-t} B_j),
\end{aligned} \tag{2.10}$$

where $r' = \min\left\{\frac{t-s}{t-1/2}, \frac{s-1/2}{t-1/2}\right\}$.

Remark 2.9 If $A_j, B_j \in \mathbb{B}(\mathscr{H})_+$ $(1 \le j \le n)$ and either $1 \ge t \ge s > \frac{1}{2}$ or $0 \le t \le s < \frac{1}{2}$, then

$$\sum_{j=1}^n (A_j \sharp_s B_j) \circ \sum_{j=1}^n (A_j \sharp_{1-s} B_j) + \sum_{j=1}^n (A_j \sharp B_j) \circ \sum_{j=1}^n (A_j \sharp B_j) - 2\sum_{j=1}^n (A_j \sharp_{\frac{3-2s}{4}} B_j) \circ \sum_{j=1}^n (A_j \sharp_{\frac{1+2s}{4}} B_j)$$

and

$$\left(\frac{t-s}{t-1/2}\right)\left(\sum_{j=1}^n (A_j \sharp_s B_j) \circ \sum_{j=1}^n (A_j \sharp_{1-s} B_j) - \sum_{j=1}^n (A_j \sharp B_j) \circ \sum_{j=1}^n (A_j \sharp B_j)\right)$$

are positive operators. Hence, inequality (2.10) is another refinement of the second inequality of inequalities (1.2) and (1.3).

If we put $B_j = I$ $(1 \le j \le n)$ in Theorem 2.8, then we get the next result.

Corollary 2.10 *Let* $A_j \in \mathbb{B}(\mathscr{H})_+$ $(1 \le j \le n)$ *and either* $1 \ge t \ge s > \frac{1}{2}$ *or* $0 \le t \le s < \frac{1}{2}$. *Then*

$$\begin{aligned}
&\sum_{j=1}^{n} A_j^{1-s} \circ \sum_{j=1}^{n} A_j^{s} \\
&\quad + \left(\frac{t-s}{t-1/2}\right)\left(\sum_{j=1}^{n} A_j^{1-s} \circ \sum_{j=1}^{n} A_j^{s} - \sum_{j=1}^{n} A_j^{\frac{1}{2}} \circ \sum_{j=1}^{n} A_j^{\frac{1}{2}}\right) \\
&\quad + r'\Big(\sum_{j=1}^{n} A_j^{1-s} \circ \sum_{j=1}^{n} A_j^{s} + \sum_{j=1}^{n} A_j^{\frac{1}{2}} \circ \sum_{j=1}^{n} A_j^{\frac{1}{2}} \\
&\quad - 2\sum_{j=1}^{n} A_j^{\frac{1+2s}{4}} \circ \sum_{j=1}^{n} A_j^{\frac{3-2s}{4}}\Big) \\
&\le \sum_{j=1}^{n} A_j^{1-t} \circ \sum_{j=1}^{n} A_j^{t},
\end{aligned}$$

where $r' = \min\left\{\frac{t-s}{t-1/2}, \frac{s-1/2}{t-1/2}\right\}$.

If in Theorem 2.8 we replace A_j, B_j, s, t, by $x_j, y_j, \frac{s+1}{2}, \frac{t+1}{2}$ $(1 \le j \le n)$, respectively, then we obtain another refinement of inequality (1.1).

Corollary 2.11 *Let* x_j, y_j $(1 \le j \le n)$ *be positive numbers and either* $-1 \le t \le s < 0$ *or* $1 \ge t \ge s > 0$. *Then*

$$\begin{aligned}
&\sum_{j=1}^{n} x_j^{\frac{1+s}{2}} y_j^{\frac{1-s}{2}} \sum_{j=1}^{n} x_j^{\frac{1-s}{2}} y_j^{\frac{1+s}{2}} \\
&\quad + \left(\frac{t-s}{t}\right)\left(\sum_{j=1}^{n} x_j^{\frac{1+s}{2}} y_j^{\frac{1-s}{2}} \sum_{j=1}^{n} x_j^{\frac{1-s}{2}} y_j^{\frac{1+s}{2}} - \left(\sum_{j=1}^{n} x_j^{\frac{1}{2}} y_j^{\frac{1}{2}}\right)^2\right) \\
&\quad + r'\Big(\sum_{j=1}^{n} x_j^{\frac{1+s}{2}} y_j^{\frac{1-s}{2}} \sum_{j=1}^{n} x_j^{\frac{1-s}{2}} y_j^{\frac{1+s}{2}} + \left(\sum_{j=1}^{n} x_j^{\frac{1}{2}} y_j^{\frac{1}{2}}\right)^2 \\
&\quad - 2\sum_{j=1}^{n} x_j^{\frac{2+s}{4}} y_j^{\frac{2-s}{4}} \sum_{j=1}^{n} x_j^{\frac{2-s}{4}} y_j^{\frac{2+s}{4}}\Big) \\
&\le \sum_{j=1}^{n} x_j^{\frac{1+t}{2}} y_j^{\frac{1-t}{2}} \sum_{j=1}^{n} x_j^{\frac{1-t}{2}} y_j^{\frac{1+t}{2}},
\end{aligned}$$

where $r' = \min\left\{\frac{t-s}{t}, \frac{s-1}{t-1}\right\}$.

3 Some Reverses of the Callebaut Type Inequality

In [9], the authors showed a reverse of the Young inequality as follows:

$$\nu a + (1-\nu)b \leq K\left(\sqrt{\frac{a}{b}}, 2\right)^{-r'} a^{\nu} b^{1-\nu} + s\left(\sqrt{a} - \sqrt{b}\right)^2, \tag{3.1}$$

in which $a, b > 0$, $\nu \in [0, 1] - \left\{\frac{1}{2}\right\}$, $r = \min\{\nu, 1-\nu\}$, $r' = \min\{2r, 1-2r\}$ and $s = \max\{\nu, 1-\nu\}$. Applying (3.1), we have the next result.

Lemma 3.1 *Let $a, b > 0$ and $\nu \in [0, 1] - \left\{\frac{1}{2}\right\}$. Then*

$$a + b \leq K\left(\sqrt{\frac{a}{b}}, 2\right)^{-r'} \left(a^{\nu} b^{1-\nu} + a^{1-\nu} b^{\nu}\right) + 2s\left(\sqrt{a} - \sqrt{b}\right)^2,$$

where $r = \min\{\nu, 1-\nu\}$, $r' = \min\{2r, 1-2r\}$ and $s = \max\{\nu, 1-\nu\}$. In particular, If $\nu \in [0, \frac{1}{2})$, then

$$a + a^{-1} \leq K(a, 2)^{-r'} \left(a^{1-2\nu} + a^{-(1-2\nu)}\right) + 2(1-\nu)\left(a^{\frac{1}{2}} - a^{\frac{-1}{2}}\right)^2. \tag{3.2}$$

Now, utilizing inequality (3.2) and the same argument in the proof of Lemma 2.2, we can accomplish the corresponding result:

Lemma 3.2 *Let $0 < m' \leq B \leq m < M \leq A \leq M'$ and either $1 \geq t \geq s > \frac{1}{2}$ or $0 \leq t \leq s < \frac{1}{2}$. Then*

$$A^t \otimes B^{1-t} + A^{1-t} \otimes B^t \leq K\left(\frac{M^{2t-1}}{m^{2t-1}}, 2\right)^{-r'} \left(A^s \otimes B^{1-s} + A^{1-s} \otimes B^s\right)$$
$$+ \left(\frac{s-1/2}{t-1/2}\right)\left(A^t \otimes B^{1-t} + A^{1-t} \otimes B^t - 2(A^{\frac{1}{2}} \otimes B^{\frac{1}{2}})\right),$$

where $r' = \min\left\{\frac{t-s}{t-1/2}, \frac{s-1/2}{t-1/2}\right\}$.

As a consequence of Lemma 3.2, we have the following result:

Theorem 3.3 *Let $0 < m' \leq B_j \leq m < M \leq A_j \leq M'$ $(1 \leq j \leq n)$. Then*

$$\sum_{j=1}^{n}(A_j \sharp_t B_j) \circ \sum_{j=1}^{n}(A_j \sharp_{1-t} B_j) \leq K\left(\frac{M^{2t-1}}{m^{2t-1}}, 2\right)^{-r'} \sum_{j=1}^{n}(A_j \sharp_s B_j) \circ \sum_{j=1}^{n}(A_j \sharp_{1-s} B_j)$$
$$+ \left(\frac{s-1/2}{t-1/2}\right)\left(\sum_{j=1}^{n}(A_j \sharp_t B_j) \circ \sum_{j=1}^{n}(A_j \sharp_{1-t} B_j) - \sum_{j=1}^{n}(A_j \sharp B_j) \circ \sum_{j=1}^{n}(A_j \sharp B_j)\right)$$

for either $1 \geq t \geq s > \frac{1}{2}$ or $0 \leq t \leq s < \frac{1}{2}$ and $r' = \min\left\{\frac{t-s}{t-1/2}, \frac{s-1/2}{t-1/2}\right\}$.

Remark 3.4 If we put $t = 1$ and $1 \geq s > \frac{1}{2}$ in Theorem 3.3, then we get a reverse of the third inequality of (1.2):

$$\left(\sum_{j=1}^{n} A_j\right) \circ \left(\sum_{j=1}^{n} B_j\right) \leq K\left(\frac{M}{m}, 2\right)^{-r'} \sum_{j=1}^{n}(A_j \natural_s B_j) \circ \sum_{j=1}^{n}(A_j \natural_{1-s} B_j)$$
$$+ (2s-1)\left(\left(\sum_{j=1}^{n} A_j\right) \circ \left(\sum_{j=1}^{n} B_j\right) - \sum_{j=1}^{n}(A_j \natural B_j) \circ \sum_{j=1}^{n}(A_j \natural B_j)\right),$$

where $r' = \min\{2-2s, 2s-1\}$.

If we replace A_j, B_j, s, t, by $x_j, y_j, \frac{s+1}{2}, \frac{t+1}{2}$ $(1 \leq j \leq n)$ in Theorem 3.3, respectively, then we obtain a reverse of the second inequality of (1.1).

Corollary 3.5 *Let* $0 < m' \leq y_j \leq m < M \leq x_j \leq M'$ $(1 \leq j \leq n)$. *Then*

$$\sum_{j=1}^{n} x_j^{\frac{1+t}{2}} y_j^{\frac{1-t}{2}} \sum_{j=1}^{n} x_j^{\frac{1-t}{2}} y_j^{\frac{1+t}{2}} \leq K\left(\frac{M^{2t-1}}{m^{2t-1}}, 2\right)^{-r'} \sum_{j=1}^{n} x_j^{\frac{1+s}{2}} y_j^{\frac{1-s}{2}} \sum_{j=1}^{n} x_j^{\frac{1-s}{2}} y_j^{\frac{1+s}{2}}$$
$$+ \left(\frac{s-1/2}{t-1/2}\right)\left(\sum_{j=1}^{n} x_j^{\frac{1+t}{2}} y_j^{\frac{1-t}{2}} \sum_{j=1}^{n} x_j^{\frac{1-t}{2}} y_j^{\frac{1+t}{2}} - \left(\sum_{j=1}^{n} x_j^{\frac{1}{2}} y_j^{\frac{1}{2}}\right)^2\right)$$

for either $1 \geq t \geq s > \frac{1}{2}$ *or* $0 \leq t \leq s < \frac{1}{2}$ *and* $r' = \min\left\{\frac{t-s}{t-1/2}, \frac{s-1/2}{t-1/2}\right\}$.

Proposition 3.6 *Let* $0 < m' \leq B_j \leq m < M \leq A_j \leq M'$ $(1 \leq j \leq n)$ *and either* $1 \geq t \geq s > \frac{1}{2}$ *or* $0 \leq t \leq s < \frac{1}{2}$. *Then*

$$K\left(h^{2t-1}, 2\right)^{r'} \sum_{j=1}^{n}(A_j \natural_s B_j) \circ \sum_{j=1}^{n}(A_j \natural_{1-s} B_j) + \left(\frac{t-s}{t-1/2}\right)\left(\sqrt{h} - \sqrt{\frac{1}{h}}\right)^2$$
$$\leq \sum_{j=1}^{n}(A_j \natural_t B_j) \circ \sum_{j=1}^{n}(A_j \natural_{1-t} B_j)$$
$$\leq K\left(h^{2t-1}, 2\right)^{-r'} \sum_{j=1}^{n}(A_j \natural_s B_j) \circ \sum_{j=1}^{n}(A_j \natural_{1-s} B_j)$$
$$+ \left(\frac{s-1/2}{t-1/2}\right)\left(\sqrt{h'} - \sqrt{\frac{1}{h'}}\right)^2,$$

where $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ *and* $r' = \min\left\{\frac{t-s}{t-1/2}, \frac{s-1/2}{t-1/2}\right\}$.

Proof Since the function $f(a) = a - \frac{1}{a}$ is increasing on $(0, \infty)$, we have

$$\left(\sqrt{h} - \sqrt{\frac{1}{h}}\right)^2 \leq \left(\sqrt{a} - \sqrt{\frac{1}{a}}\right)^2 \leq \left(\sqrt{h'} - \sqrt{\frac{1}{h'}}\right)^2 \quad (h \leq a \leq h').$$

Applying inequalities (2.2), (3.2) and the same argument in the proof of Theorem 2.3, we get the desired result.

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