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Norm inequalities involving a special class of functions for sector matrices

Davood Afraz[1], Rahmatollah Lashkaripour[1*] and Mojtaba Bakherad[1]

*Correspondence:
lashkari@hamoon.usb.ac.ir
[1]Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

Abstract

In this paper, we present some unitarily invariant norm inequalities for sector matrices involving a special class of functions. In particular, if $Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$ is a $2n \times 2n$ matrix such that numerical range of Z is contained in a sector region S_α for some $\alpha \in [0, \frac{\pi}{2})$, then, for a submultiplicative function h of the class $\mathcal{C}$ and every unitarily invariant norm, we have

$$\left\| h(|Z_{ij}|^2) \right\| \le \left\| h^r(\sec(\alpha)|Z_{11}|) \right\|^{\frac{1}{r}} \left\| h^s(\sec(\alpha)|Z_{22}|) \right\|^{\frac{1}{s}},$$

where r and s are positive real numbers with $\frac{1}{r} + \frac{1}{s} = 1$ and $i, j = 1, 2$. We also extend some unitarily invariant norm inequalities for sector matrices.

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1 Introduction and preliminaries

Let $\mathcal{M}_n$ be the algebra of all $n \times n$ complex matrices. For $Z \in \mathcal{M}_n$, the conjugate transpose of Z is denoted by Z^*. A complex matrix $Z \in \mathcal{M}_{2n}$ can be partitioned as a 2×2 block matrix

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}, \tag{1}$$

where $Z_{ij} \in \mathcal{M}_n$ $(i, j = 1, 2)$. For $Z \in \mathcal{M}_n$, let $Z = \mathcal{R}e(Z) + i\mathcal{I}m(Z)$ be the Cartesian decomposition of Z, where the Hermitian matrices $\mathcal{R}e(Z) = \frac{Z+Z^*}{2}$ and $\mathcal{I}m(Z) = \frac{Z-Z^*}{2i}$ are called the real and imaginary parts of Z, respectively. We say that a matrix $Z \in \mathcal{M}_n$ is positive semidefinite if $z^*Zz \ge 0$ for all complex numbers z. For $Z \in \mathcal{M}_n$, let $s_1(Z) \ge s_2(Z) \ge \cdots \ge s_n(Z)$ denote the singular values of Z, i.e. the eigenvalues of the positive semidefinite matrix $|Z| = (Z^*Z)^{\frac{1}{2}}$ arranged in a decreasing order and repeated according to multiplicity. Note that $s_j(Z) = s_j(Z^*) = s_j(|Z|)$ for $j = 1, 2, \ldots, n$. A norm $\|\cdot\|$ on $\mathcal{M}_n$ is said to be unitarily invariant if $\|UZV\| = \|Z\|$ for every $Z \in \mathcal{M}_n$ and for every unitary $U, V \in \mathcal{M}_n$. For $Z \in \mathcal{M}_n$ and $p > 0$, let $\|Z\|_p = (\sum_{j=1}^n s_j^p(Z))^{\frac{1}{p}}$. This defines the Schatten p-norm (quasinorm) for $p \ge 1$ $(0 < p < 1)$. It is clear that the Schatten p-norm is an unitarily invariant norm. The

w-norm of a matrix $Z \in \mathcal{M}_n$ is defined by $\|Z\|_w = \sum_{j=1}^n w_j s_j(Z)$, where $w = (w_1, w_2, \dots, w_n)$ is a decreasing sequence of nonnegative real numbers.

In this paper, we assume that all functions are continuous. It is known that if $Z \in \mathcal{M}_n$ is positive semidefinite and h is a nonnegative increasing function on $[0,\infty)$, then $h(s_j(Z)) = s_j(h(Z))$ for $j = 1, 2, \dots, n$. For positive semidefinite $X, Y \in \mathcal{M}_n$ and a nonnegative increasing function h on $[0,\infty)$, if $s_j(X) \le s_j(Y)$ for $j = 1, 2, \dots, n$, then $\|h(X)\| \le \|h(Y)\|$, where $\|\cdot\|$ is a unitarily invariant norm. For more information, see [4, 18] and references therein.

We say that a matrix Z is accretive (respectively dissipative) if in the Cartesian decomposition $Z = X + iY$, the matrix X (respectively Y) is positive semidefinite. If both X and Y are positive semidefinite, Z is called accretive–dissipative.

Another important class of matrices, which is related to the class of accretive–dissipative matrices, is called sector matrices. To introduce this class, let $\alpha \in [0, \frac{\pi}{2})$ and S_α be a sector defined in the complex plane by

$$S_\alpha = \bigl\{ z \in C : \mathcal{R}e(z) \ge 0, \bigl|\mathcal{I}m(z)\bigr| \le \tan(\alpha)\mathcal{R}e(z) \bigr\}.$$

For $Z \in \mathcal{M}_n$, the numerical range of Z is defined by

$$W(A) = \bigl\{ z^* Z z : z \in C, \|z\| = 1 \bigr\}.$$

A matrix whose its numerical range is contained in a sector region S_α for some $\alpha \in [0, \frac{\pi}{2})$, is called a sector matrix. It follows from the definition of sector matrices that Z is positive semidefinite if and only if $W(Z) \subseteq S_0$ and also Z is accretive–dissipative if and only if $W(e^{\frac{-i\pi}{4}} Z) \subseteq S_{\frac{\pi}{4}}$. Moreover, if $W(Z) \subseteq S_\alpha$, then Z is invertible with $\mathcal{R}e(Z) > 0$ and therefore Z is accretive. For more on sector matrices see [3, 6, 7, 11–15, 17, 19–22] and the references therein. For $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in R^n$ with nonnegative components, if $\sum_{j=1}^k x_j \le \sum_{j=1}^k y_j$ ($\prod_{j=1}^k x_j \le \prod_{j=1}^k y_j$) for $k = 1, 2, \dots, n$, then we say that x is weakly (weakly log) majorized by y and denoted by $x \prec_\omega y (x \prec_{\omega \log} y)$. It is known that weak log majorization implies weak majorization. A nonnegative function h on the interval $[0,\infty)$ is said to be submultiplicative if $h(ab) \le h(a)h(b)$ whenever $a, b \in [0,\infty)$.

Gumus et al. [8] introduced the special class $\mathcal{C}$ involving all nonnegative increasing functions h on $[0,\infty)$ satisfying the following condition: If $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ are two decreasing sequences of nonnegative real numbers such that $\prod_{j=1}^k x_j \le \prod_{j=1}^k y_j$ ($k = 1, 2, \dots, n$), then $\prod_{j=1}^k h(x_j) \le \prod_{j=1}^k h(y_j)$ ($k = 1, 2, \dots, n$).

Note that the power function $h(t) = t^p$ ($p > 0$) belongs to class $\mathcal{C}$. For more information about the class $\mathcal{C}$ see [8] and the references therein. For the positive semidefinite matrix $\left(\begin{smallmatrix} X & Z \\ Z^* & Y \end{smallmatrix}\right) \in \mathcal{M}_{2n}$, one proved [8] that, if $h \in \mathcal{C}$ is a submultiplicative function, then

$$\bigl\| h\bigl(|Z|^2\bigr) \bigr\| \le \bigl\| h^r(X) \bigr\|^{\frac{1}{r}} \bigl\| h^s(Y) \bigr\|^{\frac{1}{s}}, \qquad (2)$$

where r and s are positive real numbers with $\frac{1}{r} + \frac{1}{s} = 1$. Furthermore, for accretive–dissipative matrix $Z \in \mathcal{M}_{2n}$ partitioned as in (1), one showed the following unitarily invariant norm inequalities:

$$\bigl\| h\bigl(|Z_{12}|^2\bigr) + h\bigl(\bigl|Z_{21}^*\bigr|^2\bigr) \bigr\| \le \bigl\| h^r\bigl(2|Z_{11}|\bigr) \bigr\|^{\frac{1}{r}} \bigl\| h^s\bigl(2|Z_{22}|\bigr) \bigr\|^{\frac{1}{s}}, \qquad (3)$$

where $h \in \mathcal{C}$ is a submultiplicative convex function and

$$\left\| h\left(|Z_{12}|^2\right) + h\left(\left|Z_{21}^*\right|^2\right) \right\| \le 4 \left\| h^r\left(|Z_{11}|\right) \right\|^{\frac{1}{r}} \left\| h^s\left(|Z_{22}|\right) \right\|^{\frac{1}{s}}, \tag{4}$$

where $h \in \mathcal{C}$ is a submultiplicative concave function such that r and s are positive real numbers with $\frac{1}{r} + \frac{1}{s} = 1$. Moreover, for a sector matrix $Z \in \mathcal{M}_{2n}$ partitioned as in (1), Zhang [22] proved the following inequality:

$$\|Z_{12}\|^2 \le \sec^2(\alpha) \|Z_{11}\| \|Z_{22}\| \tag{5}$$

for any unitarily invariant norm and $\alpha \in [0, \frac{\pi}{2})$. Alakhrass [1] extended inequality (5) to

$$\left\| |Z_{12}|^p \right\| \le \sec^p(\alpha) \left\| Z_{11}^{\frac{pr}{2}} \right\|^{\frac{1}{r}} \left\| Z_{22}^{\frac{ps}{2}} \right\|^{\frac{1}{s}}, \tag{6}$$

where r, s and p are positive numbers in which $\frac{1}{r} + \frac{1}{s} = 1$ and $\alpha \in [0, \frac{\pi}{2})$.

In [8], the authors presented some Schatten p-norm inequalities for accretive–dissipative matrices $Z \in \mathcal{M}_{2n}$ partitioned as in (1), which compared the off-diagonal blocks of Z to its diagonal blocks as follows:

$$\|Z_{12}\|_p^p + \|Z_{21}\|_p^p \le 2^{p-1} \|Z_{11}\|_p^{\frac{p}{2}} \|Z_{22}\|_p^{\frac{p}{2}} \quad (p \ge 2) \tag{7}$$

and

$$\|Z_{12}\|_p^p + \|Z_{21}\|_p^p \le 2^{3-p} \|Z_{11}\|_p^{\frac{p}{2}} \|Z_{22}\|_p^{\frac{p}{2}} \quad (0 < p \le 2). \tag{8}$$

Let Z_{ij} $(1 \le i, j \le n)$ be square matrices of the same size such that the block matrix

$$Z = \begin{pmatrix} Z_{11} & Z_{12} & \cdots & Z_{1n} \\ Z_{21} & Z_{22} & \cdots & Z_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ Z_{n1} & Z_{n2} & \cdots & Z_{nn} \end{pmatrix} \tag{9}$$

be accretive–dissipative. For such matrices, Kittaneh and Sakkijha [10] showed that

$$\sum_{i \ne j} \|Z_{ij}\|_p^p \le (n-1) 2^{p-2} \sum_{i=1}^{n} \|Z_{ii}\|_p^p \quad (p \ge 2) \tag{10}$$

and

$$\sum_{i \ne j} \|Z_{ij}\|_p^p \le (n-1) 2^{2-p} \sum_{i=1}^{n} \|Z_{ii}\|_p^p \quad (0 \le p \le 2). \tag{11}$$

Mao and Liu [17] showed the inequality

$$\sum_{i \ne j} \|Z_{ij}\|_p^p \le (n-1) 2^{\frac{p}{2}} \sum_{i=1}^{n} \|Z_{ii}\|_p^p \quad (p > 0), \tag{12}$$

where for $0 < p \leq \frac{4}{3}$ and $p \geq 4$, this inequality improved inequalities (10) and (11). Lin and Fu [16], extended the above inequalities for sector matrices as follows:

$$\sum_{i \neq j} \|Z_{ij}\|_p^p \leq (n-1)\sec^p(\alpha) \sum_{i=1}^{n} \|Z_{ii}\|_p^p \quad (p > 0), \tag{13}$$

in which $\alpha \in [0, \frac{\pi}{2})$.

In the present paper, we establish some unitarily invariant norm inequalities for sector matrices involving the functions of class $\mathcal{C}$. For instance, we extend inequalities (2) and (6) to sector matrices and the class $\mathcal{C}$ (Theorem 4). Moreover, we improve inequalities (3) and (4) to sector matrices. Also, we prove inequality (13) for all unitarily invariant norm and function of the class $\mathcal{C}$.

2 Main result

In the following, we give some lemmas which are needed to prove our main statements.

Lemma 1 ([9, p. 207]) *Let $X, Y, Z \in \mathcal{M}_n$, and r, s be positive real numbers with $\frac{1}{r} + \frac{1}{s} = 1$. Then*

$$\|X\|_w \leq \|Y\|_w^{\frac{1}{r}} \|Z\|_w^{\frac{1}{s}},$$

where $w = (w_1, w_2, \ldots, w_n)$ is a decreasing sequence of nonnegative real numbers if and only if

$$\|X\| \leq \|Y\|^{\frac{1}{r}} \|Z\|^{\frac{1}{s}}$$

for every unitarily invariant norm $\|\cdot\|$.

Lemma 2 ([1, Theorem 3.2]) *Suppose that $Z \in \mathcal{M}_{2n}$ partitioned as in (1) such that $W(Z) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then*

$$\prod_{m=1}^{k} s_m(Z_{ij}) \leq \prod_{l=1}^{k} \sec(\alpha) s_m^{\frac{1}{2}}\big(\mathcal{R}e(Z_{ii})\big) s_m^{\frac{1}{2}}\big(\mathcal{R}e(Z_{jj})\big) \quad (i,j = 1,2),$$

where $k = 1, 2, \ldots, n$.

Lemma 3 ([5, p. 73]) *Let $Z \in \mathcal{M}_n$. Then*

$$\lambda_j\big(\mathcal{R}e(Z)\big) \leq s_j(Z) \quad (j = 1, 2, \ldots, n).$$

Consequently, $\|\mathcal{R}e(Z)\| \leq \|Z\|$ for every unitarily invariant norm $\|\cdot\|$ on $\mathcal{M}_n$.

In the sequel, we give some unitarily invariant norm inequalities for sector matrices regarding of special class $\mathcal{C}$. Furthermore, in some special cases those results reduce to previous ones, which were introduced by other authors.

Theorem 4 *Let $Z \in \mathcal{M}_{2n}$ partitioned as in (1) be a sector matrix and let $h \in \mathcal{C}$ be submultiplicative and $\alpha \in [0, \frac{\pi}{2})$. If r and s are positive real numbers with $\frac{1}{r} + \frac{1}{s} = 1$, then*

$$\begin{aligned}\big\|h\big(|Z_{ij}|^2\big)\big\| &\leq \big\|h^r\big(\sec(\alpha)\mathcal{R}e(Z_{11})\big)\big\|^{\frac{1}{r}}\big\|h^s\big(\sec(\alpha)\mathcal{R}e(Z_{22})\big)\big\|^{\frac{1}{s}} \\ &\leq \big\|h^r\big(\sec(\alpha)|Z_{11}|\big)\big\|^{\frac{1}{r}}\big\|h^s\big(\sec(\alpha)|Z_{22}|\big)\big\|^{\frac{1}{s}}\end{aligned}$$

for every unitarily invariant norm $\|\cdot\|$ on $\mathcal{M}_n$ and $i,j = 1,2$.

Proof Assume that $w = (w_1, w_2, \dots, w_n)$ is a decreasing sequence of nonnegative real numbers and $k = 1, 2, \dots, n$. Then Lemma 2 implies that

$$\begin{aligned}\prod_{m=1}^{k} s_m\big(|Z_{ij}|^2\big) = \left(\prod_{m=1}^{k} s_m(Z_{ij})\right)^2 &\leq \left(\prod_{m=1}^{k} \sec(\alpha)s_m^{\frac{1}{2}}\big(\mathcal{R}e(Z_{ii})\big)s_m^{\frac{1}{2}}\big(\mathcal{R}e(Z_{jj})\big)\right)^2 \\ &= \prod_{m=1}^{k} \sec^2(\alpha)s_m\big(\mathcal{R}e(Z_{ii})\big)s_m\big(\mathcal{R}e(Z_{jj})\big),\end{aligned}$$

where $i,j = 1,2$. Therefore

$$\begin{aligned}\prod_{m=1}^{k} s_m\big(h\big(|Z_{ij}|^2\big)\big) &= \prod_{m=1}^{k} h\big(s_m\big(|Z_{ij}|^2\big)\big) \quad \text{(since } h \text{ is increasing)} \\ &\leq \prod_{m=1}^{k} h\big(\sec^2(\alpha)s_m\big(\mathcal{R}e(Z_{ii})\big)s_m\big(\mathcal{R}e(Z_{jj})\big)\big) \\ &\quad \text{(since } f \in \mathcal{C}) \\ &\leq \prod_{m=1}^{k} h\big(\sec(\alpha)s_m\big(\mathcal{R}e(Z_{ii})\big)\big)h\big(\sec(\alpha)s_m\big(\mathcal{R}e(Z_{jj})\big)\big) \\ &\quad \text{(since } h \text{ is submultiplicative)} \\ &= \prod_{m=1}^{k} s_m\big(h\big(\sec(\alpha)\mathcal{R}e(Z_{ii})\big)\big)s_m\big(h\big(\sec(\alpha)\mathcal{R}e(Z_{jj})\big)\big).\end{aligned}$$

Since $w = (w_1, w_2, \dots, w_n)$ is a decreasing sequence of nonnegative real numbers, it follows that

$$\prod_{m=1}^{k} w_m s_m\big(h\big(|Z_{ij}|^2\big)\big) \leq \prod_{m=1}^{k} w_m s_m\big(h\big(\sec(\alpha)\mathcal{R}e(Z_{ii})\big)\big)s_m\big(h\big(\sec(\alpha)\mathcal{R}e(Z_{jj})\big)\big), \tag{14}$$

where $i,j = 1,2$. Since weak log majorization implies weak majorization, inequality (14) implies that

$$\sum_{m=1}^{k} w_m s_m\big(h\big(|Z_{ij}|^2\big)\big) \leq \sum_{m=1}^{k} w_m s_m\big(h\big(\sec(\alpha)\mathcal{R}e(Z_{ii})\big)\big)s_m\big(h\big(\sec(\alpha)\mathcal{R}e(Z_{jj})\big)\big), \tag{15}$$

where $i,j = 1,2,\ldots$. Now, by applying the previous inequality and Hölder's inequality, we deduce that

$$\begin{aligned}
&\big\|h\big(|Z_{12}|^2\big)\big\|_w \\
&= \sum_{m=1}^{n} w_m s_m\big(h\big(|Z_{12}|^2\big)\big) \\
&\leq \sum_{m=1}^{n} w_m s_m\big(h\big(\sec(\alpha)\mathcal{R}e(Z_{11})\big)\big) s_m\big(h\big(\sec(\alpha)\mathcal{R}e(Z_{22})\big)\big) \\
&\quad \text{(by inequality (15))} \\
&= \sum_{m=1}^{n} w_m^{\frac{1}{r}} s_m\big(h\big(\sec(\alpha)\mathcal{R}e(Z_{11})\big)\big) w_m^{\frac{1}{s}} s_m\big(h\big(\sec(\alpha)\mathcal{R}e(Z_{22})\big)\big) \\
&\leq \left(\sum_{m=1}^{n} w_m s_m^r\big(h\big(\sec(\alpha)\mathcal{R}e(Z_{11})\big)\big)\right)^{\frac{1}{r}} \left(\sum_{m=1}^{n} w_m s_m^s\big(h\big(\sec(\alpha)\mathcal{R}e(Z_{22})\big)\big)\right)^{\frac{1}{s}} \\
&\quad \text{(by Hölder's inequality)} \\
&= \left(\sum_{m=1}^{n} w_m s_m\big(h^r\big(\sec(\alpha)\mathcal{R}e(Z_{11})\big)\big)\right)^{\frac{1}{r}} \left(\sum_{m=1}^{n} w_m s_m\big(h^s\big(\sec(\alpha)\mathcal{R}e(Z_{22})\big)\big)\right)^{\frac{1}{s}} \\
&= \big\|h^r\big(\sec(\alpha)\mathcal{R}e(Z_{11})\big)\big\|_w^{\frac{1}{r}} \big\|h^s\big(\sec(\alpha)\mathcal{R}e(Z_{22})\big)\big\|_w^{\frac{1}{s}}.
\end{aligned} \tag{16}$$

If we replace $w_m^{\frac{1}{r}}$ with $w_m^{\frac{1}{s}}$ in the third equality, then by a similar process we obtain

$$\big\|h\big(|Z_{21}|^2\big)\big\|_w \leq \big\|h^r\big(\sec(\alpha)\mathcal{R}e(Z_{11})\big)\big\|_w^{\frac{1}{r}} \big\|h^s\big(\sec(\alpha)\mathcal{R}e(Z_{22})\big)\big\|_w^{\frac{1}{s}} \tag{17}$$

for all decreasing sequences $w = (w_1, w_2, \ldots, w_n)$ of nonnegative real numbers. It follows from Lemma 1 and inequalities (16) and (17) that

$$\begin{aligned}
\big\|h\big(|Z_{ij}|^2\big)\big\| &\leq \big\|h^r\big(\sec(\alpha)\mathcal{R}e(Z_{11})\big)\big\|^{\frac{1}{r}} \big\|h^s\big(\sec(\alpha)\mathcal{R}e(Z_{22})\big)\big\|^{\frac{1}{s}} \\
&\leq \big\|h^r\big(\sec(\alpha)|Z_{11}|\big)\big\|^{\frac{1}{r}} \big\|h^s\big(\sec(\alpha)|Z_{22}|\big)\big\|^{\frac{1}{s}} \quad (i,j = 1,2).
\end{aligned}$$

□

Remark 5 If $Z \in \mathcal{M}_{2n}$ is positive semidefinite, i.e. $W(Z) \subseteq S_0$, then Theorem 4 reduces to inequality (2). Applying Theorem 4 for $h(t) = t^{\frac{p}{2}}$ ($p > 0$), we get inequality (6). Therefore Theorem 4 is an extension of inequality (2) and inequality (6).

Corollary 6 *Suppose $Z \in \mathcal{M}_{2n}$ partitioned as in (1) is accretive–dissipative and $h \in \mathcal{C}$ is submultiplicative. If r and s are positive real numbers with $\frac{1}{r} + \frac{1}{s} = 1$, then*

$$\big\|h\big(|Z_{ij}|^2\big)\big\| \leq \big\|h^r\big(\sqrt{2}\mathcal{R}e(Z_{11})\big)\big\|^{\frac{1}{r}} \big\|h^s\big(\sqrt{2}\mathcal{R}e(Z_{22})\big)\big\|^{\frac{1}{s}} \quad (i,j = 1,2),$$

where $\|\cdot\|$ is a unitarily invariant norm.

Proof Since Z is accretive–dissipative, i.e. $W(e^{\frac{-i\pi}{4}} Z) \subseteq S_{\frac{\pi}{4}}$ and $\sec(\frac{\pi}{4}) = \sqrt{2}$, by applying Theorem 4, we get the statement. □

Corollary 7 ([2, Theorem 4.2]) *Let $Z \in \mathcal{M}_{2n}$ partitioned as in* (1) *such that $W(Z) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then*

$$\begin{aligned}\big\| |Z_{12}|^p \big\|^2 &\le \sec^{2p}(\alpha) \big\| Z_{11}^p \big\| \big\| Z_{22}^p \big\| \\ &\le \sec^{2p}(\alpha) \big\| |Z_{11}|^p \big\| \big\| |Z_{22}|^p \big\| \quad (p > 0)\end{aligned}$$

for every unitarily invariant norm.

Proof Applying Theorem 4 for $r = 2$, $s = 2$ and $h(t) = t^{\frac{p}{2}}$ $(p > 0)$, we get

$$\begin{aligned}\big\| |Z_{12}|^p \big\|^2 &\le \sec^{2p}(\alpha) \big\| \mathcal{R}e(Z_{11})^p \big\| \big\| \mathcal{R}e(Z_{22})^p \big\| \\ &\le \sec^{2p}(\alpha) \big\| Z_{11}^p \big\| \big\| Z_{22}^p \big\| \\ &\le \sec^{2p}(\alpha) \big\| |Z_{11}|^p \big\| \big\| |Z_{22}|^p \big\| \quad (p > 0).\end{aligned}$$ ☐

Corollary 8 ([22, Theorem 3.2]) *Let $Z \in \mathcal{M}_{2n}$ partitioned as in* (1) *such that $W(Z) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then*

$$\begin{aligned}\max\big\{\|Z_{12}\|^2, \|Z_{21}\|^2\big\} &\le \sec^2(\alpha) \big\| \mathcal{R}e(Z_{11}) \big\| \big\| \mathcal{R}e(Z_{22}) \big\| \\ &\le \sec^2(\alpha) \|Z_{11}\| \|Z_{22}\| \end{aligned} \tag{18}$$

for every unitarily invariant norm.

Proof Applying Theorem 4 for $r = 2$, $s = 2$ and $h(t) = \sqrt{t}$, we get

$$\big\| |Z_{12}| \big\| = \|Z_{12}\| \le \big\| \sec(\alpha) \mathcal{R}e(Z_{11}) \big\|^{\frac{1}{2}} \big\| \sec(\alpha) \mathcal{R}e(Z_{22}) \big\|^{\frac{1}{2}}.$$

Therefore

$$\|Z_{12}\|^2 \le \sec^2(\alpha) \big\| \mathcal{R}e(Z_{11}) \big\| \big\| \mathcal{R}e(Z_{22}) \big\| \le \sec^2(\alpha) \|Z_{11}\| \|Z_{22}\|.$$

Similarly, we have

$$\begin{aligned}\|Z_{21}\|^2 &\le \sec^2(\alpha) \big\| \mathcal{R}e(Z_{11}) \big\| \big\| \mathcal{R}e(Z_{22}) \big\| \\ &\le \sec^2(\alpha) \|Z_{11}\| \|Z_{22}\|.\end{aligned}$$

The above inequalities imply the expected result. ☐

Corollary 9 ([22]) *Let $Z \in \mathcal{M}_{2n}$ partitioned as in* (1) *such that $W(Z) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then, for any unitarily invariant norm, we have*

$$\begin{aligned}2\|Z_{12}\| \|Z_{21}\| &\le \|Z_{12}\|^2 + \|Z_{21}\|^2 \\ &\le 2 \sec^2(\alpha) \|Z_{11}\| \|Z_{22}\|.\end{aligned}$$

Proof By using the arithmetic–geometric mean inequality and inequality (18), we have

$$\begin{aligned} 2\|Z_{12}\|\|Z_{21}\| &\le \|Z_{12}\|^2 + \|Z_{21}\|^2 \\ &\le 2\max\left\{\|Z_{12}\|^2, \|Z_{21}\|^2\right\} \\ &\le 2\sec^2(\alpha)\|Z_{11}\|\|Z_{22}\|. \end{aligned}$$

□

Remark 10 Assume that h is a nonnegative increasing function on $[0,\infty)$. Since $s_m(|Z_{ij}|^2) = s_m(|Z_{ij}^*|^2)$ for $m = 1,2,\dots,n$ and $i,j = 1,2$, we have

$$h\big(s_m\big(|Z_{ij}|^2\big)\big) = s_m\big(h\big(|Z_{ij}|^2\big)\big) = s_m\big(h\big(\big|Z_{ij}^*\big|^2\big)\big) = h\big(s_m\big(\big|Z_{ij}^*\big|^2\big)\big)$$

for $m = 1,2,\dots,n$ and $i,j = 1,2$. Therefore $\|h(|Z_{ij}|^2)\| = \|h(|Z_{ij}^*|^2)\|$.

Theorem 11 *Suppose that $Z \in \mathcal{M}_{2n}$ partitioned as in (1) is a sector matrix and $h \in \mathcal{C}$ is submultiplicative convex. If r and s are positive real numbers with $\frac{1}{r} + \frac{1}{s} = 1$, then*

$$\begin{aligned} \big\|h\big(|Z_{12}|^2\big) + h\big(\big|Z_{21}^*\big|^2\big)\big\| &\le \big\|h^r\big(\sqrt{2}\sec(\alpha)\mathcal{R}e(Z_{11})\big)\big\|^{\frac{1}{r}}\big\|h^s\big(\sqrt{2}\sec(\alpha)\mathcal{R}e(Z_{22})\big)\big\|^{\frac{1}{s}} \\ &\le \big\|h^r\big(\sqrt{2}\sec(\alpha)|Z_{11}|\big)\big\|^{\frac{1}{r}}\big\|h^s\big(\sqrt{2}\sec(\alpha)|Z_{22}|\big)\big\|^{\frac{1}{s}}, \end{aligned}$$

where $\alpha \in [0, \frac{\pi}{2})$.

Proof Applying the triangle inequality, Remark 10 and Theorem 4, we have

$$\begin{aligned} \big\|h\big(|Z_{12}|^2\big) + h\big(\big|Z_{21}^*\big|^2\big)\big\| &\le \big\|h\big(|Z_{12}|^2\big)\big\| + \big\|h\big(\big|Z_{21}^*\big|^2\big)\big\| \\ &= \big\|h\big(|Z_{12}|^2\big)\big\| + \big\|h\big(|Z_{21}|^2\big)\big\| \\ &\le 2\big\|h^r\big(\sec(\alpha)\mathcal{R}e(Z_{11})\big)\big\|^{\frac{1}{r}}\big\|h^s\big(\sec(\alpha)\mathcal{R}e(Z_{22})\big)\big\|^{\frac{1}{s}}. \end{aligned}$$

It is well known that, if h is a convex function, then $h(\lambda Z) \ge \lambda h(Z)$ for $Z \in \mathcal{M}_n$ and $\lambda \ge 1$. Since $\sec(\alpha) \ge 1$ ($\alpha \in [0, \frac{\pi}{2})$), we have

$$\begin{aligned} \big\|h\big(|Z_{12}|^2\big) + h\big(\big|Z_{21}^*\big|^2\big)\big\| &\le \big\|h^r\big(\sqrt{2}\sec(\alpha)\mathcal{R}e(Z_{11})\big)\big\|^{\frac{1}{r}}\big\|h^s\big(\sqrt{2}\sec(\alpha)\mathcal{R}e(Z_{22})\big)\big\|^{\frac{1}{s}} \\ &\le \big\|h^r\big(\sqrt{2}\sec(\alpha)|Z_{11}|\big)\big\|^{\frac{1}{r}}\big\|h^s\big(\sqrt{2}\sec(\alpha)|Z_{22}|\big)\big\|^{\frac{1}{s}}. \end{aligned}$$

□

Remark 12 Note that, if $Z \in \mathcal{M}_{2n}$ is accretive–dissipative, i.e. $W(e^{\frac{-i\pi}{4}}Z) \subseteq S_{\frac{\pi}{4}}$, then Theorem 11 reduces to inequality (3).

Theorem 13 *Assume that $Z \in \mathcal{M}_{2n}$ partitioned as in (1) is a sector matrix and $h \in \mathcal{C}$ is submultiplicative concave. If r and s are positive real numbers with $\frac{1}{r} + \frac{1}{s} = 1$, then*

$$\begin{aligned} \big\|h\big(|Z_{12}|^2\big) + h\big(\big|Z_{21}^*\big|^2\big)\big\| &\le 2\sec^2(\alpha)\big\|h^r\big(\mathcal{R}e(Z_{11})\big)\big\|^{\frac{1}{r}}\big\|h^s\big(\mathcal{R}e(Z_{22})\big)\big\|^{\frac{1}{s}} \\ &\le 2\sec^2(\alpha)\big\|h^r\big(|Z_{11}|\big)\big\|^{\frac{1}{r}}\big\|h^s\big(|Z_{22}|\big)\big\|^{\frac{1}{s}} \end{aligned}$$

for every unitarily invariant norm $\|\cdot\|$ and $\alpha \in [0, \frac{\pi}{2})$.

Proof Applying the triangle inequality, Remark 10 and Theorem 4, we have

$$\begin{aligned}\|h(|Z_{12}|^2)+h(|Z_{21}^*|^2)\| &\le \|h(|Z_{12}|^2)\| + \|h(|Z_{21}^*|^2)\| \\ &= \|h(|Z_{12}|^2)\| + \|h(|Z_{21}|^2)\| \\ &\le 2\|h^r(\sec(\alpha)\mathcal{R}e(Z_{11}))\|^{\frac{1}{r}}\|h^s(\sec(\alpha)\mathcal{R}e(Z_{22}))\|^{\frac{1}{s}}.\end{aligned}$$

Since h is concave, it follows that $h(\lambda Z)\le \lambda h(Z)$ for $Z\in\mathcal{M}_n$ and $\lambda\ge 1$. Since $\sec(\alpha)\ge 1$ for $\alpha\in[0,\frac{\pi}{2})$,

$$\begin{aligned}\|h(|Z_{12}|^2)+h(|Z_{21}^*|^2)\| &\le 2\sec^2(\alpha)\|h^r(\mathcal{R}e(Z_{11}))\|^{\frac{1}{r}}\|h^s(\mathcal{R}e(Z_{22}))\|^{\frac{1}{s}} \\ &\le 2\sec^2(\alpha)\|h^r(|Z_{11}|)\|^{\frac{1}{r}}\|h^s(|Z_{22}|)\|^{\frac{1}{s}}.\end{aligned}$$ □

Remark 14 If $Z\in\mathcal{M}_{2n}$ is accretive–dissipative, i.e. $W(e^{\frac{-i\pi}{4}}Z)\subseteq S_{\frac{\pi}{4}}$, then Theorem 13 reduces to inequality (4).

Theorem 15 *Assume that $Z\in\mathcal{M}_{2n}$ partitioned as in (1) is a sector matrix, $h\in\mathcal{C}$ is submultiplicative and $\alpha\in[0,\frac{\pi}{2})$. If p is positive real number, then*

$$\|h(|Z_{12}|^2)\|^p + \|h(|Z_{21}|^2)\|^p \le 2\|h^2(\sec(\alpha)|Z_{11}|)\|^{\frac{p}{2}}\|h^2(\sec(\alpha)|Z_{22}|)\|^{\frac{p}{2}}$$

for every unitarily invariant norm $\|\cdot\|$. In particular, we have

$$\|h(|Z_{12}|^2)\|_p^p + \|h(|Z_{21}|^2)\|_p^p \le 2\|h^2(\sec(\alpha)|Z_{11}|)\|_p^{\frac{p}{2}}\|h^2(\sec(\alpha)|Z_{22}|)\|_p^{\frac{p}{2}}.$$

Proof Theorem 4 for $r=s=2$, implies that

$$\|h(|Z_{ij}|^2)\| \le \|h^2(\sec(\alpha)|Z_{11}|)\|^{\frac{1}{2}}\|h^2(\sec(\alpha)|Z_{22}|)\|^{\frac{1}{2}} \quad (i,j=1,2). \tag{19}$$

By taking the power p of both sides of inequality (19), we have

$$\|h(|Z_{ij}|^2)\|^p \le \|h^2(\sec(\alpha)|Z_{11}|)\|^{\frac{p}{2}}\|h^2(\sec(\alpha)|Z_{22}|)\|^{\frac{p}{2}} \quad (i,j=1,2).$$

Therefore, we have

$$\|h(|Z_{12}|^2)\|^p + \|h(|Z_{21}|^2)\|^p \le 2\|h^2(\sec(\alpha)|Z_{11}|)\|^{\frac{p}{2}}\|h^2(\sec(\alpha)|Z_{22}|)\|^{\frac{p}{2}}.$$ □

Corollary 16 ([16, Theorem 2.8]) *Let $Z\in\mathcal{M}_{2n}$ be partitioned as in (1) such that $W(Z)\subseteq S_\alpha$ for some $\alpha\in[0,\frac{\pi}{2})$. Then, for any unitarily invariant norm, we have*

$$\|Z_{12}\|^p + \|Z_{21}\|^p \le 2\sec^p(\alpha)\|Z_{11}\|^{\frac{p}{2}}\|Z_{22}\|^{\frac{p}{2}} \quad (p>0).$$

In particular, we have

$$\|Z_{12}\|_p^p + \|Z_{21}\|_p^p \le 2\sec^p(\alpha)\|Z_{11}\|_p^{\frac{p}{2}}\|Z_{22}\|_p^{\frac{p}{2}} \quad (p>0).$$

Proof Applying Theorem 15, for $h(t) = \sqrt{t}$, we have

$$\|Z_{12}\|^p + \|Z_{21}\|^p \le 2\sec^p(\alpha)\|Z_{11}\|^{\frac{p}{2}}\|Z_{22}\|^{\frac{p}{2}} \quad (p > 0).$$

By showing the particular case, by using the Schatten p-norm, we have the statement. □

In the sequel, we extend our results to $n \times n$ block matrices as introduced in (9).

Theorem 17 *Suppose that Z is a sector matrix represented as in* (9), $h \in \mathcal{C}$ *is submultiplicative and* $\alpha \in [0, \frac{\pi}{2})$. *If p is positive real number, then*

$$\sum_{i \ne j} \big\| h\big(|Z_{ij}|^2\big) \big\|^p \le (n-1) \sum_{i=1}^{n} \big\| h^2\big(\sec(\alpha)|Z_{ii}|\big) \big\|^p \tag{20}$$

for every unitarily invariant norm $\|\cdot\|$. *In particular, we have*

$$\sum_{i \ne j} \big\| h\big(|Z_{ij}|^2\big) \big\|_p^p \le (n-1) \sum_{i=1}^{n} \big\| h^2\big(\sec(\alpha)|Z_{ii}|\big) \big\|_p^p.$$

Proof Since Z is a sector matrix, so every principal submatrix of Z is also a sector matrix, it follows that $\begin{pmatrix} Z_{ii} & Z_{ij} \\ T_{ji} & Z_{jj} \end{pmatrix}$ is a sector matrix. Now, applying Theorem 15 for $\begin{pmatrix} Z_{ii} & Z_{ij} \\ Z_{ji} & Z_{jj} \end{pmatrix}$, we get

$$\big\| h\big(|Z_{ij}|^2\big) \big\|^p + \big\| h\big(|Z_{ji}|^2\big) \big\|^p \le 2 \big\| h^2\big(\sec(\alpha)|Z_{ii}|\big) \big\|^{\frac{p}{2}} \big\| h^2\big(\sec(\alpha)|Z_{jj}|\big) \big\|^{\frac{p}{2}}$$

for $i \ne j$. By using the arithmetic–geometric mean inequality, we have

$$\big\| h\big(|Z_{ij}|^2\big) \big\|^p + \big\| h\big(|Z_{ji}|^2\big) \big\|^p \le \big\| h^2\big(\sec(\alpha)|Z_{ii}|\big) \big\|^p + \big\| h^2\big(\sec(\alpha)|Z_{jj}|\big) \big\|^p$$

for $i \ne j$. Adding the previous inequalities for $i, j = 1, 2, \dots, n$, we get

$$\sum_{i \ne j} \big\| h\big(|Z_{ij}|^2\big) \big\|^p \le (n-1) \sum_{i=1}^{n} \big\| h^2\big(\sec(\alpha)|Z_{ii}|\big) \big\|^p.$$

□

Corollary 18 ([16, Theorem 2.9]) *Let Z be a sector matrix as represented in* (9) *and* $\alpha \in [0, \frac{\pi}{2})$. *Then*

$$\sum_{i \ne j} \|Z_{ij}\|^p \le (n-1)\sec^p(\alpha) \sum_{i=1}^{n} \|Z_{ii}\|^p \quad (p > 0), \tag{21}$$

for any unitarily invariant norm. In particular, we have

$$\sum_{i \ne j} \|Z_{ij}\|_p^p \le (n-1)\sec^p(\alpha) \sum_{i=1}^{n} \|Z_{ii}\|_p^p \quad (p > 0).$$

Proof Applying Theorem 17, for $h(t) = \sqrt{t}$, we have

$$\sum_{i\neq j} \|Z_{ij}\|^p \leq (n-1)\sec(\alpha) \sum_{i=1}^{n} \|Z_{ii}\|^p \quad (p > 0).$$

For the particular case, we take the Schatten p-norm. □

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