Chapter 1

Positive and Negative dependence orderings

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1.0.1 Introduction

Notions of positive dependence of two random variables X_1 and X_2 have been introduced in the literatures in an effort to mathematically describe the property that large (respectively, small) values of X_1 tend to go together with large (respectively, small) values of X_2. Many of the notions of positive dependence are defined by means of some comparison of the joint distribution of X_1 and X_2 with their distribution under the theoretical assumption that X_1 and X_2 are independence. Often such a comparison can be extended to general pairs of bivariate distributions with given marginals. This fact led researchers to introduce various notions of positive dependence orders. These orders are designed to compare the strength of the positive dependence of the two underlying bivariate distributions. In this chapter we describe some such notions.

in many sections of this chapter we first describe a positive dependence order which compares two bivariate random vectors (or distributions). When the order can be extended to general n-dimensional ($n > 2$) random vectors, we will describe the extension in a later of that section.

Most of orders that we describe in this chapter are defined on the Frechet class $\mathcal{F}(F_1, F_2)$ of bivariate distributions with fixed marginals F_1 and F_2. The upper bound of this class is the distribution defined by $F_U(x) = \min\{F_1(x), F_2(x)\}$ (whose probability mass is the concentrated on the set $\{(x_1, x_2) : F_1(x_1) = F_2(x_2)\}$). The lower bound of this class is the distribution defined by $F_L(x) = \max\{F_1(x) + F_2(x) - 1, 0\}$ (whose probability mass is the concentrated on the set $\{(x_1, x_2) : F_1(x_1) + F_2(x_2) = 1\}$)

1.0.2 Axioms for a bivariate dependence ordering

In this subsection, we list properties or axioms that an ordering of distributions should have in order that higher in the ordering means more positive dependence.Let F and G are two bivariate distributions in class of Frechet. Desirable properties or axioms are:

- P1. (concordance), $F \leq G$ implies $F(x, y) \leq G(x, y)$ for all real values x, y.
- P2. (transitivity). $F \leq G$ and $G \leq H$ imply $F \leq H$.
- P3.(reflexivity) $F \leq F$.
- P4. (equivalency) $F \leq G$ and $G \leq F$ imply $F = G$.
- P5.(bounds) $F_L \leq F \leq F_U$, where F_L and F_U are lower and upper bounds of class of Frechet respectively.
- P6.(invariance to limit in the distribution) $F_n \leq G_n, n \geq 1$ and $F_n \to F, \quad G_n \to G$ as $n \to \infty$, imply $F \leq G$.
- P7. (invariance to increasing transforms) $(X_1, X_2) \leq (Y_1, Y_2)$ in all notions implies $(\phi(X_1), X_2) \leq (\phi(Y_1), Y_2)$ for all strictly increasing functions ϕ. and $(\varphi(Y_1), Y_2) \leq (\varphi(X_1), X_2)$ for all decreasing functions φ where (X_1, X_2) and (Y_1, Y_2) have the distributions F and G respectively.

- P8.(invariance to order of indices) $(X_1, X_2) \leq (Y_1, Y_2)$ implies $(X_2, X_1) \leq (Y_2, Y_1)$.

An ordering that satisfies the night properties is called a bivariate positive dependence ordering.

1.1 PQD order

Let (X_1, X_2) be a bivariate random vector with distribution function F, and let (Y_1, Y_2) be another bivariate random vector with distribution function G, such that $F, G \in \mathcal{F}(F_1, F_2)$, for some univariate distribution functions F_1 and F_2. If for all real values x_1 and x_2,

$$F(x_1, x_2) \leq G(x_1, x_2) \quad (1)$$

then we say that (X_1, X_2) is smaller than (Y_1, Y_2) in the PQD order (denoted by $(X_1, X_2) \leq_{PQD} (Y_1, Y_2)$). Sometimes it will be useful to write this as $F \leq_{PQD} G$. Using the assumption that F and G have the same univariate marginals, it is easy to see that for all real values x_1 and x_2, (1) is equivalent to

$$\bar{F}(x_1, x_2) \leq \bar{G}(x_1, x_2)$$

Corollary 1 *i*). F is PQD $\Leftrightarrow F^I \leq_{PQD} F$
ii) Let (X_1, X_2) be a bivariate random vector with distribution function F, and let (Y_1, Y_2) be another bivariate random vector with distribution function G, such that $F, G \in \mathcal{F}(F_1, F_2)$. Then

$$(X_1, X_2) \leq_{PQD} (Y_1, Y_2) \Rightarrow Cov(X_1, X_2) \leq Cov(Y_1, Y_2) \ \ and \ \ \rho(X_1, X_2) \leq \rho(Y_1, Y_2).$$

Proof The proof of Part (i) is obvious. (ii) Using the Hoeffding"s Lemma we have

$$\begin{aligned} Cov(X_1, X_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F(x, y) - F_1(x)F_2(y)]dxdy \\ &\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[G(x, y) - F_1(x)F_2(y)]dxdy = Cov(Y_1, Y_2). \end{aligned}$$

and therefore $Var(X_i) = Var(Y_i), i = 1, 2$ we have that $\rho(X_1, X_2) \leq \rho(Y_1, Y_2)$.
Yanagimoto and Okamoto (1969) have shown that some other correlation measures, such as Kendall's tau, Spearman's rho and Blomquist's q are preserved under the PQD order.
Corollary Let (X_1, X_2) be a bivariate random vector with distribution function F, and let (Y_1, Y_2) be another bivariate random vector with distribution function G, such that $F, G \in \mathcal{F}(F_1, F_2)$ and $(X_1, X_2) \leq_{PQD} (Y_1, Y_2)$. Then
(i) For all real values x, y $\bar{F}(x, y) \leq \bar{G}(x, y)$.
(ii) For all real value x_1, $E[X_2|X_1 > x_1] \leq E[Y_2|Y_1 > x_1]$, this condition can be used to define a positive dependence stochastic order. Such an order is discussed in Muliere and Petrone

(1992).
(iii) For every distribution $F \in \mathcal{F}(F_1, F_2)$ we have

$$F_L \leq_{PQD} F \leq_{PQD} F_U.$$

1.1.1 Closure properties

A powerful closure property of the PQD order is given in the next theorem.
Theorem 1 Suppose that the four random vectors (X_1, X_2), (Y_1, Y_2), (U_1, U_2) and (V_1, V_2) satisfy

$$(X_1, X_2) \leq_{PQD} (Y_1, Y_2) \;\; and \;\; (U_1, U_2) \leq_{PQD} (V_1, V_2)$$

and suppose that (X_1, X_2) and (U_1, U_2) are independent, and also that (Y_1, Y_2) and (V_1, V_2) are independent. Then for all increasing functions ϕ and φ,

$$(\phi(X_1, U_1), \varphi(X_2, U_2)) \leq_{PQD} (\phi(Y_1, V_1), \varphi(Y_2, V_2)).$$

Corollary Under the assumption above theorem we have
(i) $(X_1 + U_1, X_2 + U_2) \leq_{PQD} (Y_1 + V_1, Y_2 + V_2)$
(ii)For all increasing functions ϕ and φ it follows that

$$(X_1, X_2) \leq_{PQD} (Y_1, Y_2) \Rightarrow (\phi(X_1), \varphi(X_2)) \leq_{PQD} (\phi(Y_1), \varphi(Y_2))$$

The closure properties that are stated in the next theorem are easy to verify.
Theorem 2$i)$ Let $\{(X_1^{(n)}, X_2^{(n)})\}$ and $\{(Y_1^{(n)}, Y_2^{(n)})\}$ be two sequences of random vectors such that $(X_1^{(n)}, X_2^{(n)}) \rightarrow_{st} (X_1, X_2)$ and $(Y_1^{(n)}, Y_2^{(n)}) \rightarrow_{st} (Y_1, Y_2)$ as $n \rightarrow \infty$. If $(X_1^{(n)}, X_2^{(n)}) \leq_{PQD} (Y_1^{(n)}, Y_2^{(n)})$ for all $n \geq 1$ then $(X_1, X_2) \leq_{PQD} (Y_1, Y_2)$.
$ii)$ Let (X_1, X_2), (Y_1, Y_2) and Θ be random vectors such that $[(X_1, X_2)|\Theta = \theta] \leq_{PQD} [(Y_1, Y_2)|\Theta = \theta]$ for all θ in the support of Θ. Then $(X_1, X_2) \leq_{PQD} (Y_1, Y_2)$. That is, the PQD order is closed under mixtures.
Example 1 Let ϕ and ψ be two Laplace transforms of positive random variables. Then F and G, defined by

$$F(x, y) = \phi(\phi^{-1}(x) + \phi^{-1}(y)), \quad (x, y) \in [0, 1]^2,$$

and

$$G(x, y) = \psi(\psi^{-1}(x) + \psi^{-1}(y)), \quad (x, y) \in [0, 1]^2,$$

are bivariate distribution functions with uniform $[0, 1]$ marginals (such as F and G are called Archimedean copulas). Let (X_1, X_2) and (Y_1, Y_2) be distributed according to F and G respectively. Then $(X_1, X_2) \leq_{PQD} (Y_1, Y_2)$ if and only if $\psi^{-1} o \phi$ is superadditive (that is for all $x, y \geq 0$, $\psi^{-1}\phi(x + y) \geq \psi^{-1}\phi(x) + \psi^{-1}\phi(y)$). Also, if $\phi^{-1} o \psi$ has a completely monotone derivative, then $(X_1, X_2) \leq_{PQD} (Y_1, Y_2)$.

1.2 NQD order

In this section we review researches on NQD order such as Ebrahimi (1982) that studied the ordering of NQD.Moreover we will study some stochastic orders for concepts of negative dependence if it is possible.

1.3 POD order

Let $X = (X_1, X_2, ..., X_n)$ be a random vector with distribution function F and survival function $\bar{F}$. Let $Y = (Y_1, Y_2, ..., Y_n)$ be another random vector with distribution G and survival function $\bar{G}$. if for all x,

$$F(x) \leq G(x) \quad and \quad \bar{F}(x) \leq \bar{G}(x), \qquad (1)$$

then we say that X is smaller then Y in the positive orthant dependence (POD) order [denoted by $X \leq_{POD} Y$] from (1) it follows that only random vectors with same univariate marginals can be compared in the POD order. ¿From (1)it follows that

$$X \leq_{POD} Y \Leftrightarrow \{X \leq_{uo} Y \;\; and \;\; X \leq_{lo} Y\}. \qquad (2)$$

An extension of Theorem 4.1.1 to the general multivariate case is the following. The proof of Theorem 4.2.1 is a strainghtforward extension of the proof of Theorem 4.1.1 and therefore it is omitted.

Theorem 1 Suppose that the four random vectors $X = (X_1, X_2, ..., X_n)$, $Y = (Y_1, Y_2, ..., Y_n)$, $U = (U_1, U_2, ..., U_n)$ and $V = (V_1, V_2, ..., V_n)$ satisfy

$$X \leq_{POD} Y \quad and \quad U \leq_{POD} V \qquad (3)$$

and suppose that X and U are independent,and also that Y and V are independent.Then for all increasing functions $\phi_i, \; i = 1, 2, ..., n$

$$(\phi(X_1, U_1), ..., \phi(X_n, U_n) \leq_{POD} (\phi(Y_1, V_1), ..., \phi(Y_n, V_n).$$

Corollary 1 Under the assumptions of Theorem 4.2.1
i)

$$X + U \leq_{POD} Y + V,$$

that is the PQD order is closed under convolutions.
ii) For all increasing functions $\phi_i, \; i = 1, 2, ..., n$, $X \leq_{POD} Y$ implies that

$$(\phi(X_1), \phi(X_2), ..., \phi(X_n)) \leq_{POD} (\phi(Y_1), \phi(Y_2), ..., \phi(Y_n)).$$

The closure properties that are stated in the next Theorem are easy to verify.
Theorem 2 *i*) Let $X_1, X_2, ..., X_m$ be a set of independent random vectors where the dimension of $X_i, \;\; is \;\; k_i, i = 1, 2, ..., m$. Let $Y_1, Y_2, ..., Y_m$ be another set of independent random

vectors where the dimension of $Y_i, \quad is \quad k_i, i = 1, 2, ..., m$. If $X_i \leq_{POD} Y_i$ for $i = 1, 2, ..., m$, then

$$(X_1, X_2, ..., X_m) \leq_{POD} (Y_1, Y_2, ..., Y_m).$$

That is , the POD order is closed under conjunctions.
ii) Let $X = (X_1, X_2, ..., X_n)$ and $Y = (Y_1, Y_2, ..., Y_n)$ be two n-dimensional random vectors. If $X \leq_{POD} Y$ then $X_I \leq_{POD} Y_I$ for each $I \subseteq \{1, 2, ..., n\}$. That is, the POD order is closed under marginalization.
iii) Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be two sequences of random vectors such that $X_n \rightarrow_{st} X$ and $Y_n \rightarrow_{st} Y$ as $n \rightarrow \infty$. If $X_n \leq_{POD} Y_n$ for all $n \geq 1$ then $X \leq_{POD} Y$.
iv) Let X, Y and Θ be random vectors such that $[X|\Theta = \theta] \leq_{POD} [Y|\Theta = \theta]$ for all θ in the support of Θ. Then $X \leq_{POD} Y$. That is, the POD order is closed under mixtures.
Corollary 2 Under the assumptions of Theorem 4.2.2, if

$$(X_1, X_2, ..., X_n) \leq_{POD} (Y_1, Y_2, ..., Y_n)$$

then for all $i \neq j$, we have that

$$Cov(X_i, X_j) \leq Cov(Y_i, Y_j)$$

and since the univariate marginals of X and Y are equal, it follows that for all $i \neq j$

$$\rho(X_i, X_j) \leq \rho(Y_i, Y_j)$$

Joe(1997) has shown that some multivariate versions of the correlation measures Kendall's tah, Spearman's rho and Blomquist's qu are monotone with respect to the POD order.
Another preservation property of the POD order is described in the next Theorem. In the following Theorem we define $\sum_{j=1}^{0} x_j \equiv 0$ for any sequence $\{x_j, j \geq 1\}$.
Theorem 3 Let $X_j = (X_{j,1}, X_{j,2}, ..., X_{j,m}), j = 1, 2, ...$ be a sequence of nonnegative random vectors, and let $M = (M_1, M_2, ..., M_m)$ and $N = (N_1, N_2, ..., N_m)$ be two vectors of nonnegative integer-valued random variables. Assume that both M and N are independent of the X_j's. If $M \leq_{POD} N$, then

$$(\sum_{j=1}^{M_1} X_{j,1}, \sum_{j=1}^{M_2} X_{j,2}, ..., \sum_{j=1}^{M_m} X_{j,m}) \leq_{POD} (\sum_{j=1}^{N_1} X_{j,1}, \sum_{j=1}^{N_2} X_{j,2}, ..., \sum_{j=1}^{N_m} X_{j,m}).$$

Consider now, n families of univariate distribution functions $\{G_\theta^{(i)}, \theta \in \chi_i\}$ where χ_i is a subset of the real line $\mathcal{R}$, $i = 1, 2, ..., n$. Let $X_i(\theta)$ denote a random variable with distribution function $G_\theta^{(i)}, i = 1, 2, ..., n$.Below we give a result which provides comparisons of two random vectors, with distribution functions of the form (6.B.18), in the POD order.
Theorem 4 Let $\{G_\theta^{(i)}, \theta \in \chi_i\}$ be n families of univariate distribution functions as above. Let Θ_1 and Θ_2 be two random vectors with supports in $\prod_{i=1}^{n} \chi_i$ and distribution functions

F_1 and F_2, respectively. Let Y_1 and Y_2 be two random vectors with distribution functions H_1 and H_2 given by

$$H_j(y_1, y_2, ..., y_n) = \int_{\chi_1} \int_{\chi_2} \cdots \int_{\chi_n} \prod_{i=1}^{n} G_\theta^{(i)}(y_i) dF_j(\theta_1, \theta_2, ..., \theta_n),$$

$$(y_1, y_2, ..., y_n) \in \mathcal{R}^n, j = 1, 2, ...$$

If $\Theta_1 \leq_{POD} \Theta_2$ then $Y_1 \leq_{POD} Y_2$.

Example 1 Let X be an n-dimensional random vector with a density f of the form

$$f(x) = |\Sigma|^{-1/2} g(x\Sigma^{-1}x),$$

where $\Sigma = (\sigma_{ij})$ is a positive definite $n \times n$ matrixs, and g satisfies $\int_0^\infty r^{n-1} g(r^2) dr < \infty$. Such density functions are called elliptically contoured. Let Y be and n-dimensional random vector with a density function h of the form

$$h(x) = |\Lambda|^{-1/2} g(x\Lambda^{-1}x),$$

where $\Lambda = (\lambda_{ij})$ is a positive definite $n \times n$ matrix. If $\sigma_{ii} = \lambda_{ii}, i = 1, 2, ..., n$, and $\sigma_{ij} \leq \lambda_{ij}, 1 \leq i < j \leq n$, then $X \leq_{POD} Y$. In particular, multivariate normal random vectors with means 0 and the same variances are ordered in the POD order if their covariances are pointwise order.

1.4 Supermodular order

The supermodular order, which is described in this section, is a sufficient condition that implies the PQD order, but it is also of independent interest.

Definition 1.4.1. A non negative function $f(x)$ is subadditive (superadditive)if for all x, y,

$$f(x + y) \leq (\geq) f(x) + f(y)$$

Remark If f is twice differentiable, then
$i)$ f is superadditive if and only if $\frac{\partial^2 f}{\partial x \partial y} \geq 0$.
$ii)$ f is subadditive if and only if $\frac{\partial^2 f}{\partial x \partial y} \leq 0$.

Definition 1.4.2. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be supermodular if for any $x, y \in \mathbb{R}^n$ it satisfies

$$\phi(x) + \phi(y) \leq \phi(x \wedge y) + \phi(x \vee y),$$

where the operators $\wedge$ and $\vee$ denote coordinatewise minimum and maximum, respectively.

Note that if $\phi : \mathbb{R}^n \to \mathbb{R}$ is supermodular, then the function ψ, defined by $\psi(x_1, x_2, \ldots, x_n) = \phi(g_1(x_1), g_2(x_2), \ldots, g_n(x_n))$, is also supermodular, whenever $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, n$, are all increasing or are all decreasing.

Definition 1.4.3. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors such that $E[\phi(\mathbf{X})] \leq E[\phi(\mathbf{Y})]$ for all supermodular functions $\phi : \mathbb{R}^n \to \mathbb{R}$, provided the expectations exist. Then $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the supermodular order (denoted by $\mathbf{X} \leq_{sm} \mathbf{Y}$).

Since the functions $\phi_x = I_{\{y:y>x\}}$ and $\psi_x = I\{y : y \leq x\}$ are supermodular for each fixed x, it is immediate that

$$\mathbf{X} \leq_{sm} \mathbf{Y} \Rightarrow \mathbf{X} \leq_{PQD} \mathbf{Y}. \tag{1.1}$$

When n = 2 we have that

$$(X_1, X_2) \leq_{sm} (Y_1, Y_2) \Leftrightarrow (X_1, X_2) \leq_{PQD} (Y_1, Y_2) \tag{1.2}$$

¿From (2.1) it is seen that if $\mathbf{X} \leq_{sm} \mathbf{Y}$, then $\mathbf{X}$ and $\mathbf{Y}$ must have the same univariate marginals.
Some closure properties of the supermodular order are described in the next theorem.

Theorem 1.4.4. *(a) Let $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_n)$ be two n-dimensional random vectors. If $(X_1, X_2, \ldots, X_n) \leq_{sm} (Y_1, Y_2, \ldots, Y_n)$, then*

$$(g_1(X_1), g_2(X_2), \ldots, g_n(X_n)) \leq_{sm} (g_1(Y_1), g_2(Y_2), \ldots, g_n(Y_n))$$

whenever $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, n$, are all increasing or are all decreasing.
(b) Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m$ be a set of independent random vectors where the dimension of $\mathbf{X}_i$ is k_i, $i = 1, 2, \ldots, m$. Let $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_m$ be another set of independent random vectors where the dimension of $\mathbf{Y}_i$ is k_i, $i = 1, 2, \ldots, m$. If $\mathbf{X}_i \leq_{sm} \mathbf{Y}_i$ for $i = 1, 2, \ldots, m$, then

$$(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m) \leq_{sm} (\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_m).$$

That is, the supermodular order is closed under conjunctions.
(c) Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ be two n-dimensional random vectors. If $\mathbf{X} \leq_{sm} \mathbf{Y}$, then $\mathbf{X}_I \leq_{sm} \mathbf{Y}_I$ for each $I \subseteq \{1, 2, \ldots, n\}$.
That is, the supermodular order is closed under marginalization.
(d) Let $\mathbf{X}$, $\mathbf{Y}$, and Θ be random vectors such that $[X|\Theta = \theta] \leq_{sm} [Y|\Theta = \theta]$ for all θ in the support of Θ. Then $\mathbf{X} \leq_{sm} \mathbf{Y}$. That is, the supermodular order is closed under mixtures.
(e) Let $\{\mathbf{X}_j, j = 1, 2, \ldots\}$ and $\{\mathbf{Y}_j, j = 1, 2, \ldots\}$ be two sequences of random vectors such that $\mathbf{X}_j \to_{st} \mathbf{X}$ and $\mathbf{Y}_j \to_{st} \mathbf{Y}$ as $j \to \infty$, where $\to_{st}$ denotes convergence in distribution. If $\mathbf{X}_j \leq_{sm} \mathbf{Y}_j$, $j = 1, 2, \ldots,$ then $\mathbf{X} \leq_{sm} \mathbf{Y}$
. Proof. Part (a) follows from the fact that a composition of a supermodular function with coordinatewise functions, that are all increasing or are all decreasing, is a supermodular function. In order to see part (b) let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two independent random vectors, and let $\mathbf{Y}_1$ and $\mathbf{Y}_2$ be two other independent random vectors. Suppose that $\mathbf{X}_1 \leq_{sm} \mathbf{Y}_1$ and that

$\mathbf{X}_2 \leq_{sm} \mathbf{Y}_2$. *Then, for any supermodular function* ϕ *(of the proper dimension) we have that*

$$\begin{aligned} E\phi(\mathbf{X}_1, \mathbf{X}_2) &= E[E\phi(\mathbf{X}_1, \mathbf{X}_2)|\mathbf{X}_2] \\ &\leq E[E\phi(\mathbf{Y}_1, \mathbf{X}_2)|\mathbf{X}_2] \\ &= E\phi(\mathbf{Y}_1, \mathbf{X}_2) \\ &\leq E\phi(\mathbf{Y}_1, \mathbf{Y}_2), \end{aligned}$$

where the first inequality follows from the fact that $\phi(\mathbf{x}_1, \mathbf{x}_2)$ *is supermodular in* $\mathbf{x}_1$ *when* $\mathbf{x}_2$ *is fixed, and the second inequality follows in a similar manner. Part (b) of Theorem 7 follows from the above by induction.*
Parts (c) and (d) are easy to prove.□

¿From parts (a) and (d) of Theorem 7 we obtain the following corollary.

Corollary 1.4.5. *If* $\mathbf{X} \leq_{sm} \mathbf{Y}$, *then for each* $i \neq j$, $(X_i, X_j) \leq_{sm} (Y_i, Y_j)$ *if and only if* $(X_i, X_j) \leq_{PQD} (Y_i, Y_j)$.

Corollary 1.4.6. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ *be two random vectors such that* $\mathbf{X} \leq_{sm} \mathbf{Y}$ *, and let* $\mathbf{Z}$ *be an m-dimensional random vector which is independent of* $\mathbf{X}$ *and* $\mathbf{Y}$ *. Then*

$$(h_1(X_1, \mathbf{Z}), h_2(X_2, \mathbf{Z}), \ldots, h_n(X_n, \mathbf{Z})) \leq_{sm} (h_1(Y_1, \mathbf{Z}), h_2(Y_2, \mathbf{Z}), \ldots, h_n(Y_n, \mathbf{Z})),$$

whenever $h_i(x, z)$, $i = 1, 2, \ldots, n$, *are all increasing or are all decreasing in* x *for every* z.

Example 1.4.7. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors such that $\mathbf{X} \leq_{sm} \mathbf{Y}$, and let $\mathbf{Z}$ be an n-dimensional random vector which is independent of $\mathbf{X}$ and $\mathbf{Y}$. Then from Corollary 3 it follows that

$$\mathbf{X} \wedge \mathbf{Z} \leq_{sm} \mathbf{Y} \wedge \mathbf{Z},$$

and that

$$\mathbf{X} + \mathbf{Z} \leq_{sm} \mathbf{Y} + \mathbf{Z}.$$

By applying Corollary 3 twice (letting $\mathbf{Z}$ there be an n-dimensional random vector, and letting each h_i depend only on its first argument and on the ith component of the second argument, $i = 1, 2, \ldots, n$), we get the following result.

Theorem 1.4.8. *Let* $\mathbf{X}_j = (X_{j,1}, X_{j,2}, \ldots, X_{j,m})$, $j = 1, 2, \ldots,$ *be a sequence of nonnegative random vectors, and let* $\mathbf{M} = (M_1, M_2, \ldots, M_m)$ *and* $\mathbf{N} = (N_1, N_2, \ldots, N_m)$ *be two vectors of nonnegative integer-valued random variables. Assume that both* $\mathbf{M}$ *and* $\mathbf{N}$ *are independent of the* $\mathbf{X}_j$*,s. If* $\mathbf{M} \leq_{sm} \mathbf{N}$, *then*

$$\left(\sum_{j=1}^{M_1} X_{j,1}, \sum_{j=1}^{M_2} X_{j,2}, \ldots, \sum_{j=1}^{M_m} X_{j,m} \right) \leq_{sm} \left(\sum_{j=1}^{N_1} Xj, 1, \sum_{j=1}^{N_2} X_{j,2}, \ldots, \sum_{j=1}^{N_m} X_{j,m} \right).$$

Proof. Let ϕ be a supermodular function. Conditioning on the possible realizations of $(\mathbf{X}_1, \mathbf{X}_2, \dots)$ we can write

$$E\left[\phi\left(\sum_{j=1}^{M_1} X_{j,1}, \sum_{j=1}^{M_2} X_{j,2}, \dots, \sum_{j=1}^{M_m} X_{j,m}\right)\right]$$
$$= E\left\{E\left[\phi\left(\sum_{j=1}^{M_1} X_{j,1}, \sum_{j=1}^{M_2} X_{j,2}, \dots, \sum_{j=1}^{M_m} X_{j,m}\right) \mid (\mathbf{X}_1, \mathbf{X}_2, \dots)\right]\right\}.$$

Now, it is easy to see that for any realization $(\mathbf{x}_1, \mathbf{x}_2, \dots)$ of $(\mathbf{X}_1, \mathbf{X}_2, \dots)$, the function ψ, defined by $\psi(n_1, n_2, \dots, n_m) = \phi(\sum_{j=1}^{n_1} x_{j,1}, \sum_{j=1}^{n_2} x_{j,2}, \dots, \sum_{j=1}^{n_m} x_{j,m})$, is supermodular. Therefore, since $\mathbf{M} \leq_{sm} \mathbf{N}$, we have that

$$E\left[\phi\left(\sum_{j=1}^{M_1} X_{j,1}, \sum_{j=1}^{M_2} X_{j,2}, \dots, \sum_{j=1}^{M_m} X_{j,m}\right) \mid (\mathbf{X}_1, \mathbf{X}_2, \dots) = (\mathbf{x}_1, \mathbf{x}_2, \dots)\right]$$
$$\leq E\left[\phi\left(\sum_{j=1}^{N_1} X_{j,1}, \sum_{j=1}^{N_2} X_{j,2}, \dots, \sum_{j=1}^{N_m} X_{j,m}\right) \mid (\mathbf{X}_1, \mathbf{X}_2, \dots) = (\mathbf{x}_1, \mathbf{x}_2, \dots)\right],$$

and thus

$$E\left[\phi\left(\sum_{j=1}^{M_1} X_{j,1}, \sum_{j=1}^{M_2} X_{j,2}, \dots, \sum_{j=1}^{M_m} X_{j,m}\right)\right]$$
$$\leq E\left\{E\left[\phi\left(\sum_{j=1}^{N_1} X_{j,1}, \sum_{j=1}^{N_2} X_{j,2}, \dots, \sum_{j=1}^{N_m} X_{j,m}\right) \mid (\mathbf{X}_1, \mathbf{X}_2, \dots)\right]\right\}$$
$$= E\left[\phi\left(\sum_{j=1}^{N_1} X_{j,1}, \sum_{j=1}^{N_2} X_{j,2}, \dots, \sum_{j=1}^{N_m} X_{j,m}\right)\right]. \square$$

Theorem 1.4.9. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors. If* $\mathbf{X} \leq_{sm} \mathbf{Y}$ *, then* $\phi(\mathbf{X}) \leq_{icx} \phi(\mathbf{Y})$ *for any increasing supermodular function* $\phi : \mathbb{R}_n \to \mathbb{R}$.

Theorem 1.4.10. *Let* (X_1, X_2) *and* (Y_1, Y_2) *be two random vectors. If* $(X_1, X_2) \leq_{sm} (Y_1, Y_2)$ *(that is,* $(X_1, X_2) \leq_{PQD} (Y_1, Y_2)$*;), then*

$$Y_1 - Y_2 \leq_{cx} X_1 - X_2.$$

Proof. Let ϕ *be a univariate convex function. Then the function* ψ*, defined by*

$$\psi(x_1, x_2) = -\phi(x_1 - x_2),$$

is easily seen to be supermodular ($\frac{\partial^2 \psi}{\partial x_1 \partial x_2} \geq 0$)*. Thus* $E\phi(Y_1 - Y_2) \leq E\phi(X_1 - X_2)$*. This proves the inequality.* $\square$

Example 1.4.11. Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be two sequences of random variables. Let N_1 and N_2 be two independent and identically distributed positive integer-valued random variables independent of the X_i's and of the Y_i's. Then

$$\sum_{i=1}^{N_1} X_i + \sum_{i=1}^{N_2} Y_i \leq_{cx} \sum_{i=1}^{N_1} (X_i + Y_i).$$

Solution.(i) (Theorem 11)$\Rightarrow$ $(N_1, N_2) \leq_{sm} (N_2, N_2)$ and $(N_1, N_2) \leq_{sm} (N_1, N_1)$
(ii)(Theorem 8) $\Rightarrow \big(\sum_{i=1}^{N_1} X_i, \sum_{i=1}^{N_2} Y_i\big) \leq_{sm} \big(\sum_{i=1}^{N_1} X_i, \sum_{i=1}^{N_1} Y_i\big)$
(iii)(Theorem 9)$\Rightarrow \sum_{i=1}^{N_1} X_i + \sum_{i=1}^{N_1} Y_i \leq_{cx} \sum_{i=1}^{N_1} (X_i + Y_i).$

Example 1.4.12. Let $\mathbf{X}$ be a multivariate normal random vector with mean vector $\mathbf{0}$ and variance-covariance matrix $\mathbf{\Sigma}$, and let $\mathbf{Y}$ be a multivariate normal random vector with mean vector $\mathbf{0}$ and variance-covariance matrix $\mathbf{\Sigma} + \mathbf{D}$, where $\mathbf{D}$ is a matrix with zero diagonal elements such that $\mathbf{\Sigma} + \mathbf{D}$ is nonnegative definite. Then $\mathbf{X} \leq_{sm} \mathbf{Y}$ if, and only if, all the entries of $\mathbf{D}$ are nonnegative.

Theorem 1.4.13. *Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random vector and let F_{X_i} be the marginal distribution of $X_i n$. Then, for a uniform $[0,1]$ random variable U we have that*

$$\mathbf{X} \leq_{sm} \big(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\big),$$

and therefore

$$\mathbf{X} \leq_{PQD} \big(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\big).$$

Proof. Suppose $n = 2$,
$F_U(x, y) = \min\{F_1(x), F_2(y)\}$
$\big(F_1^{-1}(u), F_2^{-1}(u)\big)$

$$\begin{aligned} G(x,y) &= P\left\{F_1^{-1}(U) \leq x, F_2^{-1}(U) \leq y\right\} = P\left\{U \leq F_1(x), U \leq F_2(y)\right\} \\ &= P\left\{U \leq \min\{F_1(x), F_2(y)\}\right\} = P\{U \leq F_U\} = F_U \end{aligned}$$

$$\Rightarrow (X_1, X_2) \leq_{PQD} \big(F_1^{-1}(u), F_2^{-1}(u)\big) \Leftrightarrow (X_1, X_2) \leq_{sm} \big(F_1^{-1}(u), F_2^{-1}(u)\big) . \square$$

Corollary 1.4.14. *particular, if the X_i,s in Theorem 11, marginally, have the same (univariate) distribution function, then*

$$\mathbf{X} \leq_{sm} (X_1, X_1, \ldots, X_1),$$

and therefore

$$\mathbf{X} \leq_{PQD} (X_1, X_1, \ldots, X_1).$$

Corollary 1.4.15. *Using the notation of Theorem 11, that*

$$X_1 + X_2 + \cdots + X_n \leq_{cx} F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \cdots + F_{X_n}^{-1}(U).$$

Theorem 1.4.16. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ *be a random vector, and let* $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ *be a vector of independent random variables such that, marginally,* $X_i =_{st} Y_i$*,* $i = 1, 2, \ldots, n$*.*
(a) If $X_1, X_2, \ldots, X_n$ *are weakly positively associated, then* $\mathbf{X} \geq_{sm} \mathbf{Y}$*.*
(b)If $X_1, X_2, \ldots, X_n$ *are negatively associated, then* $\mathbf{X} \leq_{sm} \mathbf{Y}$ *.*

1.5 Positive orthant orders

In this section we study some stochastic orders of positive dependence that arise when the underlying random vectors are ordered with respect to some multivariate hazard rate stochastic orders, and have the same univariate marginal distributions. A pair of such orders is studied in Section 2. After giving the definitions and some basic properties, we show how the orders can be studied by restricting them to copulae. We then give a number of examples, and study ordering of parametric families by using in orders. A pair of stronger positive dependence orders is introduced and studied in Section 3.

1.6 The weak orthant ratio orders

Let X = $(X_1, ..., X_n)$ and Y = $(Y_1, ..., Y_n)$ be two random vectors with respective distribution functions F and G, and with survival functions $\bar{F}$ and $\bar{G}$. We suppose that F and G belong to the same Fr'echet class; that is, have the same univariate marginals.

Definition 1. We say that X is smaller than Y in the lower orthant decreasing ratio order (denoted by $X \leq_{lodr} Y$ or $F \leq_{lodr} G$) if

$$F(y)G(x) \geq F(x)G(y), \quad x \leq y. \frac{G(x)}{F(x)} \quad \searrow \; in \; x \in \{x : G(x) > 0\}, \tag{1.3}$$

where in (2.2) we use the convention $a/0 \equiv \infty$ whenever $a > 0$. Note that (2.2) can be written equivalently as

$$\frac{F(x-u)}{F(x)} \leq \frac{G(x-u)}{G(x)}, \qquad u \geq 0, \; x \in \{x : F(x) > 0\} \cap \{x : G(x) > 0\}, \tag{1.4}$$

and it is also equivalent to

$$[X - x | X \leq x] \geq_{lo} [Y - x | Y \leq x], \qquad x \in \{x : F(x) > 0\} \cap \{x : G(x) > 0\}. \tag{1.5}$$

Note that if in (2.1) $y \to \infty$ then it follows that $\{x : F(x) > 0\} \subseteq \{x : G(x) > 0\}$. Thus, in (2.3) and (2.4) we can formally replace the expression $\{x : F(x) > 0\} \cap \{x : G(x) > 0\}$ by the simpler expression $\{x : F(x) > 0\}$.

Definition 2. We say that X is smaller than Y in the upper orthant increasing ratio order (denoted by $X \leq_{uoir} Y$ or $F \leq_{uoir} G$) if

$$\bar{F}(y)\bar{G}(x) \leq \bar{F}(x)\bar{G}(y), \quad x \leq y. \tag{1.6}$$

This is equivalent to

$$\frac{\bar{G}(x)}{\bar{F}(x)} \quad \nearrow \ in\ x \in \{x : \bar{G}(x) > 0\}, \tag{1.7}$$

where here, again, we use the convention $a/0 \equiv \infty$ whenever $a > 0$. Note that the above can be written equivalently as

$$\frac{\bar{F}(x-u)}{\bar{F}(x)} \leq \frac{\bar{G}(x-u)}{\bar{G}(x)}, \quad u \geq 0,\ x \in \{x : \bar{F}(x) > 0\} \cap \{x : \bar{G}(x) > 0\}, \tag{1.8}$$

and it is also equivalent to

$$[X - x|X > x] \leq_{uo} [Y - x|Y > x], \quad x \in \{x : \bar{F}(x) > 0\} \cap \{x : \bar{G}(x) > 0\}. \tag{1.9}$$

Formally the expression $\{x : \bar{F}(x) > 0\} \cap \{x : \bar{G}(x) > 0\}$ in (2.7) and (2.8) can be replaced by the simpler expression $\{x : \bar{F}(x) > 0\}$.

Definition 3. We say that X is smaller than Y in the multivariate hazard rate order (denoted by $X \leq_{hr} Y$) if

$$\bar{F}(x)\bar{G}(y) \leq \bar{F}(x \wedge y)\bar{G}(x \vee y), \quad \forall\, x, y \in R^n. \tag{1.10}$$

and we say that X is smaller than Y in the weak multivariate hazard rate order (denoted by $X \leq_{whr} Y$) if

$$\frac{\bar{G}(x)}{\bar{F}(x)} \quad \nearrow \ in\ x \in \{x : \bar{G}(x) > 0\}, \tag{1.11}$$

where in (2.10) we use the convention $a/0 \equiv \infty$ whenever $a > 0$.

Note that (2.10) can be written equivalently as

$$\bar{F}(y)\bar{G}(x) \leq \bar{F}(x)\bar{G}(y), \quad x \leq y. \tag{1.12}$$

Thus, from (2.9) and (2.11) it follows that

$$X \leq_{hr} Y \Rightarrow X \leq_{whr} Y. \tag{1.13}$$

We note that if X and Y have the same marginals, then $X \leq_{uoir} Y$ if, and only if, $X \leq_{whr} Y$.

Remark 1: Suppose that the $X \leq_{lodr} Y$, if in (2.4) $y \to \infty$ then it follows that for all x, $F(x) \leq G(x)$. Similarly, if $X \leq_{uoir} Y$, then from (2.5) follows that for all x, $\bar{F}(x) \leq \bar{G}(x)$. Thus we have that

$$(X \leq_{lodr} Y \text{ and } X \leq_{uoir} Y) \Rightarrow X \leq_{POD} Y. \tag{1.14}$$

The two orders $\leq_{lodr}$ and $\leq_{uoir}$ are closely related, as is indicated in the next result.

Theorem 1 Let X = $(X_1, ..., X_n)$ and Y = $(Y_1, ..., Y_n)$ be two random vectors in the same Fr'echet class.

1. If $X \leq_{lodr} Y$, then $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{uoir} (\phi_1(Y_1), \phi_2(Y_2),$ $..., \phi_n(Y_n))$ for any decreasing functions $\phi_1, \phi_2, ..., \phi_n$. Conversely, if $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{u}$ $(\phi_1(Y_1), \phi_2(Y_2), ..., \phi_n(Y_n))$ for some strictly decreasing functions $\phi_1, \phi_2, ..., \phi_n$, then $X \leq_{lodr} Y$.

2. If $X \leq_{uoir} Y$, then $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{lodr} (\phi_1(Y_1), \phi_2(Y_2),$ $..., \phi_n(Y_n))$ for any decreasing functions $\phi_1, \phi_2, ..., \phi_n$. Conversely, if $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{lo}$ $(\phi_1(Y_1), \phi_2(Y_2), ..., \phi_n(Y_n))$ for some strictly decreasing functions $\phi_1, \phi_2, ..., \phi_n$, then $X \leq_{uoir} Y$.

The next result is similar to Theorem 1, but it involves increasing, rather than decreasing, functions. It shows that the orders $\leq_{lodr}$ and $\leq_{uoir}$ are closed under componentwise increasing transformations.

Theorem 2 Let X = $(X_1, ..., X_n)$ and Y = $(Y_1, ..., Y_n)$ be two random vectors in the same Fr'echet class.

1. If $X \leq_{lodr} Y$, then $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{lodr} (\phi_1(Y_1), \phi_2(Y_2),$ $..., \phi_n(Y_n))$ for any increasing functions $\phi_1, \phi_2, ..., \phi_n$. Conversely, if $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{lo}$ $(\phi_1(Y_1), \phi_2(Y_2), ..., \phi_n(Y_n))$ for some strictly increasing functions $\phi_1, \phi_2, ..., \phi_n$, then $X \leq_{lodr} Y$.

2. If $X \leq_{uoir} Y$, then $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{uoir} (\phi_1(Y_1), \phi_2(Y_2),$ $..., \phi_n(Y_n))$ for any increasing functions $\phi_1, \phi_2, ..., \phi_n$. Conversely, if $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{uo}$ $(\phi_1(Y_1), \phi_2(Y_2), ..., \phi_n(Y_n))$ for some strictly increasing functions $\phi_1, \phi_2, ..., \phi_n$, then $X \leq_{uoir} Y$.

Remark 2: Order $\leq_{uoir}$ is

1. a preorder(reflexive and transitive).

2. antisymmetric.

3. imply the POD order.

4. closed under permutation of the component.

5. closed under marginalization.
6. closed under convergence in distribution.
7. closed under componentwise strictly increasing transformation.
8. not closed under convolutions.
9. not maximal at the upper Fr'echet bound.
10. minimal at the lower Fr'echet bound in the bivariate case.

Using Theorem 1 it is seen that also the order $\leq_{lodr}$ is closed under these operations.

Postulates (1) and (2) is trivial, Postulate (3) is remark 1, Postulates (4), (5) and (6) follow from results of Shaked[2], Postulate (7) follow from theorem 2 and Postulates (8), (9) and (10) is shown in the next examples.

Example 1. Show that the order $\leq_{uoir}$ is not closed under convolutions.

Solution: Let (X_1, X_2), (Y_1, Y_2), and (Z_1, Z_2) be random vectors with probability mass functions

$x_1 \backslash x_2$	0	1
0	.1	.1
1	.1	.7

$y_1 \backslash y_2$	0	1
0	.2	0
1	0	.8

$z_1 \backslash z_2$	0	1
0	.3	0
1	0	.7

Suppose that (Z_1, Z_2) is independent of (X_1, X_2) and of (Y_1, Y_2). It is easy to see that $(X_1, X_2) \leq_{uoir} (Y_1, Y_2)$, and, obviously, $(Z_1, Z_2) \leq_{uoir} (Z_1, Z_2)$. The probability mass functions of $(U_1, U_2) = (X_1, X_2) + (Z_1, Z_2)$ and $(V_1, V_2) = (Y_1, Y_2) + (Z_1, Z_2)$ are

u_1 \ u_2	0	1	2
0	.03	.03	0
1	.03	.28	.07
2	0	.07	.49

v_1 \ v_2	0	1	2
0	.06	0	0
1	0	.38	0
2	0	0	.56

and it is easily seen that $(U_1, U_2) \not\leq_{uoir} (V_1, V_2)$. ¿From Theorem 1, with $\phi_1(x) = \phi_2(x) = -x$, it also follows that the order $\leq_{lodr}$ is not closed under convolutions. $\triangle$

Example 1.6.1. Suppose that (X_1, X_2) has distribution function F, such that $F \in M(F_1, F_2)$, then

$$F^- \leq_{lodr} F \text{ and } F^- \leq_{uoir} F,$$

where F^- is lower bound in Fr'echet class.

Solution. We will give the solution only for the order $\leq_{uoir}$; the solution for the order $\leq_{lodr}$ is similar. Fix $(x_1, x_2) \leq (y_1, y_2)$. We want to show that

$$\begin{aligned} &max\{\bar{F}_1(y_1) + \bar{F}_2(y_2) - 1, 0\}\bar{F}(x_1, x_2) \\ &\leq max\{\bar{F}_1(x_1) + \bar{F}_2(x_2) - 1, 0\}\bar{F}(y_1, y_2). \end{aligned}$$

If $\bar{F}_1(y_1) + \bar{F}_2(y_2) - 1 \leq 0$ then relation is trivially true. If $\bar{F}_1(y_1) + \bar{F}_2(y_2) - 1 > 0$ then we have

$$[1 - \bar{F}_1(y_1) - \bar{F}_2(y_2)]\bar{F}(x_1, x_2) \geq [1 - \bar{F}_1(x_1) - \bar{F}_2(x_2)]\bar{F}(y_1, y_2). \tag{1.15}$$

Note that $1 - \bar{F}_1(z_1) - \bar{F}_2(z_2) = F(z_1, z_2) - \bar{F}(z_1, z_2)$ for any (z_1, z_2). Plugging this in (2.14) and simplifying, it is seen that (2.14) is equivalent to

$$F(y_1, y_2)\bar{F}(x_1, x_2) \geq F(x_1, x_2)\bar{F}(y_1, y_2),$$

which is trivially true. $\triangle$

In the next example it is shown that if $F \in M(F_1, F_2)$ then it does not necessarily follow that $F \leq_{lodr} F^+$ or that $F \leq_{uoir} F^+$, where F^+ is the upper Fr'echet bound.

Example 3. Let (X_1, X_2) and (Y_1, Y_2) have distribution functions F and G. If (X_1, X_2) and (Y_1, Y_2) are bounded from below, and if for some $(x_1, x_2) \leq (y_1, y_2)$ we have $\bar{F}(y_1, y_2) = \bar{G}(y_1, y_2) > 0$ and $\bar{F}(x_1, x_2) \neq \bar{G}(x_1, x_2)$, then (X_1, X_2) and (Y_1, Y_2) are not comparable with respect to the order $\leq_{uoir}$. This follows from (2.6). Similarly, if (X_1, X_2) and (Y_1, Y_2) are bounded from above, and if for some $(x_1, x_2) \geq (y_1, y_2)$ we have $F(y_1, y_2) = G(y_1, y_2) > 0$ and $F(x_1, x_2) \neq G(x_1, x_2)$, then (X_1, X_2) and (Y_1, Y_2) are not comparable with respect to the order $\leq_{lodr}$. This follows from (2.2). In particular, let (X_1, X_2), with distribution function F, take on the values (1, 1), (1, 2), (2, 1), (3, 3) with probabilities 1/5, 1/5, 1/5, and 2/5. Then $\bar{F^+}(1,1) \neq \bar{F}(1,1)$ and $\bar{F^+}(2,2) = \bar{F}(2,2)$, so F^+ and F are not comparable with respect to the order $\leq_{uoir}$. Taking here G to be the distribution function of $(-X_1, -X_2)$, it can be verified that G^+ and G are not comparable with respect to the order $\leq_{lodr}$. $\triangle$

Theorem 3 Let X and Y have, respectively, distribution functions F and G in $\Gamma_n(F_1, F_2, ..., F_n)$. Then $X \leq_{lodr} Y[X \leq_{uoir} Y]$ if, and only if, there exist copulae C_F and C_G such that $C_F \leq_{lodr} C_G[C_F \leq_{uoir} C_G]$.

Proof. We give only the proof for the order $\leq_{lodr}$; the proof for the other order is similar. Suppose that $X \leq_{lodr} Y$, If F is continuous then C_F is unique and it can be constructed as follows:

$$C_F(u_1, u_2, ..., u_n) \equiv F(F_1^{-1}(u_1), F_2^{-1}(u_2), ..., F_n^{-1}(u_n)),$$

$$u_i \in [0,1], i = 1, 2, ..., n,$$

For $i = 1, 2, ..., n$ the function F_i^{-1} is strictly increasing function. Therefore, by Theorem 2, $C_F \leq_{lodr} C_G$.

Conversely, suppose that there exist copulae C_F and C_G such that $C_F \leq_{lodr} C_G$. Then from (2.2) we have that

$$\frac{C_G(u)}{C_F(u)} \quad \searrow \ in \ u \in \{u : C_G(u) > 0\}. \tag{1.16}$$

Substitute $u_i = F_i(x_i), i = 1, 2, ..., n$, in (2.15) to obtain (2.2). $\square$

Theorem Let U and V be two random vectors whose distribution functions are copulae. Then

$$U \leq_{lodr} V \Leftrightarrow 1 - U \leq_{uoir} 1 - V,$$

$$U \leq_{uoir} V \Leftrightarrow 1 - U \leq_{lodr} 1 - V$$

Proof. By Theorem 1 and using from $\phi_i(x_i) = 1 - x_i, i = 1, 2, ..., n$, it is straightforward. $\square$

We close this section with two examples of distributions that are ordered with respect to the orders $\leq_{lodr}$ and $\leq_{uoir}$ (For more examples see Colangelo et al.[1]).

Example 4. Let $F^{\perp}$ and F^+ denote, respectively, the distribution functions corresponding to the independence case, and to the upper Fr'echet bound, in the class $M(F_1, F_2, ..., F_n)$; that is,

$F^{\perp}(x_1, x_2, ..., x_n) = \Pi_{i=1}^{n} F_i(x_i)$ and $F^{+}(x_1, x_2, ..., x_n) = min_{1\le i\le n}\{F_1(x_1), F_2(x_2), ..., F_n(x_n)\}$. Then

$$F^{\perp} \le_{lodr} F^{+} \quad and \quad F^{\perp} \le_{uoir} F^{+}.$$

In order to see it we may assume, by Theorem 3, that $F_i, i = 1, 2, ..., n$, are all uniform[0, 1] distribution functions. First we show that $F^{\perp} \le_{lodr} F^{+}$. That is, we need to verify, for $x \le y \in [0, 1]^n$, that

$$\left[\prod_{i=1}^{n} y_i\right] \min_{1\le i\le n}\{x_1, x_2, ..., x_n\} \ge \left[\prod_{i=1}^{n} x_i\right] \min_{1\le i\le n}\{y_1, y_2, ..., y_n\}$$

and this is straightforward. The inequality above is reversed if we replace x_i by $1 - x_i$ and y_i by $1 - y_i$, but still require $x \le y \in [0, 1]^n$,. This shows that $F^{\perp} \le_{uoir} F^{+}$. △

Remark 3. Let $\{F_\alpha\}$ be a parametric family of n-dimensional distributions, all in the same Fr'echet class, where the parameter space is a subset of R. Then $F_\alpha \le_{lodr} F_\beta$ for all $\alpha \le \beta$ if, and only if,

$$\frac{F_\beta(x_1, x_2, ..., x_n)}{F_\alpha(x_1, x_2, ..., x_n)} \quad \searrow \ in\ x_1, x_2, ..., x_n\ when\ \alpha \le \beta,$$

that is (if the partial derivatives below exists), if, and only if,

$$\frac{\partial}{\partial x_i} log F_\alpha(x_1, x_2, ..., x_n) \quad \searrow \ in\ \alpha\ for\ i = 1, 2, ..., n. \tag{1.17}$$

Similarly, $F_\alpha \le_{uoir} F_\beta$ if, and only if,

$$\frac{\partial}{\partial x_i} log \bar{F}_\alpha(x_1, x_2, ..., x_n) \quad \nearrow \ in\ \alpha\ for\ i = 1, 2, ..., n \tag{1.18}$$

(this really means that $\bar{F}_\alpha(x_1, x_2, ..., x_n)$ is TP_2 (totally positive of order 2) in (α, x_i) for $i = 1, 2, ..., n$).

Example 5.$(Ali.Mikhail.Haq)$. Consider the family of bivariate copulae C_α defined by

$$C_\alpha(u, v) = \frac{uv}{1 - \alpha(1 - u)(1 - v)}, (u, v) \in (0, 1)^2,$$

where $|\alpha| \le 1$. Denote the corresponding survival copulae by D_α. Their survival functions are given by

$$D_\alpha(u, v) = C_\alpha(1 - u, 1 - v) = \frac{(1 - u)(1 - v)}{1 - \alpha uv}, (u, v) \in (0, 1)^2.$$

We will show that $C_\alpha \le_{lodr} C_\beta$ whenever $-1 \le \alpha \le \beta \le 1$ In order to see it we compute

$$\frac{\partial}{\partial u} log C_\alpha(u, v) = \frac{1 - \alpha(1 - v)}{u(1 - \alpha(1 - u)(1 - v))}$$

and this is decreasing in $\alpha \in [0, 1]$. Similarly, $\frac{\partial}{\partial v} log C_\alpha(u, v)$ is decreasing in $\alpha \in [0, 1]$. The claim thus follows from (2.16). The inequality $C_\alpha \le_{lodr} C_\beta, -1 \le \alpha \le \beta \le 1$. Using Theorem 4 we also see that $D_\alpha \le_{uoir} D_\beta$ whenever $-1 \le \alpha \le \beta \le 1$. △

1.7 The strong orthant ratio orders

Let X = $(X_1, ..., X_n)$ and Y = $(Y_1, ..., Y_n)$ be two random vectors with respective distribution functions F and G, and with survival functions $\bar{F}$ and $\bar{G}$. We suppose that F and G belong to the same Fr'echet class; that is, have the same univariate marginals.

Definition 4. We say that X is smaller than Y in the strong lower orthant decreasing ratio order (denoted by $X \leq_{slodr} Y$ or $F \leq_{slodr} G$) if

$$F(x)G(y) \leq F(x \vee y)G(y \wedge x), \quad x, y \in R^n. \tag{1.19}$$

Definition 5. We say that X is smaller than Y in the strong upper orthant increasing ratio order (denoted by $X \leq_{suoir} Y$ or $F \leq_{suoir} G$) if

$$\bar{F}(x)\bar{G}(y) \leq \bar{F}(x \wedge y)\bar{G}(y \vee x), \quad x, y \in R^n. \tag{1.20}$$

We note that if X and Y have the same marginals, then $X \leq_{suoir} Y$ if, and only if, $X \leq_{hr} Y$.

Remark 4: By choosing $x \leq y$ in (3.1) we get (2.1), and by choosing $x \geq y$ in (3.2) we get (2.5), that is,

$$X \leq_{slodr} Y \Rightarrow X \leq_{lodr} Y \quad and \quad X \leq_{suoir} Y \Rightarrow X \leq_{uoir} Y. \tag{1.21}$$

Thus the orders $\leq_{slodr}$ and $\leq_{suoir}$ are often useful as a tool to identify random vectors that are ordered with respect to the orders $\leq_{lodr}$ and $\leq_{uoir}$.

The two orders $\leq_{slodr}$ and $\leq_{suoir}$ are closely related, and are preserved under componentwise increasing transformations, as is indicated in the next analog of Theorems 1 and 2.

Theorem 1 Let X = $(X_1, ..., X_n)$ and Y = $(Y_1, ..., Y_n)$ be two random vectors in the same Fr'echet class.

1. If $X \leq_{slodr} Y$, then $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{suoir} (\phi_1(Y_1), \phi_2(Y_2), ..., \phi_n(Y_n))$ for any decreasing functions $\phi_1, \phi_2, ..., \phi_n$. Conversely, if $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{s}$ $(\phi_1(Y_1), \phi_2(Y_2), ..., \phi_n(Y_n))$ for some strictly decreasing functions $\phi_1, \phi_2, ..., \phi_n$, then $X \leq_{slodr} Y$.

2. If $X \leq_{suoir} Y$, then $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{slodr} (\phi_1(Y_1), \phi_2(Y_2), ..., \phi_n(Y_n))$ for any decreasing functions $\phi_1, \phi_2, ..., \phi_n$. Conversely, if $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{sl}$ $(\phi_1(Y_1), \phi_2(Y_2), ..., \phi_n(Y_n))$ for some strictly decreasing functions $\phi_1, \phi_2, ..., \phi_n$, then $X \leq_{suoir} Y$.

3. If $X \leq_{slodr} Y$, then $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{slodr} (\phi_1(Y_1), \phi_2(Y_2), ..., \phi_n(Y_n))$ for any increasing functions $\phi_1, \phi_2, ..., \phi_n$. Conversely, if $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{sl}$

$(\phi_1(Y_1), \phi_2(Y_2), ..., \phi_n(Y_n))$ for some strictly increasing functions $\phi_1, \phi_2, ..., \phi_n$, then $X \leq_{slodr} Y$.

4. If $X \leq_{suoir} Y$, then $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{suoir} (\phi_1(Y_1), \phi_2(Y_2), ..., \phi_n(Y_n))$ for any increasing functions $\phi_1, \phi_2, ..., \phi_n$. Conversely, if $(\phi_1(X_1), \phi_2(X_2), ..., \phi_n(X_n)) \leq_{su}$ $(\phi_1(Y_1), \phi_2(Y_2), ..., \phi_n(Y_n))$ for some strictly increasing functions $\phi_1, \phi_2, ..., \phi_n$, then $X \leq_{suoir} Y$.

The orders $\leq_{slodr}$ and $\leq_{suoir}$ satisfy most of the postulates given in Section 2(see Colangelo et al.[1] and Shaked[2]). For example the order $\leq_{suoir}$ is closed under permutation of the component, marginalization, and convergence in distribution. Using Theorem 5 it is seen that also the order $\leq_{slodr}$ is closed under these operations.

The converses of the implications in (3.3) are not true in general. However, under an additional assumption they are valid; these are given in the following theorem.

Theorem Let X and Y be two random vectors in the same Fr'echet class with respective distribution functions F and G, and respective survival functions $\bar{F}$ and $\bar{G}$.

1. If F and/or G are/is MTP_2, then $X \leq_{lodr} Y \Rightarrow X \leq_{slodr} Y$.

2. If $\bar{F}$ and/or $\bar{G}$ are/is MTP_2, then $X \leq_{uoir} Y \Rightarrow X \leq_{suoir} Y$.

Proof. First we prove part (2) when $\bar{G}$ is MTP_2. Fix $x, y \in R^n (y \leq x)$. From $X \leq_{uoir} Y$ it follows that

$$\begin{aligned} \bar{F}(x)\bar{G}(y) \leq \bar{F}(y)\bar{G}(x) \quad &\Rightarrow \quad \bar{F}(x)\bar{G}(x \wedge y) \leq \bar{F}(x \wedge y)\bar{G}(x) \\ &\Rightarrow \quad \frac{\bar{G}(x \wedge y)}{\bar{G}(x)} \leq \frac{\bar{F}(x \wedge y)}{\bar{F}(x)} \end{aligned}$$

and from the MTP_2 property of $\bar{G}$ it follows that

$$\bar{G}(x)\bar{G}(y) \leq \bar{G}(x \wedge y)\bar{G}(x \vee y) \Rightarrow \frac{\bar{G}(y)}{\bar{G}(x \vee y)} \leq \frac{\bar{G}(x \wedge y)}{\bar{G}(x)}$$

Multiplication of these two inequalities yields

$$\bar{F}(x)\bar{G}(y) \leq \bar{F}(x \wedge y)\bar{G}(y \vee x),$$

that is $X \leq_{suoir} Y$. $\square$

1.8 The convex order

1.8.1 Definition and equivalent conditions

Let X and Y be two random variables such that

$$E[\phi(X)] \leq E[\phi(Y)] \quad for\ all\ convex\ functions\ \phi : \mathcal{R} \rightarrow \mathcal{R} \tag{1.22}$$

provided the expectations exist. Then X is said to be smaller than Y in the convex order (denoted as $X \leq_{cx} Y$). Roughly speaking, convex functions are functions that take on their (relatively) larger values over regions of the form $(-\infty, a) \cup (b, +\infty)$ for $a < b$. Therefore, if (1.1) holds, then Y is more likely to take on "extreme" values than X. That is, Y is "more variable" than X. It should be mentioned here that in (1.1) it is sufficient to consider only functions ϕ that are convex on the union of the supports of X and Y rather than over the whole real line; we will not keep repeating this point throughout this section.
One can also define a concave order by requiring (1.1) to hold for all concave functions ϕ (denoted as $X \leq_{cv} Y$). However, $X \leq_{cv} Y$ if, and only if, $Y \leq_{cx} X$. Therefore, it is not necessary to have a separate discussion for the concave order.
Note that the functions ϕ_1 and ϕ_2, defined by $\phi_1(x) = x$ and $\phi_2(x) = -x$, are both convex. Therefore, from (1.1) it easily follows that

$$X \leq_{cx} Y \Rightarrow E[X] = E[Y] \tag{1.23}$$

provided the expectations exist. Later it will be helpful to observe that if $E[X] = E[Y]$, then

$$\int_{-\infty}^{+\infty} [F(u) - G(u)]du = \int_{-\infty}^{+\infty} [\bar{F}(u) - \bar{G}(u)]du = 0 \tag{1.24}$$

provided the integrals exist, where $\bar{F}$ $[F]$ and $\bar{G}$ $[G]$ are the survival [distribution] functions of X and Y , respectively. The function ϕ defined by $\phi(x) = x^2$, is convex. Therefore, from (1.1) and (1.2), it follows that

$$X \leq_{cx} Y \Rightarrow Var[X] \leq Var[Y], \tag{1.25}$$

whenever $Var(Y) < \infty$.
For a fixed a, the function ϕ_a, defined by $\phi_a(x) = (x - a)_+$, and the function φ_a, defined by $\varphi_a = (a - x)_+$, are both convex. Therefore, if $X \leq_{cx} Y$, then

$$E[(X - a)_+] \leq E[(Y - a)_+] \quad for\ all\ a \tag{1.26}$$

and

$$E[(a - X)_+] \leq E[(a - Y)_+] \quad for\ all\ a \tag{1.27}$$

provided the expectations exist. Alternatively, using a simple integration by parts, it is seen that (1.5) and (1.6) can be rewritten as

$$\int_x^\infty \bar{F}(u)du \leq \int_x^\infty \bar{G}(u)du \; for \; all \;\; x \tag{1.28}$$

and

$$\int_{-\infty}^x F(u)du \leq \int_{\infty}^x G(u)du \; for \; all \;\; x \tag{1.29}$$

provided the integrals exist.

In fact, when $E[X] = E[Y]$, (1.7) is equivalent to $X \leq_{cx} Y$. To see this equivalence, note that every convex function can be approximated by (that is, is a limit of) positive linear combinations of the functions ϕ_a's, for various choices of a's, and of the function $\phi(x) = -x$. By (1.7), $E[\phi_a(X)] \leq E[\phi_a(Y)]$ for all a's, and this fact, together with the equality of the means of X and Y , implies (1.1). We thus have proved the first part of the following result. The other part is proven similarly.

Theorem 1 Let X and Y be two random variables such that $E[X] = E[Y]$. Then

(a) $X \leq_{cx} Y$ if, and only if, (1.7) holds.

(b) $X \leq_{cx} Y$ if, and only if, (1.8) holds.

By adding a to both sides of the inequality in (1.5), it is seen that (1.5) can be rewritten as

$$E[\max\{X, a\}] \leq E[\max\{Y, a\}] \; for \; all \;\; a. \tag{1.30}$$

Thus, when $E[X] = E[Y]$, then (1.9) is equivalent to $X \leq_{cx} Y$. In a similar manner (1.6) can be rewritten. The following theorem provides another characterization of the convex order.

Theorem 2 Let X and Y be two random variables such that $E[X] = E[Y]$. Then $X \leq_{cx} Y$ if, and only if,

$$E|X - a| \leq E|Y - a| \; for \; all \;\; a \in \mathcal{R}. \tag{1.31}$$

*Proof.*Clearly, if $X \leq_{cx} Y$, then (1.10) holds. So suppose that (1.10) holds. Without loss of generality it can be assumed that $E[X] = E[Y] = 0$. A straightforward computation gives

$$E|X - a| = a + 2\int_a^\infty \bar{F}(u)du = -a + 2\int_{-\infty}^a F(u)du. \tag{1.32}$$

The result now follows from (1.7) or (1.8). □

Theorem 3 The random variables X and Y satisfy $X \leq_{cx} Y$ if, and only if, there exist two random variables $\hat{X}$ and $\hat{Y}$, defined on the same probability space, such that

$$\hat{X} =_{st} X, \hat{Y} =_{st} Y,$$

and $\{\hat{X}, \hat{Y}\}$ is a martingale, that is,

$$E[\hat{Y}|\hat{X}] = \hat{X} \; a.s. \tag{1.33}$$

Proof.

$$E[\phi(X)] = E[\phi(\hat{X})] = E\phi(E[\hat{Y}|\hat{X}]) \leq E\{E[\phi(\hat{Y})|\hat{X}]\} = E[\phi(\hat{Y})] = E[\phi(Y)],$$

which is (1.1). □

Remark 1 If $Y_1 \geq_{cx} X_1$ and $Y_2 \geq_{cx} X_2$,then $E[\max\{Y_1; Y_2\}] \geq E[\max\{X_1; X_2\}]$ that $(Y_1; Y_2)$ and $(X_1; X_2)$ are pairs of independent random variables.

Theorem 4 Suppose $(X_1, X_2, \dots, X_n)$ and $(Y_1, Y_2, \dots, Y_n)$ are sets of mutually independent random variables. If:
$(i) Y_i \geq_{cx} X_i$ for all i, and
$(ii) f$ is a component-wise convex function,
then $E[f(Y_1, Y_2, \dots, Y_n)] \geq E[f(X_1, X_2, \dots, X_n)]$.

Theorem 5 Suppose $Y_1 \geq_{cx} X_1$ and$Y_2 \geq_{cx} X_2$. Then $E[\max(Y_1, Y_2)] \geq E[\max(X_1, X_2)]$, if $Cov(Y_1, Y_2) < Cov(X_1, X_2)$ and (Y_1, Y_2), (X_1, X_2) are both negatively dependent pairs of random variables.

Corollary 1 Suppose $Y_1 \geq_{cx} X_1$ and $Y_2 \geq_{cx} X_2$. Then $E[\max(Y_1, Y_2)] \geq E[\max(X_1, X_2)]$, if (Y_1, Y_2) is negatively dependent and (X_1, X_2) is independent.

Theorem 6 Suppose $Y_i \geq_{cx} X_i$ for each $i = 1, 2, \dots, N$. Then $E[\max(Y_1, Y_2, \dots, Y_N)] \geq E[\max(X_1, X_2, \dots, X_N)]$ if $(Y_1, Y_2, \dots, Y_N)$ is negatively dependent and $(X_1, X_2, \dots, X_N)$ is positively dependent.

Remark 2 Suppose $Y_i \geq_{cx} X_i$ for each $i = 1, 2, \dots, N$. Then $E[\max(Y_1, Y_2, \dots, Y_N)] \geq E[\max(X_1, X_2, \dots, X_N)]$ if $(Y_1, Y_2, \dots, Y_N) \leq_{NQD} (X_1, X_2, \dots, X_N)$.

1.9 LTD, RTI and SI orders

For any random vector (X_1, X_2) with distribution function $F \in \mathcal{F}(F_1, F_2)$ we define the conditional distribution function F_x^L by

$$F_{x_1}^L(x_2) = P[X_2 \leq x_2 | X_1 \leq x_1], \qquad (1)$$

for all x_1 for which this conditional distribution is well defined. Barlow and Proschan (1972) defined F (or X_1 and X_2) to be left tail decreasing (LTD) if for all $x_1 \leq x_1', \ and \ x_2$

$$F_{x_1}^L(x_2) \geq F_{x_1'}^L(x_2)$$

or,equivalently,if $x_1 \leq x_1', \ and \ u \in [0, 1]$

$$(F_{x_1}^L)^{-1}(u) \leq (F_{x_1'}^L)^{-1}(u), \quad (2)$$

Note that when $(F_{x_1}^L)^{-1}(u)$ is continuous in u for all x_1 then (2) can be equivalently written as

$$F_{x_1'}^L[(F_{x_1}^L)^{-1}(u)] \leq u, \quad for \ \ all \ x_1 \leq x_1' \ \ and \ u \in [0, 1]. \qquad (3)$$

This notion to the following definition. Let (X_1, X_2) be a bivariate random vector with distribution function $F \in \mathcal{F}(F_1, F_2)$ and let (Y_1, Y_2) be another bivariate random vector with distribution $G \in \mathcal{F}(F_1, F_2)$ suppose that for any $x_1 \le x_1'$ we have

$$(F^L_{x_1})^{-1}(u) \le (F^L_{x_1'})^{-1}(v) \Rightarrow (G^L_{x_1})^{-1}(u) \le (G^L_{x_1'})^{-1}(v), u, v \in [0,1] \quad (4)$$

Then we say that (X_1, X_2) is smaller than (Y_1, Y_2) in the LTD order [denoted by or $(X_1, X_2) \le_{LTD}$ (Y_1, Y_2) or $F \le_{LTD} G$.
Note that for all $x_1 \le x_1'$ and $u \in [0,1]$ (4) can be equivalently written as

$$G^L_{x_1'}[(G^L_{x_1})^{-1}(u)] \le F^L_{x_1'}[(F^L_{x_1})^{-1}(u)], \qquad (5)$$

It can be shown that if $F^L_{x_1}(x_2)$ and $G^L_{x_1}(x_2)$ are continuous in x_2 for all x_1 then $(X_1, X_2) \le_{LTD}$ (Y_1, Y_2) if,and only if,for any $x_1 \le x_1'$

$$F^L_{x_1}(x_2) \ge G^L_{x_1}(x_2') \Rightarrow F^L_{x_1'}(x_2) \ge G^L_{x_1'}(x_2'), \quad for \quad any \quad x-2, x-2' \qquad (6)$$

Note that for all $x_1 \le x_1'$ and x_2 (6) can be equivalently written as

$$(G^L_{x_1})^{-1}[F^L_{x_1}(x_2)] \le (G^L_{x_1'})^{-1}[F^L_{x_1'}(x_2)]$$

that is, $(G^L_{x_1})^{-1}[F^L_{x_1}(x_2)]$ is increasing in x_1 for all x_2.
in the continuous case,it is immediate from (3) and (5) that F is LTD if,and only if,

$$F^I \le_{LTD} F,$$

this is true also when F is not continuous.
Theorem 1 Let (X_1, X_2) and (Y_1, Y_2) be two random vectors with distribution functions $F, G \in \mathcal{F}(F_1, F_2)$ such that $F^L_{x_1}(x_2)$ and $G^L_{x_1}(x_2)$ are continuous in x_2 for all x_1, then

$$(X_1, X_2) \le_{LTD} (Y_1, Y_2) \Rightarrow (X_1, X_2) \le_{PQD} (Y_1, Y_2)$$

The LTD order is not symmetric in the sense that $(X_1, X_2) \le_{LTD} (Y_1, Y_2)$ does not necessarily imply that $(Y_1, Y_2) \le_{LTD} (X_1, X_2)$. However, it satisfies the following closure under monotone transformations property.
Theorem 2.Let (X_1, X_2) and (Y_1, Y_2) be two random vectors with distribution functions in the same frechet class. if $(X_1, X_2) \le_{LTD} (Y_1, Y_2)$ then $(\phi(X_1), \psi(X_2)) \le_{LTD} (\phi(Y_1), \psi(Y_2))$ for all increasing functions ϕ and ψ.
Example 1.Let $\phi_\theta(t) = (1-t^\theta)^{\frac{1}{\theta}}, t \in [0,1], \theta \in (0,1)$. Then the function C_θ defined as

$$C_{\phi_\theta}(x, y) = \phi_\theta^{-1}(\phi_\theta(x) + \phi_\theta(y)), \quad x, y \in [0,1].$$

is a bivariate distribution function with uniform $[0,1]$ marginals (it is a particular Archimedean copula). if $\theta_1 \le \theta_2$ then $C_{\phi_{\theta_1}} \le_{LTD} C_{\phi_{\theta_2}}$. An order that is similar to the LTD order,but which is based on conditioning on right tails,rather than on left tails,is described next. for any random vector (X_1, X_2) with distribution function $F \in \mathcal{F}(F_1, F_2)$ we define the conditional distribution F_x^R by

$$F_{x_1}^R(x_2) = P[X_2 \leq x_2 | X_1 > x_1]$$

for all x_1 for which this conditional distribution is well defined.Barlow and Proschan [36] defined F (or X_1 and X_2) to be right tail increasing [RTI]if

$$F_{x_1}^R(x_2) \geq F_{x_1'}^R(x_2), \forall x_1 \leq x_1' \ and \ x_2,$$

or, equivalently,if

$$(F_{x_1}^R)^{-1}(u) \leq (F_{x_1'}^R)^{-1}(u), \quad for \ \ all \ \ x_1 \leq x_1' \ \ and \ \ u \in [0,1].$$

when $(F_{x_1}^R)^{-1}(u)$ is continuous in u for all x_1 then the above inequality can be written as

$$F_{x_1'}^R(F_{x_1}^R)^{-1}(u) \leq u \quad for \ \ all \ \ x_1 \leq x_1' \ \ and \ \ u \in [0,1].$$

this notion leads to the following definition. Let (X_1, X_2) be a bivariate random vector with distribution function $F \in \mathcal{F}(F_1, F_2)$ and let (Y_1, Y_2) be another bivariate random vector with distribution function $G \in \mathcal{F}(F_1, F_2)$ .Suppose that for any $x_1 \leq x_1'$ we have

$$(F_{x_1}^R)^{-1}(u) \leq (F_{x_1'}^R)^{-1}(v) \Rightarrow (G_{x_1}^R)^{-1}(u) \leq (G_{x_1'}^R)^{-1}(v) \quad for \ \ all \ \ v, u \in [0,1].$$

Then we say that (X_1, X_2) is smaller than (Y_1, Y_2) in the RTI order [denoted by $(X_1, X_2) \leq_{RTI} (Y_1, Y_2)$ or $F \leq_{RTI} G$ In analogy to [9.C.5] we note that [9.C.11] can be written as

$$G_{x_1'}^R(G_{x_1}^R)^{-1}(u) \leq F_{x_1'}^R(F_{x_1}^R)^{-1}(u) \quad for \ \ all \ \ x_1 \leq x_1' \ \ and \ \ u \in [0,1]$$

It can be shown that if $F_{x_1}^R(x_2)$ and $G_{x_1}^R(x_2)$ are continuous in x_2 for all x_1 then $(X_1, X_2) \leq_{RTI} (Y_1, Y_2)$ if,and only if,for any $x_1 \leq x_1'$,

$$F_{x_1}^R(x_2) \geq G_{x_1}^R(x_2') \Rightarrow F_{x_1'}^R(x_2) \geq G_{x_1'}^R(x_2'), \qquad [9.C.13]$$

for any x_2 and x_2'. Note that [9.C.13] can be written as

$$(G_{x_1}^R)^{-1}[F_{x_1}^R(x_2)] \leq (G_{x_1'}^R)^{-1}[F_{x_1'}^R(x_2)] \quad for \ \ all \ \ x_1 \leq x_1' \ \ and \ \ x_2,$$

that is,$(G_{x_1}^R)^{-1}[F_{x_1}^R(x_2)]$ is increasing in x_1 for all x_2. In the continuous case,it is immediate from [9.C.10]and [9.C.12] that F is RTI if,and only if, $F^I \leq_{RTI} F$, where F^I is independence case,but this is true also when F is not continuous. The following result is an analog of Theorem 9.C.I,its proof is similar to the proof of that theorem,and is therefore omitted.

Theorem 3 Let (X_1, X_2) and (Y_1, Y_2) be two random vectors with distribution functions $F, G \in \mathcal{F}(F_1, F_2)$ such that $F_{x_1}^R(x_2)$ and $G_{x_1}^R(x_2)$ are continuous in x_2 for all x_1. Then

$$(X_1, X_2) \leq_{RTI} (Y_1, Y_2) \Rightarrow (X_1, X_2) \leq_{PQD} (Y_1, Y_2).$$

The RTI order is not symmetric in the sense that $(X_1, X_2) \leq_{RTI} (Y_1, Y_2)$ does not necessarily imply that $(X_2, X_1) \leq_{RTI} (Y_2, Y_1)$ However,it satisfies the following closure under monotone transformations property.

Theorem 4. Let (X_1, X_2) and (Y_1, Y_2) be two random vectors with distribution functions in the same Frechet class.If $(X_1, X_2) \leq_{RTI} (Y_1, Y_2)$ then $(\phi(X_1), \varphi(X_2) \leq_{RTI} (\phi(Y_1), \varphi(Y_2)$ for all increasing functions ϕ and φ.

The LTD and RTI orders are related to each other as follows. **Theorem 5.** Let (X_1, X_2) and (Y_1, Y_2) be two random vectors in the same Frechet class.

(a) If $(X_1, X_2) \leq_{LTD} (Y_1, Y_2)$ then $(\phi_1(X_1), \phi_2(X_2)) \leq_{RTI} (\phi_1(Y_1), \phi_2(Y_2))$ For any decreasing functions ϕ_1 and ϕ_2 Conversely,if $(\phi_1(X_1), \phi_2(X_2)) \leq_{RTI} (\phi_1(Y_1), \phi_2(Y_2))$ for some strictly decreasing functions ϕ_1 and ϕ_2 then $(X_1, X_2) \leq_{LTD} (Y_1, Y_2)$.

(b) If $(X_1, X_2) \leq_{RTI} (Y_1, Y_2)$ then $(\phi_1(X_1), \phi_2(X_2) \leq_{LTD} (\phi_1(Y_1), \phi_2(Y_2)$ for any decreasing functions ϕ_1 and ϕ_2 Conversely,if $(\phi_1(X_1), \phi_2(X_2) \leq_{LTD} (\phi_1(Y_1), \phi_2(Y_2)$ for some strictly decreasing functions ϕ_1 and ϕ_2 then $(X_1, X_2) \leq_{RTI} (Y_1, Y_2)$.

The orders $\leq_{slodr}$ and $\leq_{suodr}$ imply the LTD and RTI orders under some regularity conditions.This is shown in the next result.

Theorem 6. Let F and G be in the Frechet class $\mathcal{F}(F_1, F_2)$. Assume that,for every x,the conditional distributions and F_x^L and F_x^R [9.C.8] are strictly increasing and continuous on their supports.Then

$$F \leq_{slodr} G \Rightarrow F \leq_{LTD} G \ \ and \ \ F \leq_{suodr} G \Rightarrow F \leq_{RTI} G.$$

In light of Theorem 6 it is of interest to note that the [weak] orthant ratio orders $\leq_{lodr}$ and $\leq_{uoir}$ do not imply the orders $\leq_{LTD}$ and $\leq_{RTI}$ respectively.Counterexamples can be found in the literature.

An order that is of the same type as the LTD and RTI orders is the one that we study next. For any random vector (X_1, X_2) with distribution function $F \in \mathcal{F}(F_1, F_2)$ let F_{x_1} denote the conditional distribution of X_2 given that $X_1 = x_1$. Lehmann (1966) defined F[or X_1 and X_2] to be positive regression dependent [PRD]if X_2 is stochastically increasing in X_1 ,that is,if for all $x_1 \leq x_1'$ and x_2

$$F_{x_1}(x_2) \geq F_{x_1'}(x_2),$$

or,equivalently,if for all $x_1 \leq x_1'$ and $u \in [0, 1]$

$$F_{x_1}^{-1}(u) \geq F_{x_1'}^{-1}(u).$$

Note that when $F_{x_1}^{-1}(u)$ is continuous in u for all x_1 then the above inequality can be written as

$$F_{x_1'}(F_{x_1}^{-1}(u)) \leq u \ \ for \ \ all \ \ x_1 \leq x_1' \ \ and \ \ u \in [0, 1].$$

This notion leads to the following definition.

Definition 1 Let (X_1, X_2) be a bivariate random vector with distribution function $F \in \mathcal{F}(F_1, F_2)$ and let (Y_1, Y_2) be a bivariate random vector with distribution function $G \in \mathcal{F}(F_1, F_2)$. Suppose that for any $x_1 \leq x_1'$ we have

$$F_{x_1}^{-1}(u) \leq F_{x_1'}^{-1}(v) \Rightarrow G_{x_1}^{-1}(u) \leq G_{x_1'}^{-1}(v) \quad for \quad all \quad u, v \in [0, 1].$$

Then we say that (X_1, X_2) is smaller than (Y_1, Y_2) in the PRD order [denoted by] $(X_1, X_2) \leq_{PRD} (Y_1, Y_2)$ or $F \leq_{PRD} G$.

Note that the above statement can be written as

$$G_{x_1'}(G_{x_1}^{-1}(u)) \leq F_{x_1'}(F_{x_1}^{-1}(u)) \quad for \quad all \quad x_1 \leq x_1' \quad and \quad u \in [0, 1].$$

It can be shown that if $F_{x_1}(x_2)$ and $G_{x_1}(x_2)$ are continuous in x_2 for all x_1 then $(X_1, X_2) \leq_{PRD} (Y_1, Y_2)$ if,and only if,for any $x_1 \leq x_1'$, and for any x_2 and x_2'

$$F_{x_1}(x_2) \geq G_{x_1}(x_2') \Rightarrow F_{x_1'}(x_2) \geq G_{x_1'}(x_2').$$

Note that the above relation can be written as

$$G_{x_1}^{-1}(F_{x_1}(x_2)) \leq G_{x_1'}^{-1}(F_{x_1'}(x_2)) \quad for \quad all \quad x_1 \leq x_1' \quad and \quad x_2,$$

that is, $G_{x_1}^{-1}(F_{x_1}(x_2))$ is increasing in x_1 for all x_2.

In the continuous case,it is immediate from the above relations that F is PRD if,and only if, $F^I \leq_{PRD} F$, where F^I is defined as independence case, but this is true also when F is not continuous.

The next result shows the relationship between the PRD,LTD,and RTI orders. we do not give the proof of it here.

Theorem 7. Let (X_1, X_2) and (Y_1, Y_2) be two random vectors with absolutely continuous distribution functions $F, G \in \mathcal{F}(F_1, F_2)$. Then

$$(X_1, X_2) \leq_{PRD} (Y_1, Y_2) \Rightarrow (X_1, X_2) \leq_{LTD} (Y_1, Y_2)$$

and

$$(X_1, X_2) \leq_{PRD} (Y_1, Y_2) \Rightarrow (X_1, X_2) \leq_{RTI} (Y_1, Y_2)$$

The PRD order is not symmetric in the sense that $(X_1, X_2) \leq_{PRD} (Y_1, Y_2)$ does not necessarily imply that $(X_2, X_1) \leq_{PRD} (Y_2, Y_1)$. However,it satisfies the following closure under monotone transformations property.

Theorem 8. Let (X_1, X_2) and (Y_1, Y_2) be two random vectors. If $(X_1, X_2) \leq_{PRD} (Y_1, Y_2)$ then $(\phi(X_1), \varphi(X_2)) \leq_{PRD} (\phi(Y_1), \varphi(Y_2))$ for all increasing functions ϕ and φ.

Example 2. Let U and V be any independent random variables,each having a continuous distribution.Define

$$X = U, Y_\rho = \rho U + (1 - \rho^2)^{1/2} V, \quad for \quad all \quad -1 \leq \rho \leq 1.$$

Then $(X, Y_{\rho_1}) \leq_{PRD} (X, Y_{\rho_2})$ whenever $\rho_1 \leq \rho_2$. A bivariate normal distribution is a particular case of this example when U and V are normally distributed.

Example 3.Let U and V be any independent random variables,each having a continuous distribution.Define

$$X = U, Y_\alpha = \alpha U + V, -\infty \leq \alpha \leq \infty.$$

Then $(X, Y_{\alpha_1}) \leq_{PRD} (X, Y_{\alpha_2})$ whenever $\alpha_1 \leq \alpha_2$.

Example 4.Let U and V be any independent random variables,each having a continuous distribution,such that U is distributed on $(0, 1)$,while V is nonnegative.Define

$$X = U, Y_\alpha = (1 + \alpha U)V, \alpha \geq -1$$

Then $(X, Y_{\alpha_1}) \leq_{PRD} (X, Y_{\alpha_2})$ whenever $\alpha_1 \leq \alpha_2$.

1.10 PLRD order (Kimeldorf and Sampson (1987))

Let the random variables X and Y have the joint distribution F. for any intervals I and J of the real line, let us denote $I \leq J$ if $x \in I$ and $y \in J$ imply that $x \leq y$. For any two intervals I and J of the real line denote $\mu(I, J) = P[X \in I, Y \in J]$. Block, Savits and Shaked (1981) essentially defined μ to be positive likelihood ration dependent if

$$\mu(I_1, J_1)\mu(I_2, J_2) \geq \mu(I_1, J_2)\mu(I_2, J_1), \quad whenever \quad I_1 \leq I_2, \quad J_1 \leq J_2.$$

In fact Block, Savits and Shaked (1981) called μ totally positive of order 2 (TP_2) if the above inequality holds. When F is a distribution and f is a density function it is equivalent to the condition that f is TP_2, that is for all $x_1 \leq x_2$ and $y_1 \leq y_2$,

$$f(x_1, y_1)f(x_2, y_2) \geq f(x_1, y_2)f(x_2, y_1).$$

This is the same as the condition for the positive dependence notion that Lehmann (1966) called PLRD.This notion leads naturally to the order that is described bellow. Let (X_1, X_2) be a bivariate random vector with distribution function F, and let (Y_1, Y_2) be another bivariate random vector with distribution function G. Suppose that for all $I_1 \leq I_2, J_1 \leq J_2$

$$F(I_1, J_1)F(I_2, J_2)G(I_1, J_2)G(I_2, J_1) \leq F(I_1, J_2)F(I_2, J_1)G(I_1, J_1)G(I_2, J_2)$$

Then we say that (X_1, X_2) is smaller that (Y_1, Y_2) in the PLRD order denoted by

$$(X_1, X_2) \leq_{PLRD} (Y_1, Y_2), \quad or \quad F \leq_{PLRD} G.$$

Since only random vectors with the same univariate margins can be compared in the PLRD order, we will implicity assume this is equivalent to

$$f(x_1, y_1)f(x_2, y_2)g(x_1, y_2)g(x_2, y_1) \leq f(x_1, y_2)f(x_2, y_1)g(x_1, y_1)g(x_2, y_2)$$

and this equivalent to

$$\frac{f(x_1,y_1)f(x_2,y_2)}{f(x_1,y_2)f(x_2,y_1)} \le \frac{g(x_1,y_1)g(x_2,y_2)}{g(x_1,y_2)g(x_2,y_1)}$$

for all $x_1 \le x_2, \quad y_1 \le y_2$. If $\frac{\partial^2}{\partial x \partial y} f$ and $\frac{\partial^2}{\partial x \partial y} g$ then it is equivalent to

$$f^2 \triangle_g - g^2 \triangle_f \ge 0,$$

where $\triangle_f = f \frac{\partial^2}{\partial x \partial y} f - \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$ and $\triangle_g = g \frac{\partial^2}{\partial x \partial y} g - \frac{\partial g}{\partial x}\frac{\partial g}{\partial y}$

Remark 1 F is PLRD if and only if $F^I \le_{PLRD} F$.

Theorem 1 Let (X_1, X_2) and (Y_1, Y_2) be two random vectors with distributions F, G.Then

$$F \le_{PLRD} G \Rightarrow F \le_{PQD} G.$$

Problem(Shaked and Shantikumar, 2007). we do not know whether $F \le_{PLRD} G \Rightarrow F \le_{PRD} G$.

The folowing closure properties of the PLRD order are easy to prove.

Theorem 2 i) Let (X_1, X_2) and (Y_1, Y_2) be two random vectors such that $(X_1, X_2) \le_{PLRD} (Y_1, Y_2)$. Then $(\varphi(X_1), \psi(X_2)) \le_{PLRD} (\varphi(Y_1), \psi(Y_2))$ for all increasing functions φ and ψ.
ii)-Let $\{(X_1^{(n)}, X_2^{(n)})\}$ and $\{(Y_1^{(n)}, Y_2^{(n)})\}$ be two sequences of random vectors such that $(X_1^{(n)}, X_2^{(n)}) \to_{st} (X_1, X_2)$ and $(Y_1^{(n)}, Y_2^{(n)}) \to_{st} (Y_1, Y_2)$ as $n \to \infty$. If $(X_1^{(n)}, X_2^{(n)}) \le_{PLRD} (Y_1^{(n)}, Y_2^{(n)})$ for all $n \ge 1$ then $(X_1, X_2) \le_{PLRD} (Y_1, Y_2)$.

Remark 2 For every distribution F we have

$$F_L \le_{PLRD} F \le_{PLRD} F_U.$$

Theorem 3 Let (X_1, X_2) and (Y_1, Y_2) be two random vectors such that $(X_1, X_2) \le_{PLRD} (Y_1, Y_2)$ and $(X_1, X_2) \ge_{PLRD} (Y_1, Y_2)$. Then $(X_1, X_2) =_{st} (Y_1, Y_2)$

Example 1 Let F and G be two continuous univariate distribution functions and for $-1 \le \theta \le 1$,

$$F_\theta(x,y) = F(x)g(y)[1 - \theta(1 - \bar{F}(x))(1 - \bar{G}(y))].$$

Then $F_{\theta_1} \le_{PLRD} F_{\theta_2}$ whenever $\theta_1 \le \theta_2$.

Example 2 Let ϕ and ψ be two Laplace transforms of positive random variables and let the random vectors (X_1, X_2) and (Y_1, Y_2) be two random vectors with df's F and G as the following

$$F(x,y) = \varphi(\varphi^{-1}(x) + \varphi^{-1}(y))$$

and

$$G(x,y) = \psi(\psi^{-1}(x) + \psi^{-1}(y))$$

Then $(X_1, X_2) \le_{PLRD} (Y_1, Y_2)$ if $\varphi^{-1}\psi$ has a completely monotone derivative.

Example 3 Let (X_1, X_2) and (Y_1, Y_2) be bivariate normal random vectors with the same marginals, and with correlation coefficients ρ_X and ρ_Y, respectively. If $\rho_X \le \rho_Y$ then $(X_1, X_2) \le_{PLRD} (Y_1, Y_2)$.

1.11 Association orders

Let (X_1, X_2) be a bivariate random vector with distribution function $F \in M(F_1, F_2)$, and let (Y_1, Y_2) be another bivariate vector with distribution function $G \in M(F_1, F_2)$. Suppose that

$$(Y_1, Y_2) =_{st} (K(X_1, X_2), L(X_1, X_2)), \quad (4.5.1),$$

for some increasing functions K and L which satisfy

$$K(x_1, y_1) < K(x_2, y_2), \;\; L(x_1, y_1) > L(x_2, y_2) \Rightarrow \; x_1 < x_2, \; y_1 > y_2. \;\; (4.5.2)$$

Then we say that (X_1, X_2) is smaller that (Y_1, Y_2) in the association order (denoted by $(X_1, X_2) \leq_{assoc} (Y_1, Y_2)$ or $F \leq_{assoc} G$). Since only random vectors with the same univariate marginals are compared in the association order, we will implicitly assume this fact throughout this section. The restriction (4.5.2) on the function K and L is for the purpose of making the association order applicable in situations which are not symmetric in the X_1 and X_2 variables.[In case (4.5.2) is dropped, $(X_1, X_2) \geq_{assoc} (X_2, X_1) \geq_{assoc} (X_1, X_2)$.] If K and L are partially differentiable increasing functions, then (4.5.2) is equivalent to

$$\frac{\partial}{\partial x}K(x,y)\frac{\partial}{\partial y}L(x,y) \geq \frac{\partial}{\partial y}K(x,y)\frac{\partial}{\partial x}L(x,y) \;\; for \;\; all x, y.$$

From the fact that increasing functions of independent random variables are associated, it follows that if $F^I \leq_{assoc} F$, then F is the distribution function of associated random variables.
Theorem 1 Let (X_1, X_2) and (Y_1, Y_2) be to random vectors. If $(X_1, X_2) \leq_{assoc} (Y_1, Y_2)$, then $(\varphi(X_1), \psi(X_2)) \leq_{assoc} (\varphi(Y_1), \psi(Y_2))$ for all strictly increasing function φ and ψ.
The relationship between the association and the PQD and PRD(SI) orders is described in the next results.
Theorem 2 Let (X_1, X_2) and (Y_1, Y_2) be two random vectors with distribution functions $F, G \in \mathcal{F}(\mathcal{F}_\infty, \mathcal{F}_\in)$. Then

$$(X_1, X_2) \leq_{assoc} (Y_1, Y_2) \Rightarrow (X_1, X_2) \leq_{PQD} (Y_1, Y_2)$$

Theorem 3 Let (X_1, X_2) and (Y_1, Y_2) be two random vectors with distribution functions $F, G \in \mathcal{F}(F_1, F_2)$ such that $F_{X_2|X_1}(x_2|x_1)$ and $G_{Y_2|Y_1}(x_2|x_1)$ are continuous in x_2 for all x_1. Then

$$(X_1, X_2) \leq_{PRD} (Y_1, Y_2) \Rightarrow (X_1, X_2) \leq_{assoc} (Y_1, Y_2)$$

Example 1 Let U and V be any independent random variables. Define

$$X_\alpha = (1-\alpha)U + \alpha V, Y_\alpha = \alpha U + (1-\alpha)V, \;\; for \;\; \alpha \in [0, \frac{1}{2}.$$

Then $(X_{\alpha_1}, Y_{\alpha_1}) \leq_{assoc} (X_{\alpha_2}, Y_{\alpha_2})$ whenever $\alpha_1 \leq \alpha_2$.
Example 2 Let (X_1, X_2) and (Y_1, Y_2) have bivariate normal distributions with correlation coefficients ρ_1 and ρ_2 respectively.Then

$$(X_1, X_2) \leq_{assoc} (Y_1, Y_2) \Leftrightarrow -1 \leq \rho_1 \leq \rho_2 \leq 1.$$

1.11.1 CFG order

Caperaa, Fougeres and Genest (1997) introduced an order that is related to the association order. In order to define it we need first to introduce some notation. Let (X_1, X_2) be a random vector with continuous distribution $F \in \mathcal{F}(F_1, F_2)$. Define

$$V_F = F(X_1, X_2), \quad and \quad K_F(t) = P[V_F \leq t], \quad t \in [0, 1].$$

For example:
i $K_{F_U}(t) = t, \quad t \in [0, 1]$ where $F_U = \max\{F_1, F_2\}$.
ii $K_{F_L}(t) = 1, \quad t \in [0, 1]$, where $F_L = \min\{F_1 + F_2 - 1, 0.\}$
iii If X_1 and X_2 are independent, with distribution function $F^I \in \mathcal{F}(\mathcal{F}_\infty, \mathcal{F}_\in)$, then $K_{F^I}(t) = t - t\log(t), \quad t \in [0, 1]$.
For Part iii for all $t \in [0, 1]$ we have

$$\begin{aligned} K_{F^I}(t) &= P[F_1(X_1)F_2(X_2) \leq t] \\ &= P[-\log(F_1(X_1)) - \log(F_2(X_2)) \geq -\log(t)] \\ &= P[\Gamma(2, 1) \geq -\log(t)] = t - t\log(t). \end{aligned}$$

Definition 2 Let (X_1, X_2) and (Y_1, Y_2) be two random vectors with distribution functions $F, G \in \mathcal{F}(F_1, F_2)$. Suppose that

$$K_F(t) \geq K_G(t), \quad for \quad all\ t \in [0, 1].$$

Then say that (X_1, X_2) is smaller than (Y_1, Y_2) in the Caperaa-Fougeres-Genest order and denoted by $(X_1, X_2) \leq_{CFG} (Y_1, Y_2)$ or $F \leq_{CFG} G$.
Theorem 4(CFG, 1997) For every continuous distribution function $F \in \mathcal{F}(F_1, F_2)$ we have

$$F_L \leq_{CFG} F \leq_{CFG} F_U.$$

and under some regularity conditions we have

$$(X_1, X_2) \leq_{assoc} (Y_1, Y_2) \Rightarrow (X_1, X_2) \leq_{CFG} (Y_1, Y_2)$$

Corollary 1 i- (CFG, 1997) The order CFG dose not imply order PQD.
ii-(Nelsen, et.al.,2001) The order PQD dose not imply order CFG

1.11.2 Weak association order

Let $X = (X_1, X_2, ..., X_n)$ and $Y = (Y_1, Y_2, ..., Y_n)$ be two random vectors that have the same univariate marginals, and that satisfy

$$Cov(h_1(X_{i_1}, ..., X_{i_k}), h_2(X_{j_1}, ..., X_{j_{n-k}})) \leq Cov(h_1(Y_{i_1}, ..., Y_{i_k}), h_2(Y_{j_1}, ..., Y_{j_{n-k}}))$$

for all choices of disjoint subsets $\{i_1, i_2, ..., i_k\}$ and $\{j_1, j_2, ..., j_{n-k}\}$ of $\{1, 2, 3, ..., n\}$, and for all increasing functions h_1 and h_2 for which the above covariances are defined. Then X is said to be smaller than Y in the weak association order and denoted by $X \leq_{wassoc} Y$.
Some closure properties of the weak association order are described in the next Theorem.
Theorem 5 Let $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ..., Y_n)$ be two n-dimensional random vectors.
i- If $(X_1, X_2, ..., X_n) \leq_{wassoc} (Y_1, Y_2, ..., Y_n)$, then

$$(g_1(X_1), g_2(X_2), ..., g_n(X_n)) \leq_{wassoc} (g_1(Y_1), g_2(Y_2), ..., g_n(Y_n))$$

whenever $g_i, i = 1, 2, ..., n$ are all increasing real functions.
ii-If $(X_1, X_2, ..., X_n) \leq_{wassoc} (Y_1, Y_2, ..., Y_n)$, the $X_I \leq_{wassoc} Y_I$ for each $I \subseteq \{1, 2, 3, ..., n\}$. That is, the weak association order is closed under marginalization.
An important useful property of the weak association order is the following.
Theorem 6 Let $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ..., Y_n)$ be two n-dimensional random vectors with the same univariate marginals. Then

$$X \leq_{wassoc} Y \Rightarrow X \leq_{sm} Y.$$

Remark Note that if $X = (X_1, X_2, ..., X_n)$ is a vector of weakly associated random variables and if $Y = (Y_1, Y_2, ..., Y_n)$ is a vector of independent random variables such that, $X_i =_{st} Y_i, i = 1, 2, ..., n$, then $X \geq_{wassoc} Y$. Similarly if X is a vector of NA random variables and if Y is a vector of independent random variables such that $X_i =_{st} Y_i, i = 1, 2, ..., n$, then $X \leq_{wassoc} Y$.

1.12 PDD order

Let the random variables X_1 and X_2 have the symmetric joint distribution F. Shaked (1979) defines F (or X_1 and X_2) to be positive definite dependent (PDD) if F is a positive definite kernel on $S \times S$ where S is the support of X_1 and therefore, by symmetry, S is also the support of X_2. Shaked (1979) has shown that X_1 and X_2 are PDD if and only if for every real function ϕ

$$Cov(\phi(X_1), \phi(X_2)) \geq 0,$$

provided the covriance is well defined. This notion naturally leads to the order that is defined below.
Let (X_1, X_2) be a bivariate random vectors with distribution functions $F \in \mathcal{F}^{(\int)}(\hat{F})$, where $\mathcal{F}^{(\int)}(\hat{F})$ is the class of all the bivariate symmetric distributions with univariate marginals $\hat{F}$. Let (Y_1, Y_2) be another bivariate random vectors with distribution functions $G \in \mathcal{F}^{(\int)}(\hat{F})$. Suppose that for every real function ϕ,

$$Cov(\phi(X_1), \phi(X_2)) \leq Cov(\phi(Y_1), \phi(Y_2))$$

provided that covariances are well defined. Then we say that (X_1, X_2) is smaller than (Y_1, Y_2) in the PDD order (denoted by $(X_1, X_2) \leq_{PDD} (Y_1, Y_2)$ or $F \leq_{PDD} G$.) Since only symmetric random vectors with the same univariate marginals are compared in the PDD order, we will implicitly assume this fact throughout this section.

Since $E\phi(X_1) = E\phi(X_2) = E\phi(Y_1) = E\phi(Y_2)$ for every real function ϕ, it follows that $(X_1, X_2) \leq_{PDD} (Y_1, Y_2)$ if and only if for every real function ϕ

$$E\phi(X_1)\phi(X_2) \leq E\phi(Y_1)\phi(Y_2),$$

provided the expectations exist. Thus if $(X_1, X_2) \leq_{PDD} (Y_1, Y_2)$, then

$$P[X_1 \in A, X_2 \in A] \leq P[Y_1 \in A, Y_2 \in A]$$

for all Borel-measurable sets $A \in R$.

Another characterization of the PDD order is given in the next theorem.

Theorem 1 Let F and G be two symmetric bivariate distributions in $\mathcal{F}^{(f)}(\hat{F})$. Then $F \leq_{PDD} G$ if and only if $G(x, y) - F(x, y)$ is a positive definite kernel.

Corollary It is easily seen that F is PDD if and only if $F^I \leq_{PDD} F$.

A powerful closure property of the PDD order is described in the next Theorem.

Theorem 2 Suppose that the random vectors (X_1, X_2), (Y_1, Y_2), (U_1, U_2) and (V_1, V_2) satisfy

$$(X_1, X_2) \leq_{PDD} (Y_1, Y_2) \ \ \textit{and} \ \ (U_1, U_2) \leq_{PDD} (V_1, V_2),$$

and suppose that (X_1, X_2) and (U_1, U_2) are independent, and also that (Y_1, Y_2) and (V_1, V_2) are independent. Then for every increasing function ϕ

$$(\phi(X_1, U_1), \phi(X_2, U_2)) \leq_{PDD} (\phi(Y_1, V_1), \phi(Y_2, V_2)).$$

Corollary The PDD order closed under convolutions, that is

$$(X_1 + U_1, X_2 + U_2) \leq_{PDD} (Y_1 + V_1, Y_2 + V_2).$$

Moreover, we can check that the following closure properties.

Theorem 2*i* Let $\{(X_1^{(j)}, X_2^{(j)}), j \geq 1\}$ and $\{(Y_1^{(j)}, Y_2^{(j)}), j \geq 1\}$ be two sequences of random vectors such that $(X_1^{(j)}, X_2^{(j)} \to_{st} (X_1, X_2)$ and $(Y_1^{(j)}, Y_2^{(j)}) \to_{st} (Y_1, Y_2)$ as $j \to \infty$. If for all $j \geq 1$, $(X_1^{(j)}, X_2^{(j)}) \leq_{PDD} (Y_1^{(j)}, Y_2^{(j)})$, then

$$(X_1, X_2 \leq_{PDD} (Y_1, Y_2)$$

ii Let (X_1, X_2), (Y_1, Y_2), and Θ be random vectors such that for all θ in the support of Θ,

$$(X_1, X_2)|\Theta = \theta \leq_{PDD} (Y_1, Y_2)|\Theta = \theta$$

then $(X_1, X_2 \leq_{PDD} (Y_1, Y_2)$. That is, the order of PDD is closed under mixtures.

Example 1 Let (X_1, X_2) and (Y_1, Y_2) have exchangeable bivariate normal distributions with common marginals and correlation coefficients ρ_1 and ρ_2 respectively. If $0 \leq \rho_1 \leq \rho_2 \leq 1$ then

$$(X_1, X_2 \leq_{PDD} (Y_1, Y_2).$$

Remark If (X_1, X_2) and (Y_1, Y_2) have distributions F and G which are not symmetric, but still have the same marginals (that X_1, X_2, Y_1 and Y_2 are all identically distributed), then the PDD order can still be defined on the symmetrizations $\tilde{F}(x, y) = \frac{F9x,y)+F(y,x)}{2}$ and $\tilde{G}(x, y) = \frac{G(x,y)+G(y,x)}{2}$ of F and G.
An n-variate extension of the PDD order for the case when $n \geq 2$ is suggested by (1). Explicity, let $X = (X_1, X_2, ..., X_n)$ and $Y = (Y_1, Y_2, ..., Y_n)$ have distribution functions with common marginals. Then we can say that X less positively dependent than Y if for every nonnegative real function ϕ,

$$E \prod_{i=1}^{n} \phi(X_i) \leq E \prod_{i=1}^{n} Y_i. \quad (2)$$

Note that for this definition it is not required that X and Y have exchangeable distribution functions, it is only required that X and Y have the same common marginals.
One reason for the usefulness of inequality (2) is that it implies that

$$P[X_1 \in A, X_2 \in A, ..., X_n \in A] \leq P[Y_1 \in A, Y_2 \in A, ..., Y_n \in A]$$