Ann. Funct. Anal. 9 (2018), no. 3, 297–309
https://doi.org/10.1215/20088752-2017-0029
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

UPPER BOUNDS FOR NUMERICAL RADIUS INEQUALITIES INVOLVING OFF-DIAGONAL OPERATOR MATRICES

MOJTABA BAKHERAD[1*] and KHALID SHEBRAWI[2]

Communicated by T. Yamazaki

Abstract. In this article, we establish some upper bounds for numerical radius inequalities, including those of 2×2 operator matrices and their off-diagonal parts. Among other inequalities, it is shown that if $T = \left[\begin{smallmatrix} 0 & X \\ Y & 0 \end{smallmatrix}\right]$, then

$$\omega^r(T) \le 2^{r-2}\big\|f^{2r}\big(|X|\big) + g^{2r}\big(|Y^*|\big)\big\|^{\frac{1}{2}}\big\|f^{2r}\big(|Y|\big) + g^{2r}\big(|X^*|\big)\big\|^{\frac{1}{2}}$$

and

$$\omega^r(T) \le 2^{r-2}\big\|f^{2r}\big(|X|\big) + f^{2r}\big(|Y^*|\big)\big\|^{\frac{1}{2}}\big\|g^{2r}\big(|Y|\big) + g^{2r}\big(|X^*|\big)\big\|^{\frac{1}{2}},$$

where X, Y are bounded linear operators on a Hilbert space $\mathcal{H}$, $r \ge 1$, and f, g are nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, \infty))$. Moreover, we present some inequalities involving the generalized Euclidean operator radius of operators $T_1, \dots, T_n$.

1. Introduction and preliminaries

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. In the case when $\dim\mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field. An operator $A \in \mathbb{B}(\mathcal{H})$ is said to be a *contraction*, if $A^*A \le I$. The numerical radius of $T \in \mathbb{B}(\mathcal{H})$ is defined by

$$\omega(T) := \sup\big\{\big|\langle Tx, x\rangle\big| : x \in \mathcal{H}, \|x\| = 1\big\}.$$

Received Jun. 6, 2017; Accepted Sep. 5, 2017.
First published online Oct. 17, 2017.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47A12; Secondary 47A30, 47A63, 47B33.
Keywords. numerical radius off-diagonal part, positive operator, Young inequality, generalized Euclidean operator radius.

It is well known that $\omega(\cdot)$ defines a norm on $\mathbb{B}(\mathcal{H})$, which is equivalent to the usual operator norm. In fact, $\frac{1}{2}\|\cdot\| \le \omega(\cdot) \le \|\cdot\|$ (see [10]). An important inequality for $\omega(A)$ is the power inequality stating that $\omega(A^n) \le \omega(A)^n$ $(n = 1, 2, \ldots)$. (For further information about the properties of numerical radius inequalities, see [1], [5], [6], [12], [15], [17], and references therein.) Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, and consider the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. With respect to this decomposition, every operator $T \in \mathbb{B}(\mathcal{H})$ has a 2×2 operator matrix representation $T = [T_{ij}]$ with entries $T_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$, the space of all bounded linear operators from $\mathcal{H}_j$ to $\mathcal{H}_i$ $(1 \le i, j \le 2)$. Operator matrices provide a useful tool for studying Hilbert space operators, which have been extensively studied in the literature. Let $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_1)$, $B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$, $C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$, and $D \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_2)$. The operator $\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right]$ is called the *diagonal* part of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]$ is the *off-diagonal* part.

The classical Young inequality says that if $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ for positive real numbers a, b. In [3], the authors showed that a refinement of the scalar Young inequality as follows $(a^{\frac{1}{p}} b^{\frac{1}{q}})^m + r_0^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \le (\frac{a}{p} + \frac{b}{q})^m$, where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$ and $m = 1, 2, \ldots$. In particular, if $p = q = 2$, then

$$(a^{\frac{1}{2}} b^{\frac{1}{2}})^m + \left(\frac{1}{2}\right)^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \le 2^{-m}(a+b)^m. \tag{1.1}$$

It has been shown in [9] that if $T \in \mathbb{B}(\mathcal{H})$, then

$$\omega(T) \le \frac{1}{2}\big\||T| + |T^*|\big\|, \tag{1.2}$$

where $|T| = (T^*T)^{\frac{1}{2}}$ is the absolute value of T. Recently, in [2], the authors extended this inequality for off-diagonal operator matrices of the form $T = \left[\begin{smallmatrix} 0 & X \\ Y & 0 \end{smallmatrix}\right] \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ as follows:

$$\omega(T) \le \frac{1}{2}\big\||X| + |Y^*|\big\|^{\frac{1}{2}} \big\||X^*| + |Y|\big\|^{\frac{1}{2}}. \tag{1.3}$$

Let $T_1, T_2, \ldots, T_n \in \mathbb{B}(\mathcal{H})$. The functional ω_p of operators $T_1, \ldots, T_n$ for $p \ge 1$ is defined in [13] as follows:

$$\omega_p(T_1, \ldots, T_n) := \sup_{\|x\|=1} \Big(\sum_{i=1}^n \big|\langle T_i x, x\rangle\big|^p\Big)^{\frac{1}{p}}.$$

If $p = 2$, then we have the Euclidean operator radius of $T_1, \ldots, T_n$ which was defined in [11]. In [15], the authors showed that an upper bound for the functional ω_p,

$$\omega_p^p(T_1, \ldots, T_n) \le \frac{1}{2}\Big\|\sum_{i=1}^n \big(f^{2p}\big(|T_i|\big) + g^{2p}\big(|T_i^*|\big)\big)\Big\| - \inf_{\|x\|=1} \zeta(x),$$

where $T_i \in \mathbb{B}(\mathcal{H})$ $(i = 1, 2, \ldots, n)$, f, g are nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ $(t \in [0, \infty))$, $p \ge 1$, and

$$\zeta(x) = \frac{1}{2}\sum_{i=1}^n \big(\big\langle f^{2p}\big(|T_i|\big)x, x\big\rangle^{\frac{1}{2}} - \big\langle g^{2p}\big(|T_i^*|\big)x, x\big\rangle^{\frac{1}{2}}\big)^2.$$

In this paper, we show some inequalities involving powers of the numerical radius for off-diagonal parts of 2×2 operator matrices. In particular, we extend inequalities (1.2) and (1.3) for nonnegative continuous functions f, g on $[0,\infty)$ such that $f(t)g(t) = t(t \in [0,\infty))$. Moreover, we present some inequalities, including the generalized Euclidean operator radius ω_p.

2. Main results

To prove our first result, we need the following lemmas.

Lemma 2.1 ([7, Lemma 2.1], [16, p. 84]). *Let $X \in \mathbb{B}(\mathcal{H})$. Then*

(a) $\omega(X) = \max_{\theta\in\mathbb{R}} \|\operatorname{Re}(e^{i\theta}X)\| = \max_{\theta\in\mathbb{R}} \|\operatorname{Im}(e^{i\theta}X)\|$,
(b) $\omega\left(\left[\begin{smallmatrix} 0 & X \\ X & 0 \end{smallmatrix}\right]\right) = \omega(X)$.

The next lemma follows from the spectral theorem for positive operators and the Jensen inequality (see, e.g., [8, p. 288]).

Lemma 2.2. *Let $T \in \mathbb{B}(\mathcal{H})$, $T \geq 0$, and $x \in \mathcal{H}$ such that $\|x\| \leq 1$. Then*

(a) $\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle$ *for* $r \geq 1$,
(b) $\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r$ *for* $0 < r \leq 1$.

Lemma 2.3 ([8, Theorem 1]). *Let $T \in \mathbb{B}(\mathcal{H})$, and let $x, y \in \mathcal{H}$ be any vectors. If f, g are nonnegative continuous functions on $[0,\infty)$ which are satisfying the relation $f(t)g(t) = t(t \in [0,\infty))$, then*

$$\left|\langle Tx, y\rangle\right|^2 \leq \left\langle f^2\left(|T|\right)x, x\right\rangle\left\langle g^2\left(|T^*|\right)y, y\right\rangle.$$

Now, we are in position to demonstrate the main results of this section by using some ideas from [2, Theorem 4] and [15, Theorem 2.4].

Theorem 2.4. *Let $T = \left[\begin{smallmatrix} 0 & X \\ Y & 0 \end{smallmatrix}\right] \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, $r \geq 1$, and f, g be nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t(t \in [0,\infty))$. Then*

$$\omega^r(T) \leq 2^{r-2}\left\|f^{2r}\left(|X|\right) + g^{2r}\left(|Y^*|\right)\right\|^{\frac{1}{2}}\left\|f^{2r}\left(|Y|\right) + g^{2r}\left(|X^*|\right)\right\|^{\frac{1}{2}}$$

and

$$\omega^r(T) \leq 2^{r-2}\left\|f^{2r}\left(|X|\right) + f^{2r}\left(|Y^*|\right)\right\|^{\frac{1}{2}}\left\|g^{2r}\left(|Y|\right) + g^{2r}\left(|X^*|\right)\right\|^{\frac{1}{2}}.$$

Proof. Let $\mathbf{x} = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{H}_1 \oplus \mathcal{H}_2$ be a unit vector (i.e., $\|x_1\|^2 + \|x_2\|^2 = 1$). Then

$$\begin{aligned}
&\left|\langle T\mathbf{x}, \mathbf{x}\rangle\right|^r \\
&= \left|\langle Xx_2, x_1\rangle + \langle Yx_1, x_2\rangle\right|^r \\
&\leq \left(\left|\langle Xx_2, x_1\rangle\right| + \left|\langle Yx_1, x_2\rangle\right|\right)^r \quad \text{(by the triangular inequality)} \\
&\leq \frac{2^r}{2}\left(\left|\langle Xx_2, x_1\rangle\right|^r + \left|\langle Yx_1, x_2\rangle\right|^r\right) \quad \left(\text{by the convexity of } f(t) = t^r\right) \\
&\leq \frac{2^r}{2}\Big(\left(\left\langle f^2\left(|X|\right)x_2, x_2\right\rangle^{\frac{1}{2}}\left\langle g^2\left(|X^*|\right)x_1, x_1\right\rangle^{\frac{1}{2}}\right)^r \\
&\quad + \left(\left\langle f^2\left(|Y|\right)x_1, x_1\right\rangle^{\frac{1}{2}}\left\langle g^2\left(|Y^*|\right)x_2, x_2\right\rangle^{\frac{1}{2}}\right)^r\Big) \quad \text{(by Lemma 2.3)}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{2^r}{2}\Big(\big\langle f^{2r}\big(|X|\big)x_2,x_2\big\rangle^{\frac{1}{2}}\big\langle g^{2r}|X^*|x_1,x_1\big\rangle^{\frac{1}{2}}\\
&\quad+\big\langle f^{2r}\big(|Y|\big)x_1,x_1\big\rangle^{\frac{1}{2}}\big\langle g^{2r}\big(|Y^*|\big)x_2,x_2\big\rangle^{\frac{1}{2}}\Big)\quad\big(\text{by Lemma 2.2(a)}\big)\\
&\le \frac{2^r}{2}\Big(\big\langle f^{2r}\big(|X|\big)x_2,x_2\big\rangle+\big\langle g^{2r}\big(|Y^*|\big)x_2,x_2\big\rangle\Big)^{\frac{1}{2}}\\
&\quad\times\Big(\big\langle f^{2r}\big(|Y|\big)x_1,x_1\big\rangle\\
&\quad+\big\langle g^{2r}\big(|X^*|\big)x_1,x_1\big\rangle\Big)^{\frac{1}{2}}\quad(\text{by the Cauchy–Schwarz inequality})\\
&= \frac{2^r}{2}\Big\langle\Big(f^{2r}\big(|X|\big)+g^{2r}\big(|Y^*|\big)\Big)x_2,x_2\Big\rangle^{\frac{1}{2}}\Big\langle\Big(f^{2r}\big(|Y|\big)+g^{2r}\big(|X^*|\big)\Big)x_1,x_1\Big\rangle^{\frac{1}{2}}\\
&\le \frac{2^r}{2}\Big\|f^{2r}\big(|X|\big)+g^{2r}\big(|Y^*|\big)\Big\|^{\frac{1}{2}}\Big\|f^{2r}\big(|Y|\big)+g^{2r}\big(|X^*|\big)\Big\|^{\frac{1}{2}}\|x_1\|\|x_2\|\\
&\le \frac{2^r}{2}\Big\|f^{2r}\big(|X|\big)+g^{2r}\big(|Y^*|\big)\Big\|^{\frac{1}{2}}\Big\|f^{2r}\big(|Y|\big)+g^{2r}\big(|X^*|\big)\Big\|^{\frac{1}{2}}\Big(\frac{\|x_1\|^2+\|x_2\|^2}{2}\Big)\\
&\quad(\text{by the arithmetic-geometric mean inequality})\\
&= \frac{2^r}{4}\Big\|f^{2r}\big(|X|\big)+g^{2r}\big(|Y^*|\big)\Big\|^{\frac{1}{2}}\Big\|f^{2r}\big(|Y|\big)+g^{2r}\big(|X^*|\big)\Big\|^{\frac{1}{2}}.
\end{aligned}
$$

Hence, we get the first inequality. Now, applying this fact,

$$
\begin{aligned}
&\big|\langle T\mathbf{x},\mathbf{x}\rangle\big|^r\\
&\quad=\big|\langle Xx_2,x_1\rangle+\langle Yx_1,x_2\rangle\big|^r\\
&\quad\le\Big(\big|\langle Xx_2,x_1\rangle\big|+\big|\langle Yx_1,x_2\rangle\big|\Big)^r\quad(\text{by the triangular inequality})\\
&\quad\le\frac{2^r}{2}\Big(\big|\langle Xx_2,x_1\rangle\big|^r+\big|\langle Yx_1,x_2\rangle\big|^r\Big)\quad\big(\text{by the convexity of } f(t)=t^r\big)\\
&\quad\le\frac{2^r}{2}\Big(\Big(\big\langle f^2\big(|X|\big)x_2,x_2\big\rangle^{\frac{1}{2}}\big\langle g^2\big(|X^*|\big)x_1,x_1\big\rangle^{\frac{1}{2}}\Big)^r\\
&\qquad+\Big(\big\langle g^2\big(|Y|\big)x_1,x_1\big\rangle^{\frac{1}{2}}\big\langle f^2\big(|Y^*|\big)x_2,x_2\big\rangle^{\frac{1}{2}}\Big)^r\Big)\quad(\text{by Lemma 2.3}),
\end{aligned}
\tag{2.1}
$$

and, by a similar argument to the proof of the first inequality, we have the second inequality; this completes the proof of the theorem. □

Theorem 2.4 includes a special case, as follows.

Corollary 2.5. *Let* $T=\left[\begin{smallmatrix}0 & X\\ Y & 0\end{smallmatrix}\right]\in\mathbb{B}(\mathcal{H}_1\oplus\mathcal{H}_2)$, $0\le p\le 1$, *and* $r\ge 1$. *Then*

$$\omega^r(T)\le 2^{r-2}\big\||X|^{2rp}+|Y^*|^{2r(1-p)}\big\|^{\frac{1}{2}}\big\||Y|^{2rp}+|X^*|^{2r(1-p)}\big\|^{\frac{1}{2}}$$

and

$$\omega^r(T)\le 2^{r-2}\big\||X|^{2rp}+|Y^*|^{2rp}\big\|^{\frac{1}{2}}\big\||Y|^{2r(1-p)}+|X^*|^{2r(1-p)}\big\|^{\frac{1}{2}}.$$

Proof. The result follows immediately from Theorem 2.4 for $f(t)=t^p$ and $g(t)=t^{1-p}$ $(0\le p\le 1)$. □

Remark 2.6. Taking $f(t)=g(t)=t^{\frac{1}{2}}(t\in[0,\infty))$ and $r=1$ in Theorem 2.4, we get (see [2, Theorem 4])

$$\omega(T)\le\frac{1}{2}\big\||X|+|Y^*|\big\|^{\frac{1}{2}}\big\||Y|+|X^*|\big\|^{\frac{1}{2}},$$

where $T=\left[\begin{smallmatrix}0 & X\\ Y & 0\end{smallmatrix}\right]\in\mathbb{B}(\mathcal{H}_1\oplus\mathcal{H}_2)$.

If we put $Y = X$ in Theorem 2.4, then by using Lemma 2.1(b) we get an extension of inequality (1.2).

Corollary 2.7. *Let $X \in \mathbb{B}(\mathcal{H})$, $r \geq 1$, and f, g be nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t (t \in [0,\infty))$. Then*

$$\omega^r(X) \leq 2^{r-2}\big\|f^{2r}(|X|) + g^{2r}(|X^*|)\big\|$$

and

$$\omega^r(X) \leq 2^{r-2}\big\|f^{2r}(|X|) + f^{2r}(|X^*|)\big\|^{\frac{1}{2}}\big\|g^{2r}(|X|) + g^{2r}(|X^*|)\big\|^{\frac{1}{2}}.$$

Corollary 2.8. *Let $X, Y \in \mathbb{B}(\mathcal{H})$ and $0 \leq p \leq 1$. Then*

$$\omega^{\frac{r}{2}}(XY) \leq 2^{r-2}\big\||X|^{2rp} + |Y^*|^{2r(1-p)}\big\|^{\frac{1}{2}}\big\||Y|^{2rp} + |X^*|^{2r(1-p)}\big\|^{\frac{1}{2}}$$

and

$$\omega^{\frac{r}{2}}(XY) \leq 2^{r-2}\big\||X|^{2rp} + |Y^*|^{2rp}\big\|^{\frac{1}{2}}\big\||Y|^{2r(1-p)} + |X^*|^{2r(1-p)}\big\|^{\frac{1}{2}}$$

for $r \geq 1$.

Proof. It follows from the power inequality $\omega^{\frac{1}{2}}(T^2) \leq \omega(T)$ that

$$\omega^{\frac{1}{2}}(T^2) = \omega^{\frac{1}{2}}\left(\begin{bmatrix} XY & 0 \\ 0 & YX \end{bmatrix}\right) = \max\big\{\omega^{\frac{1}{2}}(XY), \omega^{\frac{1}{2}}(YX)\big\}.$$

The required result follows from Corollary 2.5. □

Corollary 2.9. *Let $X, Y \in \mathbb{B}(\mathcal{H})$ and $r \geq 1$. Then*

$$\|X \pm Y^*\|^r \leq 2^{2r-2}\big\||X|^r + |Y^*|^r\big\|^{\frac{1}{2}}\big\||Y|^r + |X^*|^r\big\|^{\frac{1}{2}}.$$

In particular, if X and Y are normal operators, then

$$\|X \pm Y\|^r \leq 2^{2r-2}\big\||X|^r + |Y|^r\big\|. \tag{2.2}$$

Proof. Applying Lemma 2.1(a) and Corollary 2.5 (for $p = \frac{1}{2}$), we have

$$\begin{aligned}\|X + Y^*\|^r &= \|T + T^*\|^r \\ &\leq 2^r \max_{\theta\in\mathbb{R}}\big\|\mathrm{Re}(e^{i\theta}T)\big\|^r \\ &= 2^r\omega^r(T) \\ &\leq 2^{2r-2}\big\||X|^r + |Y^*|^r\big\|^{\frac{1}{2}}\big\||Y|^r + |X^*|^r\big\|^{\frac{1}{2}},\end{aligned}$$

where $T = \left[\begin{smallmatrix} 0 & X \\ Y & 0 \end{smallmatrix}\right]$. Similarly,

$$\begin{aligned}\|X - Y^*\|^r &= \|T - T^*\|^r \\ &\leq 2^r \max_{\theta\in\mathbb{R}}\big\|\mathrm{Im}(e^{i\theta}T)\big\|^r \\ &= 2^r\omega^r(T) \\ &\leq 2^{2r-2}\big\||X|^r + |Y^*|^r\big\|^{\frac{1}{2}}\big\||Y|^r + |X^*|^r\big\|^{\frac{1}{2}}.\end{aligned}$$

Hence we get the desired result. For the particular case, note that $|Y^*| = |Y|$ and $|X^*| = |X|$. □

Remark 2.10. It should be mentioned here that inequality (2.2), which has been given earlier, is a generalized form of the well-known inequality (see [4]): if A and B are normal operators, then

$$\|X+Y\| \le \big\||X|+|Y|\big\|. \tag{2.3}$$

The normality of X and Y are necessary, which means inequality (2.3) is not true for arbitrary operators X and Y (see, e.g., [14]).

Applying inequality (1.1), we obtain the following theorem.

Theorem 2.11. *Let $T = \left[\begin{smallmatrix} 0 & X \\ Y & 0 \end{smallmatrix}\right] \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, and let f, g be nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0,\infty))$. Then for $r \ge 1$,*

$$\begin{aligned}\omega^r(T) \le\ & 2^{r-2}\big(\big\|f^{2r}(|X|)+g^{2r}(|Y^*|)\big\| + \big\|f^{2r}(|Y|)+g^{2r}(|X^*|)\big\|\big)\\ & - 2^{r-2} \inf_{\|(x_1,x_2)\|=1} \zeta(x_1,x_2),\end{aligned}$$

where

$$\zeta(x_1,x_2) = \big(\big\langle\big(f^{2r}(|X|)+g^{2r}(|Y^*|)\big)x_2,x_2\big\rangle^{\frac12} - \big\langle\big(f^{2r}(|Y|)+g^{2r}(|X^*|)\big)x_1,x_1\big\rangle^{\frac12}\big)^2.$$

Proof. Let $\mathbf{x} = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{H}_1 \oplus \mathcal{H}_2$ be a unit vector. Then

$$\begin{aligned}
&\big|\langle T\mathbf{x},\mathbf{x}\rangle\big|^r\\
&\quad = \big|\langle Xx_2,x_1\rangle + \langle Yx_1,x_2\rangle\big|^r\\
&\quad \le \big(\big|\langle Xx_2,x_1\rangle\big| + \big|\langle Yx_1,x_2\rangle\big|\big)^r \quad \text{(by the triangular inequality)}\\
&\quad \le \frac{2^r}{2}\big(\big|\langle Xx_2,x_1\rangle\big|^r + \big|\langle Yx_1,x_2\rangle\big|^r\big) \quad \big(\text{by the convexity of } f(t)=t^r\big)\\
&\quad \le \frac{2^r}{2}\big(\big\langle f^2(|X|)x_2,x_2\big\rangle^{\frac r2}\big\langle g^2(|X^*|)x_1,x_1\big\rangle^{\frac r2}\\
&\qquad + \big\langle f^2(|Y|)x_1,x_1\big\rangle^{\frac r2}\big\langle f^2(|Y^*|)x_2,x_2\big\rangle^{\frac r2}\big) \quad \text{(by Lemma 2.3)}\\
&\quad \le \frac{2^r}{2}\big(\big\langle f^{2r}(|X|)x_2,x_2\big\rangle^{\frac12}\big\langle g^{2r}(|X^*|)x_1,x_1\big\rangle^{\frac12}\\
&\qquad + \big\langle f^{2r}(|Y|)x_1,x_1\big\rangle^{\frac12}\big\langle g^{2r}(|Y^*|)x_2,x_2\big\rangle^{\frac12}\big)\\
&\quad \le \frac{2^r}{2}\big(\big\langle f^{2r}(|X|)x_2,x_2\big\rangle + \big\langle g^{2r}(|Y^*|)x_2,x_2\big\rangle\big)^{\frac12}\big(f^{2r}(|Y|)x_1,x_1\rangle\\
&\qquad + \big\langle g^{2r}(|X^*|)x_1,x_1\big\rangle\big)^{\frac12}\\
&\quad = \frac{2^r}{2}\big\langle\big(f^{2r}(|X|)+g^{2r}(|Y^*|)\big)x_2,x_2\big\rangle^{\frac12}\big\langle\big(f^{2r}(|Y|)+g^{2r}(|X^*|)\big)x_1,x_1\big\rangle^{\frac12}\\
&\quad \le \frac{2^r}{4}\big(\big\langle\big(f^{2r}(|X|)+g^{2r}(|Y^*|)\big)x_2,x_2\big\rangle + \big\langle\big(f^{2r}(|Y|)+g^{2r}(|X^*|)\big)x_1,x_1\big\rangle\big)\\
&\qquad - \frac{2^r}{4}\big(\big\langle\big(f^{2r}(|X|)+g^{2r}(|Y^*|)\big)x_2,x_2\big\rangle^{\frac12} - \big\langle\big(f^{2r}(|Y|)+g^{2r}(|X^*|)\big)x_1,x_1\big\rangle^{\frac12}\big)^2\\
&\qquad \big(\text{by inequality (1.1)}\big)\\
&\quad \le \frac{2^r}{4}\big(\big\|f^{2r}(|X|)+g^{2r}(|Y^*|)\big\| + \big\|f^{2r}(|Y|)+g^{2r}(|X^*|)\big\|\big)
\end{aligned}$$

$$\begin{aligned}&-\frac{2^r}{4}\Big(\big\langle\big(f^{2r}\big(|X|\big)+g^{2r}\big(|Y^*|\big)\big)x_2,x_2\big\rangle^{\frac{1}{2}}\\&-\big\langle\big(f^{2r}\big(|Y|\big)+g^{2r}\big(|X^*|\big)\big)x_1,x_1\big\rangle^{\frac{1}{2}}\Big)^2.\end{aligned}$$

Taking the supremum over all unit vectors $\mathbf{x}=\left[\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right]\in\mathcal{H}_1\oplus\mathcal{H}_2$ we get the desired inequality. □

If we put $Y=X$ in Theorem 2.11, then we get next result.

Corollary 2.12. *Let $X\in\mathbb{B}(\mathcal{H})$, and let f, g be nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t)=t$ $(t\in[0,\infty))$. Then for $r\geq 1$,*

$$\omega^r(X)\leq 2^{r-1}\big\|f^{2r}\big(|X|\big)+g^{2r}\big(|X^*|\big)\big\|-2^{r-2}\inf_{\|(x_1,x_2)\|=1}\zeta(x_1,x_2),$$

where

$$\zeta(x_1,x_2)=\Big(\big\langle\big(f^{2r}\big(|X|\big)+g^{2r}\big(|X^*|\big)\big)x_2,x_2\big\rangle^{\frac{1}{2}}-\big\langle\big(f^{2r}\big(|X|\big)+g^{2r}\big(|X^*|\big)\big)x_1,x_1\big\rangle^{\frac{1}{2}}\Big)^2.$$

Remark 2.13. If $\mathbf{x}=\left[\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right]\in\mathcal{H}_1\oplus\mathcal{H}_2$ is a unit vector, then by using the inequality

$$\begin{aligned}&\big|\langle T\mathbf{x},\mathbf{x}\rangle\big|^r\\&\quad=\big|\langle Xx_2,x_1\rangle+\langle Yx_1,x_2\rangle\big|^r\\&\quad\leq\big(\big|\langle Xx_2,x_1\rangle\big|+\big|\langle Yx_1,x_2\rangle\big|\big)^r\\&\quad\leq\frac{2^r}{2}\big(\big|\langle Xx_2,x_1\rangle\big|^r+\big|\langle Yx_1,x_2\rangle\big|^r\big)\\&\quad\leq\frac{2^r}{2}\big(\big\langle f^2\big(|X|\big)x_2,x_2\big\rangle^{\frac{r}{2}}\big\langle g^2\big(|X^*|\big)x_1,x_1\big\rangle^{\frac{r}{2}}\big\langle g^2\big(|X|\big)x_2,x_2\big\rangle^{\frac{r}{2}}\big\langle f^2\big(|X^*|\big)x_1,x_1\big\rangle^{\frac{r}{2}}\big)\end{aligned}$$

and the same argument in the proof if Theorem 2.11, we get the following inequality:

$$\begin{aligned}\omega^r(T)\leq{}&\frac{2^r}{4}\big(\big\|f^{2r}\big(|X|\big)+f^{2r}\big(|Y^*|\big)\big\|+\big\|g^{2r}\big(|Y|\big)+g^{2r}\big(|X^*|\big)\big\|\big)\\&-\frac{2^r}{4}\inf_{\|(x_1,x_2)\|=1}\zeta(x_1,x_2),\end{aligned}$$

where $T=\left[\begin{smallmatrix}0&X\\Y&0\end{smallmatrix}\right]\in\mathbb{B}(\mathcal{H}_1\oplus\mathcal{H}_2)$, f, g are nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t)=t$ $(t\in[0,\infty))$, $r\geq 1$, and

$$\zeta(x_1,x_2)=\Big(\big\langle\big(f^{2r}\big(|X|\big)+f^{2r}\big(|Y^*|\big)\big)x_2,x_2\big\rangle^{\frac{1}{2}}-\big\langle\big(g^{2r}\big(|Y|\big)+g^{2r}\big(|X^*|\big)\big)x_1,x_1\big\rangle^{\frac{1}{2}}\Big)^2.$$

3. Some upper bounds for ω_p

In this section, we obtain some upper bounds for ω_P. We first show the following theorem.

Theorem 3.1. *Let $\widetilde{S_i}=\left[\begin{smallmatrix}A_i&0\\0&B_i\end{smallmatrix}\right],\widetilde{T_i}=\left[\begin{smallmatrix}0&X_i\\Y_i&0\end{smallmatrix}\right]$, and $\widetilde{U_i}=\left[\begin{smallmatrix}C_i&0\\0&D_i\end{smallmatrix}\right]$ be operator matrices in $\mathbb{B}(\mathcal{H}_1\oplus\mathcal{H}_2)$ $(1\leq i\leq n)$ such that A_i,B_i,C_i, and D_i are contractions.*

Then

$$\begin{aligned}&\omega_p^p(\widetilde{S}_1^*\widetilde{T}_1\widetilde{U}_1,\dots,\widetilde{S}_n^*\widetilde{T}_n\widetilde{U}_n)\\&\quad\le 2^{p-2}\sum_{i=1}^n\big\|D_i^*f^{2p}\big(|X_i|\big)D_i+B_i^*g^{2p}\big(|Y_i^*|\big)B_i\|^{\frac12}\|C_i^*f^{2p}\big(|Y_i|\big)C_i\\&\qquad+A_i^*g^{2p}\big(|X_i^*|\big)A_i\big\|^{\frac12}\end{aligned}$$

and

$$\begin{aligned}&\omega_p^p(\widetilde{S}_1^*\widetilde{T}_1\widetilde{U}_1,\dots,\widetilde{S}_n^*\widetilde{T}_n\widetilde{U}_n)\\&\quad\le 2^{p-2}\sum_{i=1}^n\big\|D_i^*f^{2p}\big(|X_i|\big)D_i+B_i^*f^{2p}\big(|Y_i^*|\big)B_i\big\|^{\frac12}\big\|C_i^*g^{2p}\big(|Y_i|\big)C_i\\&\qquad+A_i^*g^{2p}\big(|X_i^*|\big)A_i\big\|^{\frac12},\end{aligned}$$

where $p\ge 1$ *and* f*,* g *are nonnegative continuous functions on* $[0,\infty)$ *satisfying the relation* $f(t)g(t)=t$ $(t\in[0,\infty))$*.*

Proof. For any unit vector $\mathbf{x}=\left[\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right]\in\mathcal{H}_1\oplus\mathcal{H}_2$ we have

$$\begin{aligned}&\sum_{i=1}^n\big|\langle T_i\mathbf{x},\mathbf{x}\rangle\big|^p\\&\quad=\sum_{i=1}^n\big|\langle A_i^*X_iD_ix_2,x_1\rangle+\langle B_i^*Y_iC_ix_1,x_2\rangle\big|^p\\&\quad\le\sum_{i=1}^n\Big(\big|\langle A_i^*X_iD_ix_2,x_1\rangle\big|+\big|\langle B_i^*Y_iC_ix_1,x_2\rangle\big|\Big)^p\quad\text{(by the triangular inequality)}\\&\quad\le\frac{2^p}{2}\sum_{i=1}^n\big|\langle A_i^*X_iD_ix_2,x_1\rangle\big|^p+\big|\langle B_i^*Y_iC_ix_1,x_2\rangle\big|^p\\&\qquad\big(\text{by the convexity of } f(t)=t^p\big)\\&\quad=\frac{2^p}{2}\sum_{i=1}^n\big|\langle X_iD_ix_2,A_ix_1\rangle\big|^p+\big|\langle Y_iC_ix_1,B_ix_2\rangle\big|^p\\&\quad\le\frac{2^p}{2}\sum_{i=1}^n\big\langle f^2\big(|X_i|\big)D_ix_2,D_ix_2\big\rangle^{\frac p2}\big\langle g^2\big(|X_i^*|\big)A_ix_1,A_ix_1\big\rangle^{\frac p2}\\&\qquad+\big\langle f^2\big(|Y_i|\big)C_ix_1,C_ix_1\big\rangle^{\frac p2}\big\langle g^2\big(|Y_i^*|\big)B_ix_2,B_ix_2\big\rangle^{\frac p2}\quad\text{(by Lemma 2.3)}\\&\quad\le\frac{2^p}{2}\sum_{i=1}^n\big\langle f^{2p}\big(|X_i|\big)D_ix_2,D_ix_2\big\rangle^{\frac12}\big\langle g^{2p}\big(|X_i^*|\big)A_ix_1,A_ix_1\big\rangle^{\frac12}\\&\qquad+\big\langle f^{2p}\big(|Y_i|\big)C_ix_1,C_ix_1\big\rangle^{\frac12}\big\langle g^{2p}\big(|Y_i^*|\big)B_ix_2,B_ix_2\big\rangle^{\frac12}\quad\big(\text{by Lemma 2.2(a)}\big)\\&\quad=\frac{2^p}{2}\sum_{i=1}^n\big\langle D_i^*f^{2p}\big(|X_i|\big)D_ix_2,x_2\big\rangle^{\frac12}\big\langle A_i^*g^{2p}\big(|X_i^*|\big)A_ix_1,x_1\big\rangle^{\frac12}\end{aligned}$$

$$+\left\langle C_i^* f^{2p}\big(|Y_i|\big)C_i x_1, x_1\right\rangle^{\frac{1}{2}}\left\langle B_i^* g^{2p}\big(|Y_i^*|\big)B_i x_2, x_2\right\rangle^{\frac{1}{2}}$$

$$\leq \frac{2^p}{2}\sum_{i=1}^{n}\Big(\left\langle D_i^* f^{2p}\big(|X_i|\big)D_i x_2, x_2\right\rangle + \left\langle B_i^* g^{2p}\big(|Y_i^*|\big)B_i x_2, x_2\right\rangle\Big)^{\frac{1}{2}}$$

$$\times\Big(\left\langle C_i^* f^{2p}\big(|Y_i|\big)C_i x_1, x_1\right\rangle + \left\langle A_i^* g^{2p}\big(|X_i^*|\big)A_i x_1, x_1\right\rangle\Big)^{\frac{1}{2}}$$

(by the Cauchy–Schwarz inequality)

$$= \frac{2^p}{2}\sum_{i=1}^{n}\Big(\Big\langle\big(D_i^* f^{2p}\big(|X_i|\big)D_i + B_i^* g^{2p}\big(|Y_i^*|\big)B_i\big)x_2, x_2\Big\rangle\Big)^{\frac{1}{2}}$$

$$\times\Big(\Big\langle\big(C_i^* f^{2p}\big(|Y_i|\big)C_i + A_i^* g^{2p}\big(|X_i^*|\big)A_i\big)x_1, x_1\Big\rangle\Big)^{\frac{1}{2}}$$

$$\leq \frac{2^p}{2}\sum_{i=1}^{n}\|D_i^* f^{2p}\big(|X_i|\big)D_i + B_i^* g^{2p}\big(|Y_i^*|\big)B_i\|^{\frac{1}{2}}\|C_i^* f^{2p}\big(|Y_i|\big)C_i$$

$$+ A_i^* g^{2p}\big(|X_i^*|\big)A_i\|^{\frac{1}{2}}\|x_1\|\|x_2\|$$

$$= \frac{2^p}{2}\sum_{i=1}^{n}\Big\|D_i^* f^{2p}\big(|X_i|\big)D_i + B_i^* g^{2p}\big(|Y_i^*|\big)B_i\Big\|^{\frac{1}{2}}$$

$$\times\Big\|C_i^* f^{2p}\big(|Y_i|\big)C_i + A_i^* g^{2p}\big(|X_i^*|\big)A_i\Big\|^{\frac{1}{2}}\left(\frac{\|x_1\|^2+\|x_2\|^2}{2}\right)$$

$$= \frac{2^p}{4}\sum_{i=1}^{n}\Big\|D_i^* f^{2p}\big(|X_i|\big)D_i + B_i^* g^{2p}\big(|Y_i^*|\big)B_i\Big\|^{\frac{1}{2}}\Big\|C_i^* f^{2p}\big(|Y_i|\big)C_i$$

$$+ A_i^* g^{2p}\big(|X_i^*|\big)A_i\Big\|^{\frac{1}{2}}.$$

Taking the supremum over all unit vectors $\mathbf{x} \in \mathcal{H}_1 \oplus \mathcal{H}_2$, we obtain the first inequality. Using the inequality

$$\sum_{i=1}^{n}\big|\langle T_i\mathbf{x}, \mathbf{x}\rangle\big|^p$$

$$= \sum_{i=1}^{n}\big|\langle A_i^* X_i D_i x_2, x_1\rangle + \langle B_i^* Y_i C_i x_1, x_2\rangle\big|^p$$

$$\leq \sum_{i=1}^{n}\Big(\big|\langle A_i^* X_i D_i x_2, x_1\rangle\big| + \big|\langle B_i^* Y_i C_i x_1, x_2\rangle\big|\Big)^p \quad \text{(by the triangular inequality)}$$

$$\leq \frac{2^p}{2}\sum_{i=1}^{n}\big|\langle A_i^* X_i D_i x_2, x_1\rangle\big|^p + \big|\langle B_i^* Y_i C_i x_1, x_2\rangle\big|^p$$

$\big($by the convexity of $f(t) = t^p\big)$

$$= \frac{2^p}{2}\sum_{i=1}^{n}\big|\langle X_i D_i x_2, A_i x_1\rangle\big|^p + \big|\langle Y_i C_i x_1, B_i x_2\rangle\big|^p$$

$$\leq \frac{2^p}{2}\sum_{i=1}^{n}\left\langle f^2\big(|X_i|\big)D_i x_2, D_i x_2\right\rangle^{\frac{p}{2}}\left\langle g^2\big(|X_i^*|\big)A_i x_1, A_i x_1\right\rangle^{\frac{p}{2}}$$

$$+\left\langle g^2\left(|Y_i|\right)C_ix_1, C_ix_1\right\rangle^{\frac{p}{2}}\left\langle f^2\left(|Y_i^*|\right)B_ix_2, B_ix_2\right\rangle^{\frac{p}{2}}$$
(by Lemma 2.3)

and a similar process in the proof of the first inequality, we reach the second inequality. □

In the special case of Theorem 3.1 for $A_i = B_i = C_i = D_i = I$ $(1 \leq i \leq n)$, we have the next result.

Corollary 3.2. *Let* $T_i = \left[\begin{smallmatrix} 0 & X_i \\ Y_i & 0 \end{smallmatrix}\right] \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ $(1 \leq j \leq n)$*. Then*

$$\omega_p^p(T_1, T_2, \dots, T_n) \leq 2^{p-2}\sum_{i=1}^{n}\left\|f^{2p}\left(|X_i|\right) + g^{2p}\left(|Y_i^*|\right)\right\|^{\frac{1}{2}}\left\|f^{2p}\left(|Y_i|\right) + g^{2p}\left(|X_i^*|\right)\right\|^{\frac{1}{2}}$$

and

$$\omega_p^p(T_1, T_2, \dots, T_n) \leq 2^{p-2}\sum_{i=1}^{n}\left\|f^{2p}\left(|X_i|\right) + f^{2p}\left(|Y_i^*|\right)\right\|^{\frac{1}{2}}\left\|g^{2p}\left(|Y_i|\right) + g^{2p}\left(|X_i^*|\right)\right\|^{\frac{1}{2}}$$

in which $p \geq 1$ *and* f, g *are nonnegative continuous functions on* $[0, \infty)$ *satisfying the relation* $f(t)g(t) = t$ $(t \in [0, \infty))$.

If we put $f(t) = g(t) = t^{\frac{1}{2}}$ $(t \in [0, \infty))$, then we get the next result.

Corollary 3.3. *Let* $T_i = \left[\begin{smallmatrix} 0 & X_i \\ Y_i & 0 \end{smallmatrix}\right] \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ $(1 \leq j \leq n)$*. Then*

$$\omega_p^p(T_1, T_2, \dots, T_n) \leq 2^{p-2}\sum_{i=1}^{n}\left\||X_i|^p + |Y_i^*|^p\right\|^{\frac{1}{2}}\left\||Y_i|^p + |X_i^*|^p\right\|^{\frac{1}{2}}$$

for $p \geq 1$.

Theorem 3.4. *Let* $T_i = \left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right] \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ $(1 \leq i \leq n)$ *and* $p \geq 1$*. Then*

$$\begin{aligned}&\omega_p^p(T_1, \dots, T_n)\\ &\quad \leq 2^{-p}\sum_{i=1}^{n}\left(\omega(A_i) + \omega(D_i) + \sqrt{\left(\omega(A_i) - \omega(D_i)\right)^2 + \left(\|B_i\| + \|C_i\|\right)^2}\right)^p.\end{aligned}$$

In particular,

$$\omega\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \frac{1}{2}\left(\omega(A) + \omega(D) + \sqrt{\left(\omega(A) - \omega(D)\right)^2 + \left(\|B\| + \|C\|\right)^2}\right).$$

Proof. Let $\mathbf{x} = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ be a unit vector in $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then

$$\begin{aligned}\left|\langle T_i\mathbf{x}, \mathbf{x}\rangle\right| &= \left|\left\langle \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\rangle\right|\\ &= \left|\left\langle \begin{bmatrix} A_ix_1 + B_ix_2 \\ C_ix_1 + D_ix_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\rangle\right|\\ &= \left|\langle A_ix_1, x_1\rangle + \langle B_ix_2, x_1\rangle + \langle C_ix_1, x_2\rangle + \langle D_ix_2, x_2\rangle\right|\\ &\leq \left|\langle A_ix_1, x_1\rangle\right| + \left|\langle B_ix_2, x_1\rangle\right| + \left|\langle C_ix_1, x_2\rangle\right| + \left|\langle D_ix_2, x_2\rangle\right|.\end{aligned}$$

Thus,

$$
\begin{aligned}
&\omega_p^p(T_1,\dots,T_n)\\
&\quad= \sup_{\|\mathbf{x}\|=1} \sum_{i=1}^n \big|\langle T_i\mathbf{x},\mathbf{x}\rangle\big|^p\\
&\quad\le \sup_{\|x_1\|^2+\|x_2\|^2=1} \sum_{i=1}^n \Big(\big|\langle A_ix_1,x_1\rangle\big| + \big|\langle B_ix_2,x_1\rangle\big| + \big|\langle C_ix_1,x_2\rangle\big| + \big|\langle D_ix_2,x_2\rangle\big|\Big)^p\\
&\quad\le \sum_{i=1}^n \Big(\sup_{\|x_1\|^2+\|y\|^2=1} \Big(\big|\langle A_ix_1,x_1\rangle\big| + \big|\langle B_ix_2,x_1\rangle\big| + \big|\langle C_ix_1,x_2\rangle\big| + \big|\langle D_ix_2,x_2\rangle\big|\Big)\Big)^p\\
&\quad\le \sum_{i=1}^n \Big(\sup_{\|x_1\|^2+\|x_2\|^2=1} \Big(\omega(A_i)\|x_1\|^2 + \omega(D_i)\|x_2\|^2 + \big(\|B_i\|+\|C_i\|\big)\|x_1\|\|x_2\|\Big)\Big)^p\\
&\quad= \sum_{i=1}^n \Big(\sup_{\theta\in[0,2\pi]} \Big(\omega(A_i)\cos^2\theta + \omega(D_i)\sin^2\theta + \big(\|B_i\|+\|C_i\|\big)\cos\theta\sin\theta\Big)\Big)^p\\
&\quad= 2^{-p}\sum_{i=1}^n \Big(\omega(A_i)+\omega(D_i)+\sqrt{\big(\omega(A_i)-\omega(D_i)\big)^2+\big(\|B_i\|+\|C_i\|\big)^2}\Big)^p. \qquad \square
\end{aligned}
$$

For $A_i = D_i$ and $B_i = C_i$ $(1 \le i \le n)$ we get the following result.

Corollary 3.5. *Let $T_i = \left[\begin{smallmatrix} \pm A_i & \pm B_i \\ \pm B_i & \pm A_i \end{smallmatrix}\right]$ be an operator matrix with $A_i, B_i \in \mathbb{B}(\mathcal{H})$ $(1 \le i \le n)$. Then*

$$\omega_p^p(T_1,\dots,T_n) \le \sum_{i=1}^n \big(\omega(A_i)+\|B_i\|\big)^p$$

for $p \ge 1$. In particular, if $A, B \in \mathbb{B}(\mathcal{H})$, then

$$\omega\left(\begin{bmatrix} \pm A & \pm B \\ \pm B & \pm A \end{bmatrix}\right) \le \omega(A)+\|B\|.$$

If we take $B_i = C_i = 0$ $(1 \le i \le n)$ in Theorem 3.4, then we get the following inequality.

Corollary 3.6. *Let $T_i = \left[\begin{smallmatrix} A_i & 0 \\ 0 & D_i \end{smallmatrix}\right] \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ $(1 \le i \le n)$. Then for all $p \ge 1$,*

$$\omega_p^p(T_1,\dots,T_n) \le \sum_{i=1}^n \max\big(\omega^p(A_i),\omega^p(D_i)\big).$$

For $C_i = D_i = 0$ $(1 \le i \le n)$ we obtain a result that generalizes and refines the inequality $\omega\left(\left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]\right) \le \omega(A) + \frac{\|B\|}{2}$.

Corollary 3.7. *Let $T_i = \left[\begin{smallmatrix} A_i & B_i \\ 0 & 0 \end{smallmatrix}\right] \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ $(1 \le i \le n)$ and $p \ge 1$. Then*

$$\omega_p^p(T_1,\dots,T_n) \le 2^{-p}\sum_{i=1}^n \Big(\omega(A_i)+\sqrt{\omega^2(A_i)+\|B_i\|^2}\Big)^p.$$

In particular,

$$\omega\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \le \frac{1}{2}\big(\omega(A) + \sqrt{\omega^2(A) + \|B\|^2}\big).$$

If we put $A_i = D_i = 0$ $(1 \le i \le n)$, then we deduce the following.

Corollary 3.8. *Let* $T_i = \left[\begin{smallmatrix} 0 & B_i \\ C_i & 0 \end{smallmatrix}\right] \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ $(1 \le i \le n)$ *and* $p \ge 1$. *The*

$$\omega_p^p(T_1, \dots, T_n) \le 2^{-p} \sum_{i=1}^{n} \big(\|B_i\| + \|C_i\|\big)^p.$$

In particular, if $B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ *and* $C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$, *then*

$$\omega\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \le \frac{1}{2}\big(\|B\| + \|C\|\big).$$

References

1. A. Abu-Omar and F. Kittaneh, *Estimates for the numerical radius and the spectral radius of the Frobenius companion matrix and bounds for the zeros of polynomials*, Ann. Func. Anal. **5** (2014), no. 1, 56–62. Zbl 1298.47011. MR3119112. DOI 10.15352/afa/1391614569. 298
2. A. Abu-Omar and F. Kittaneh, *Numerical radius inequalities for $n \times n$ operator matrices*, Linear Algebra Appl. **468** (2015), 18–26. Zbl 1316.47005. MR3293237. DOI 10.1016/j.laa.2013.09.049. 298, 299, 300
3. Y. Al-manasrah and F. Kittaneh, *A generalization of two refined Young inequalities*, Positivity **19** (2015), no. 4, 757–768. Zbl 1335.15029. MR3415102. DOI 10.1007/s11117-015-0326-8. 298
4. J. C. Bourin, *Matrix subadditivity inequalities and block-matrices*, Internat. J. Math. **20** (2009), no. 6, 679–691. Zbl 1181.15030. MR2541930. DOI 10.1142/S0129167X09005509. 302
5. M. Hajmohamadi, R. Lashkaripour, and M. Bakherad, *Some generalizations of numerical radius on off-diagonal part of 2×2 operator matrices*, preprint, to appear in J. Math. Inequal. 298
6. P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Grad. Texts in Math. **19**, Springer, New York, 1982. Zbl 0496.47001. MR0675952. 298
7. O. Hirzallah, F. Kittaneh, and K. Shebrawi, *Numerical radius inequalities for certain 2×2 operator matrices*, Integral Equations Operator Theory **71** (2011), no. 1, 129–149. Zbl 1238.47004. MR2822431. DOI 10.1007/s00020-011-1893-0. 299
8. F. Kittaneh, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 2, 283–293. Zbl 0655.47009. MR0944864. DOI 10.2977/prims/1195175202. 299
9. F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. **158** (2003), 11–17. Zbl 1113.15302. MR2014548. DOI 10.4064/sm158-1-2. 298
10. K. E. Gustafson and D. K. M. Rao, *Numerical Range: The Field of Values of Linear Operators and Matrices*, Springer, New York, 1996. Zbl 0874.47003. MR1417493. DOI 10.1007/978-1-4613-8498-4. 298
11. G. Popescu, *Unitary invariants in multivariable operator theory*, Mem. Amer. Math. Soc. **200** (2009), no. 941. Zbl 1180.47010. MR2519137. DOI 10.1090/memo/0941. 298
12. G. Ramesh, *On the numerical radius of a quaternionic normal operator*, Adv. Oper. Theory **2** (2017), no. 1, 78–86. Zbl 06692021. DOI 10.22034/aot.1611-1060. 298

13. M. S. Moslehian, M. Sattari, and K. Shebrawi, *Extension of Euclidean operator radius inequalities*, Math Scand. **120** (2017), no. 1, 129–144. Zbl 06697365. MR3624011. DOI 10.7146/math.scand.a-25509. 298
14. K. Shebrawi and H. Albadawi, *Numerical radius and operator norm inequalities*, J. Inequal. Appl. **2009**, art. ID 492154. Zbl 1179.47004. MR2496270. DOI 10.1155/2009/492154. 302
15. A. Sheikhhosseini, M. S. Moslehian, and Kh. Shebrawi, *Inequalities for generalized Euclidean operator radius via Young's inequality*, J. Math. Anal. Appl. **445** (2017), no. 2, 1516–1529. Zbl 1358.47010. MR3545256. DOI 10.1016/j.jmaa.2016.03.079. 298, 299
16. T. Yamazaki, *On upper and lower bounds of the numerical radius and an equality condition*, Studia Math. **178** (2007), 83–89. Zbl 1114.47003. MR2282491. DOI 10.4064/sm178-1-5. 299
17. A. Zamani, *Some lower bounds for the numerical radius of Hilbert space operators*, Adv. Oper. Theory **2** (2017), no. 2, 98–107. Zbl 06711470. DOI 10.22034/aot.1612-1076. 298

[1]Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, I.R. Iran.

E-mail address: mojtaba.bakherad@yahoo.com; bakherad@member.ams.org

[2]Department of Mathematics, Al-Balqa Applied University, Salt, Jordan.

E-mail address: khalid@bau.edu.jo; shebrawi@gmail.com