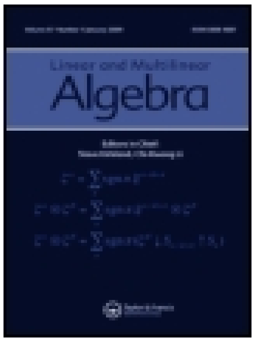

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Improvements of Berezin number inequalities

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ABSTRACT

In this paper, we generalize several Berezin number inequalities involving product of operators, which acting on a Hilbert space $\mathscr{H}(\Omega)$. Among other inequalities, it is shown that if A,B are positive operators and X is any operator, then

$$\begin{aligned}\mathbf{ber}^r(H_\alpha(A,B)) &\le \frac{\|X\|^r}{2}\left(\mathbf{ber}(A^r+B^r)-2\inf_{\|\hat{k}_\lambda\|=1}\eta(\hat{k}_\lambda)\right)\\ &\le \frac{\|X\|^r}{2}\Big(\mathbf{ber}(\alpha A^r+(1-\alpha)B^r)\\ &\quad+\mathbf{ber}((1-\alpha)A^r+\alpha B^r)-2\inf_{\|\hat{k}_\lambda\|=1}\eta(\hat{k}_\lambda)\Big),\end{aligned}$$

where $\eta(\hat{k}_\lambda)=r_0(\langle A^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2}-\langle B^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2})^2$, $r\ge 2$, $0\le\alpha\le 1$, $r_0=\min\{\alpha,1-\alpha\}$ and $H_\alpha(A,B)=(A^\alpha XB^{1-\alpha}+A^{1-\alpha}XB^\alpha)/2$.

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1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with an inner product $\langle\, .\, ,\, .\,\rangle$ and the corresponding norm $\|\,.\,\|$. In the case when $\dim\mathscr{H}=n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n\times n$ matrices with entries in the complex field. An operator $A\in\mathbb{B}(\mathscr{H})$ is called positive if $\langle Ax,x\rangle\ge 0$ for all $x\in\mathscr{H}$, and then we write $A\ge 0$.

A functional Hilbert space $\mathscr{H}=\mathscr{H}(\Omega)$ is a Hilbert space of complex valued functions on a (nonempty) set Ω, which has the property that point evaluations are continuous i.e. for each $\lambda\in\Omega$ the map $f\mapsto f(\lambda)$ is a continuous linear functional on $\mathscr{H}$. The Riesz representation theorem ensure that for each $\lambda\in\Omega$ there is a unique element $k_\lambda\in\mathscr{H}$ such that $f(\lambda)=\langle f,k_\lambda\rangle$ for all $f\in\mathscr{H}$. The collection $\{k_\lambda:\lambda\in\Omega\}$ is called the reproducing kernel of $\mathscr{H}$. If $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathscr{H}$, then the reproducing kernel of $\mathscr{H}$ is given by $k_\lambda(z)=\sum_n\overline{e_n(\lambda)}e_n(z)$; (see [1, Problem 37]). For $\lambda\in\Omega$, let $\hat{k}_\lambda=k_\lambda/\|k_\lambda\|$ be the normalized reproducing kernel of $\mathscr{H}$. For a bounded linear operator A on $\mathscr{H}$, the function $\widetilde{A}$ defined on Ω by $\widetilde{A}(\lambda)=\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle$ is the Berezin symbol of A, which firstly have been introduced by Berezin [2, 3]. The Berezin set and the Berezin

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number of the operator A are defined by

$$\mathbf{Ber}(A) := \{\widetilde{A}(\lambda) : \lambda \in \Omega\} \quad \text{and} \quad \mathbf{ber}(A) := \sup\{|\widetilde{A}(\lambda)| : \lambda \in \Omega\},$$

respectively,(see [4]).

The numerical radius of $[A \in \mathbb{B}(\mathscr{H}(\Omega))]$ is defined by $w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathscr{H}, \|x\| = 1\}$. It is clear that

$$\mathbf{ber}(A) \leq w(A) \leq \|A\| \tag{1.1}$$

for all $A \in \mathbb{B}(\mathscr{H})$. Moreover, the Berezin number of an operator A satisfies the following properties:

(i) $\mathbf{ber}(\alpha A) = |\alpha|\mathbf{ber}(A)$ for all $\alpha \in \mathbb{C}$.
(ii) $\mathbf{ber}(A + B) \leq \mathbf{ber}(A) + \mathbf{ber}(B)$.

Let $T_i \in \mathbb{B}(\mathscr{H}(\Omega))(1 \leq i \leq n)$. The generalized Euclidean Berezin number of $T_1, \ldots, T_n$ is defined in [5] as follows

$$\mathbf{ber}_p(T_1, \ldots, T_n) := \sup_{\lambda\in\Omega} \left(\sum_{i=1}^{n} \left| \langle T_i\hat{k}_\lambda, \hat{k}_\lambda\rangle \right|^p \right)^{1/p},$$

which has the following properties:

(a) $\mathbf{ber}_p(\alpha T_1, \ldots, \alpha T_n) = |\alpha|\mathbf{ber}_p(T_1, \ldots, T_n)$ for all $\alpha \in \mathbb{C}$;
(b) $\mathbf{ber}_p(T_1 + S_1, \ldots, T_n + S_n) \leq \mathbf{ber}_p(T_1, \ldots, T_n) + \mathbf{ber}_p(S_1, \ldots, S_n)$,

where $T_i, S_i \in \mathbb{B}(\mathscr{H}(\Omega))(1 \leq i \leq n)$.

Namely, the Berezin symbol have been investigated in detail for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it is widely applied in the various questions of analysis and uniquely determines the operator (i.e. for all $\lambda \in \Omega, \widetilde{A}(\lambda) = \widetilde{B}(\lambda)$ implies $A = B$). For further information about Berezin symbol we refer the reader to [5–12] and references therein.

In [13], the author showed some Berezin number inequalities as follows:

$$\mathbf{ber}(A^*XB) \leq \tfrac{1}{2}\mathbf{ber}(B^*|X|B + A^*|X^*|A),$$
$$\mathbf{ber}(AX \pm XA) \leq \mathbf{ber}^{1/2}(A^*A + AA^*)\mathbf{ber}^{1/2}(X^*X + XX^*), \tag{1.2}$$

and

$$\mathbf{ber}(A^*XB + B^*YA) \leq 2\sqrt{\|X\|\|Y\|}\mathbf{ber}^{1/2}(B^*B)\mathbf{ber}^{1/2}(AA^*) \tag{1.3}$$

for any $A, B, X, Y \in \mathbb{B}(\mathscr{H}(\Omega))$.

In this paper, we would like to state more extensions of the Berezin number inequalities. Moreover, we obtain several Berezin number inequalities for 2×2 operator matrices. For this goal we will apply some methods from [14,15].

2. main results

To prove our Berezin number inequalities, we need several well known lemmas.

The following lemma is a simple consequence of the classical Jensen and Young inequalities (see [16]).

Lemma 2.1: *Let $a, b \geq 0$, $0 \leq \alpha \leq 1$ and p,q > 1 such that $1/p + 1/q = 1$. Then*

(*a*) $a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{1/r}$ *for* $r \geq 1$;
(*b*) $ab \leq a^p/p + b^q/q \leq (a^{pr}/p + b^{qr}/q)^{1/r}$ *for* $r \geq 1$.

A refinement of the Young inequality is presented in [17] as follows:

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b - r_0 (a^{1/2} - b^{1/2})^2, \tag{2.1}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

The following lemma is known as generalized mixed schwarz inequality [18].

Lemma 2.2: *Let $T \in \mathbb{B}(\mathscr{H})$ and $x, y \in \mathscr{H}$ be any vectors.*

(*a*) *If* $0 \leq \alpha \leq 1$, *then*

$$| \langle Tx, y \rangle |^2 \leq \langle | T |^{2\alpha} x, x \rangle \langle | T^* |^{2(1-\alpha)} y, y \rangle,$$

where $|T| = (T^*T)^{1/2}$ *is the absolute value of* T.

(*b*) *If f, g are nonnegative continuous functions on* $[0, \infty)$ *which are satisfying the relation* $f(t)g(t) = t (t \in [0, \infty))$, *then*

$$| \langle Tx, y \rangle | \leq \| f(| T |) x \| \| g(| T^* |) y \|$$

for all $x, y \in \mathscr{H}$.

Lemma 2.3: *[5] Let* $A \in \mathbb{B}(\mathscr{H}_1(\Omega))$, $B \in \mathbb{B}(\mathscr{H}_2(\Omega), \mathscr{H}_1(\Omega))$, $C \in \mathbb{B}(\mathscr{H}_1(\Omega), \mathscr{H}_2(\Omega))$ *and* $D \in \mathbb{B}(\mathscr{H}_2(\Omega))$. *Then the following statements hold:*

(*a*) $\mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \leq \max\{\mathbf{ber}(A), \mathbf{ber}(D)\}$;
(*b*) $\mathbf{ber}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2}(\|B\| + \|C\|)$.

The next lemma follows from the spectral theorem for positive operators and the Jensen inequality (see [18]).

Lemma 2.4 (McCarthy inequality): *Let* $T \in \mathbb{B}(\mathscr{H}), T \geq 0$ *and* $x \in \mathscr{H}$ *be a unit vector. Then*

(*a*) $\langle Tx, x \rangle^r \leq \langle T^r x, x \rangle$ *for* $r \geq 1$;
(*b*) $\langle T^r x, x \rangle \leq \langle Tx, x \rangle^r$ *for* $0 < r \leq 1$.

Now, by applying these lemmas, we extend some Berezin number inequalities.

Theorem 2.5: *Let $A, B, X \in \mathbb{B}(\mathscr{H}(\Omega))$. Then*

(i) $\mathbf{ber}^r(A^*XB) \le \|X\|^r \mathbf{ber}((1/p)(A^*A)^{pr/2} + (1/q)(B^*B)^{qr/2})$ *for* $r \ge 0$ *and* $p,q > 1$ *with* $1/p + 1/q = 1$ *and* $pr, qr \ge 2$.
(ii) $\mathbf{ber}(A^*XB) \le \frac{1}{2}\mathbf{ber}(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A)$ *for every* $0 \le \alpha \le 1$.

Proof: If $\hat{k}_\lambda$ is the normalized reproducing kernel of $\mathscr{H}(\Omega)$, then

$$\begin{aligned}
|\langle A^*XB\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r &= |\langle XB\hat{k}_\lambda, A\hat{k}_\lambda\rangle|^r \\
&\le \|X\|^r\|A\hat{k}_\lambda\|^r\|B\hat{k}_\lambda\|^r \quad \text{(by the Cauchy-Schwarz inequality)} \\
&\le \|X\|^r\langle A\hat{k}_\lambda, A\hat{k}_\lambda\rangle^{r/2}\langle B\hat{k}_\lambda, B\hat{k}_\lambda\rangle^{r/2} \\
&\le \|X\|^r\left(\frac{1}{p}\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{pr/2} + \frac{1}{q}\langle B^*B\hat{k}_\lambda, \hat{k}_\lambda\rangle^{qr/2}\right) \quad \text{(by Lemma 2.1)} \\
&\le \|X\|^r\left(\frac{1}{p}\langle (A^*A)^{pr/2}\hat{k}_\lambda, \hat{k}_\lambda\rangle + \frac{1}{q}\langle (B^*B)^{qr/2}\hat{k}_\lambda, \hat{k}_\lambda\rangle\right) \quad \text{(by Lemma 2.4)} \\
&= \|X\|^r\left\langle \frac{1}{p}(A^*A)^{pr/2} + \frac{1}{q}(B^*B)^{qr/2}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&\le \|X\|^r\mathbf{ber}\left(\frac{1}{p}(A^*A)^{pr/2} + \frac{1}{q}(B^*B)^{qr/2}\right).
\end{aligned}$$

Therefore

$$\mathbf{ber}^r(A^*XB) \le \|X\|^r\mathbf{ber}\left(\frac{1}{p}(A^*A)^{pr/2} + \frac{1}{q}(B^*B)^{qr/2}\right),$$

and so we get the part (i). For the proof of the part (ii) we have

$$\begin{aligned}
|\langle A^*XB\hat{k}_\lambda, \hat{k}_\lambda\rangle| &= |\langle XB\hat{k}_\lambda, A\hat{k}_\lambda\rangle| \\
&\le \langle |X|^{2\alpha}B\hat{k}_\lambda, B\hat{k}_\lambda\rangle^{1/2}\langle |X^*|^{2(1-\alpha)}A\hat{k}_\lambda, A\hat{k}_\lambda\rangle^{1/2} \quad \text{(by Lemma 2.2)} \\
&\le \tfrac{1}{2}(\langle B^*|X|^{2\alpha}B\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle A^*|X^*|^{2(1-\alpha)}A\hat{k}_\lambda, \hat{k}_\lambda\rangle) \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \tfrac{1}{2}(\langle B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A\hat{k}_\lambda, \hat{k}_\lambda\rangle) \\
&\le \tfrac{1}{2}\mathbf{ber}(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A).
\end{aligned}$$

Now, the result follows by taking supremum on $\lambda \in \Omega$. ■

Theorem 2.6: *Let $A, B, X, Y \in \mathbb{B}(\mathscr{H}(\Omega))$. Then for every* $0 \le \alpha \le 1$

$$\mathbf{ber}(A^*XB + B^*YA) \le \tfrac{1}{2}\mathbf{ber}(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A + A^*|Y|^{2\alpha}A + B^*|Y^*|^{2(1-\alpha)}B). \tag{2.2}$$

Proof: Applying Lemma 2.2 and the arithmetic-geometric mean inequality, for any $\hat{k}_\lambda \in \mathscr{H}(\Omega)$, we have

$$\begin{aligned}
|\langle (A^*XB + B^*YA)\hat{k}_\lambda, \hat{k}_\lambda\rangle| &\leq |\langle A^*XB\hat{k}_\lambda, \hat{k}_\lambda\rangle| + |\langle B^*YA\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\
&= |\langle XB\hat{k}_\lambda, A\hat{k}_\lambda\rangle| + |\langle YA\hat{k}_\lambda, B\hat{k}_\lambda\rangle| \\
&\leq \langle |X|^{2\alpha} B\hat{k}_\lambda, B\hat{k}_\lambda\rangle^{1/2} \langle |X^*|^{2(1-\alpha)} A\hat{k}_\lambda, A\hat{k}_\lambda\rangle^{1/2} \\
&\quad + \langle |Y|^{2\alpha} A\hat{k}_\lambda, A\hat{k}_\lambda\rangle^{1/2} \langle |Y^*|^{2(1-\alpha)} B\hat{k}_\lambda, B\hat{k}_\lambda\rangle^{1/2} \\
&\leq \frac{1}{2}\left(\langle B^*|X|^{2\alpha} B\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle A^*|X^*|^{2(1-\alpha)} A\hat{k}_\lambda, \hat{k}_\lambda\rangle\right) \\
&\quad + \frac{1}{2}\left(\langle A^*|Y|^{2\alpha} A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + \langle B^*|Y^*|^{2(1-\alpha)} B\hat{k}_\lambda, \hat{k}_\lambda\rangle\right) \\
&= \frac{1}{2}\Big(\langle (B^*|X|^{2\alpha} B + A^*|X^*|^{2(1-\alpha)} A + A^*|Y|^{2\alpha} A \\
&\quad + B^*|Y^*|^{2(1-\alpha)} B)\hat{k}_\lambda, \hat{k}_\lambda\rangle\Big) \\
&\leq \frac{1}{2}\mathbf{ber}(B^*|X|^{2\alpha} B + A^*|X^*|^{2(1-\alpha)} A + A^*|Y|^{2\alpha} A \\
&\quad + B^*|Y^*|^{2(1-\alpha)} B).
\end{aligned}$$

Now, by taking supremum over $\lambda \in \Omega$, we get the desired inequality. ∎

Inequality (2.2) yields several Berezin number inequalities as special cases. A sample of elementary inequalities are demonstrated in the following remarks.

Remark 2.7: By letting $\alpha = \frac{1}{2}$ in inequality (2.2), we get the following inequalities:

$$\begin{aligned}
\mathbf{ber}(A^*XB + B^*YA) &\leq \frac{1}{2}\mathbf{ber}(B^*|X|B + A^*|X^*|A + A^*|Y|A + B^*|Y^*|B) \\
&\leq \frac{1}{2}\mathbf{ber}(B^*|X|B + A^*|X^*|A) + \frac{1}{2}\mathbf{ber}(A^*|Y|A + B^*|Y^*|B).
\end{aligned}$$

Remark 2.8: Putting $\alpha = \frac{1}{2}$, $A = I$ and $X = Y = A$, in inequality (2.2) we can obtain the following inequality:

$$\mathbf{ber}(AB + B^*A) \leq \frac{1}{2}\mathbf{ber}(|A| + |A^*|) + \frac{1}{2}\mathbf{ber}(B^*(|A| + |A^*|)B).$$

In the next result we find an upper bound for power of the Berezin number of $A^\alpha XB^{1-\alpha}$, in which $0 \leq \alpha \leq 1$.

Theorem 2.9: *Suppose that $A, B, X \in \mathbb{B}(\mathscr{H}(\Omega))$ such that A,B are positive. Then*

$$\boldsymbol{ber}^r(A^\alpha XB^{1-\alpha}) \leq \|X\|^r \left(\boldsymbol{ber}(\alpha A^r + (1-\alpha)B^r) - \inf_{\|\hat{k}_\lambda\|=1} \eta(\hat{k}_\lambda)\right), \tag{2.3}$$

in which $\eta(\hat{k}_\lambda) = r_0(\langle A^r\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2} - \langle B^r\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2})^2$, $r_0 = \min\{\alpha, 1-\alpha\}$, $r \geq 2$ *and* $0 \leq \alpha \leq 1$.

Proof: Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathscr{H}(\Omega)$. Then

$$\begin{aligned}
|\langle A^\alpha XB^{1-\alpha}\hat{k}_\lambda,\hat{k}_\lambda\rangle|^r &= |\langle XB^{1-\alpha}\hat{k}_\lambda, A^\alpha\hat{k}_\lambda\rangle|^r \\
&\le \|x\|^r\|B^{1-\alpha}\hat{k}_\lambda\|^r\|A^\alpha\hat{k}_\lambda\|^r \quad \text{(by the Cauchy-Schwarz inequality)} \\
&= \|X\|^r\langle B^{2(1-\alpha)}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{r/2}\langle A^{2\alpha}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{r/2} \\
&\le \|X\|^r\langle A^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\alpha}\langle B^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1-\alpha} \quad \text{(by Lemma 2.4)} \\
&\le \|X\|^r\Big(\langle(\alpha A^r+(1-\alpha)B^r)\hat{k}_\lambda,\hat{k}_\lambda\rangle - r_0(\langle A^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2} \\
&\quad -\langle B^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2})^2\Big) \text{ (by (2.1))} \\
&= \|X\|^r(\langle(\alpha A^r+(1-\alpha)B^r)\hat{k}_\lambda,\hat{k}_\lambda\rangle) - \|X\|^r r_0(\langle A^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2} \\
&\quad -\langle B^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2})^2 \\
&\le \|X\|^r\Big(\mathbf{ber}(\alpha A^r+(1-\alpha)B^r) - r_0(\langle A^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2}-\langle B^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2})^2\Big).
\end{aligned}$$

Taking the supremum over $\lambda\in\Omega$, we deduce the desired result. ■

Remark 2.10: Putting $A=B=I$ in inequality (2.3), we get a generalization of the inequality (1.1).

The Heinz mean is defined as $H_\alpha(a,b)=(a^{1-\alpha}b^\alpha+a^\alpha b^{1-\alpha})/2$ for $a,b>0$ and $0\le\alpha\le 1$. The function H_α is symmetric about the point $\alpha=1/2$ and $\sqrt{ab}\le H_\alpha(a,b)\le(a+b)/2$ for all $\alpha\in[0,1]$. For further information about the Heinz mean we refer the reader to [14, 19, 20] and references therein. In the next theorem we can obtain an upper bound for the Berezin number involving power Heinz mean.

Theorem 2.11: *Suppose that $A,B,X\in\mathbb{B}(\mathscr{H}(\Omega))$ such that A,B are positive. Then*

$$\begin{aligned}
\boldsymbol{ber}^r\Big(\frac{A^\alpha XB^{1-\alpha}+A^{1-\alpha}XB^\alpha}{2}\Big) &\le \frac{\|X\|^r}{2}\left(\boldsymbol{ber}(A^r+B^r) - 2\inf_{\|\hat{k}_\lambda\|=1}\eta(\hat{k}_\lambda)\right) \\
&\le \frac{\|X\|^r}{2}\Big(\boldsymbol{ber}(\alpha A^r+(1-\alpha)B^r)+\boldsymbol{ber}((1-\alpha)A^r+\alpha B^r) \\
&\quad -2\inf_{\|\hat{k}_\lambda\|=1}\eta(\hat{k}_\lambda)\Big),
\end{aligned}$$

in which $\eta(\hat{k}_\lambda)=r_0(\langle A^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2}-\langle B^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2})^2$, $r_0=\min\{\alpha,1-\alpha\}$, $r\ge 2$ and $0\le\alpha\le 1$.

Proof: Using Theorem 2.9 for $\hat{k}_\lambda$, which is the normalized reproducing kernel of $\mathscr{H}(\Omega)$ we have

$$\begin{aligned}
&\left|\left\langle\frac{A^\alpha XB^{1-\alpha}+A^{1-\alpha}XB^\alpha}{2}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r \\
&\quad\le\left(\frac{|\langle A^\alpha XB^{1-\alpha}\hat{k}_\lambda,\hat{k}_\lambda\rangle|+|\langle A^{1-\alpha}XB^\alpha\hat{k}_\lambda,\hat{k}_\lambda\rangle|}{2}\right)^r
\end{aligned}$$

$$\le \frac{1}{2}(|\langle A^{\alpha}XB^{1-\alpha}\hat{k}_{\lambda},\hat{k}_{\lambda}\rangle|^r + |\langle A^{1-\alpha}XB^{\alpha}\hat{k}_{\lambda},\hat{k}_{\lambda}\rangle|^r) \qquad \text{(by the convexity of } f(t)=t^r)$$

$$\le \frac{\|X\|^r}{2}(\langle \alpha A^r + (1-\alpha)B^r\hat{k}_{\lambda},\hat{k}_{\lambda}\rangle + \langle (1-\alpha)A^r + \alpha B^r\hat{k}_{\lambda},\hat{k}_{\lambda}\rangle - 2r_0(\langle A^r\hat{k}_{\lambda},\hat{k}_{\lambda}\rangle^{1/2}$$
$$- \langle B^r\hat{k}_{\lambda},\hat{k}_{\lambda}\rangle^{1/2})^2)$$
$$= \frac{\|X\|^r}{2}(\langle (A^r+B^r)\hat{k}_{\lambda},\hat{k}_{\lambda}\rangle - 2r_0(\langle A^r\hat{k}_{\lambda},\hat{k}_{\lambda}\rangle^{1/2} - \langle B^r\hat{k}_{\lambda},\hat{k}_{\lambda}\rangle^{1/2})^2)$$
$$\le \frac{\|X\|^r}{2}(\mathbf{ber}(A^r+B^r) - 2r_0(\langle A^r\hat{k}_{\lambda},\hat{k}_{\lambda}\rangle^{1/2} - \langle B^r\hat{k}_{\lambda},\hat{k}_{\lambda}\rangle^{1/2})^2).$$

Taking supremum over $\lambda \in \Omega$, we get the first inequality. For the second inequality, we have

$$\frac{\|X\|^r}{2}\mathbf{ber}(A^r+B^r) = \frac{\|X\|^r}{2}\mathbf{ber}(\alpha A^r + (1-\alpha)B^r + (1-\alpha)A^r + \alpha B^r)$$
$$\le \frac{\|X\|^r}{2}(\mathbf{ber}(\alpha A^r + (1-\alpha)B^r) + \mathbf{ber}((1-\alpha)A^r + \alpha B^r)).$$

■

3. Berezin number inequalities of off-diagonal matrices

In this section, we obtain some inequalities involving powers of the Berezin number for the off-diagonal parts of 2×2 operator matrices.

Theorem 3.1: *Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(\mathscr{H}(\Omega_1) \oplus \mathscr{H}(\Omega_2))$ and f, g be nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t(t \in [0,\infty))$. Then*

$$\mathbf{ber}^r(T) \le \max\left\{ \mathbf{ber}\left(\frac{1}{p}f^{pr}(|\,C\,|) + \frac{1}{q}g^{qr}(|\,B^*\,|)\right),\right.$$
$$\left.\mathbf{ber}\left(\frac{1}{p}f^{pr}(|\,B\,|) + \frac{1}{q}g^{qr}(|\,C^*\,|)\right)\right\}, \tag{3.1}$$

in which $r \ge 1, p \ge q > 1$ such that $1/p + 1/q = 1$ and $pr \ge 2$.

Proof: For any $(\lambda_1,\lambda_2) \in \Omega_1 \times \Omega_2$, let $\hat{k}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel in $\mathscr{H}(\Omega_1) \oplus \mathscr{H}(\Omega_2)$. Then

$$|\,\langle T\hat{k}(\lambda_1,\lambda_2),\hat{k}(\lambda_1,\lambda_2)\rangle\,|^r$$
$$\le \|\, f(|\,T\,|)\,\hat{k}(\lambda_1,\lambda_2)\,\|^r \|\, g\left(|\,T^*\,|\right)\hat{k}(\lambda_1,\lambda_2)\,\|^r \qquad \text{(by Lemma 2.2(b))}$$
$$= \langle f^2(|\,T\,|)\hat{k}(\lambda_1,\lambda_2),\hat{k}(\lambda_1,\lambda_2)\rangle^{r/2}\langle g^2(|\,T^*\,|)\hat{k}(\lambda_1,\lambda_2),\hat{k}(\lambda_1,\lambda_2)\rangle^{r/2}$$
$$\le \frac{1}{p}\left\langle f^2\left(\begin{bmatrix} |\,C\,| & 0 \\ 0 & |\,B\,| \end{bmatrix}\right)\hat{k}(\lambda_1,\lambda_2),\hat{k}(\lambda_1,\lambda_2)\right\rangle^{pr/2}$$

$$+\frac{1}{q}\left\langle g^2\left(\begin{bmatrix} | B^* | & 0 \\ 0 & | C^* | \end{bmatrix}\right)\hat{k}(\lambda_1,\lambda_2),\hat{k}(\lambda_1,\lambda_2)\right\rangle^{qr/2}$$

(by Lemma 2.1(b))

$$\le \frac{1}{p}\left\langle\begin{bmatrix} f^{pr} | C | & 0 \\ 0 & f^{pr} | B | \end{bmatrix}\hat{k}(\lambda_1,\lambda_2),\hat{k}(\lambda_1,\lambda_2)\right\rangle$$

$$+\frac{1}{q}\left\langle\begin{bmatrix} g^{qr} | B^* | & 0 \\ 0 & g^{qr} | C^* | \end{bmatrix}\hat{k}(\lambda_1,\lambda_2),\hat{k}(\lambda_1,\lambda_2)\right\rangle$$

(by Lemma 2.4(a))

$$=\left\langle\begin{bmatrix} \frac{1}{p}f^{pr}(| C |)+\frac{1}{q}g^{qr}(| B^* |) & 0 \\ 0 & \frac{1}{p}f^{pr}(| B |)+\frac{1}{q}g^{qr}(| C^* |) \end{bmatrix}\hat{k}(\lambda_1,\lambda_2),\hat{k}(\lambda_1,\lambda_2)\right\rangle.$$

Hence

$$| \langle T\hat{k}(\lambda_1,\lambda_2),\hat{k}(\lambda_1,\lambda_2)\rangle |^r$$

$$\le \mathbf{ber}\left(\left\langle\begin{bmatrix} \frac{1}{p}f^{pr}(| C |)+\frac{1}{q}g^{qr}(| B^* |) & 0 \\ 0 & \frac{1}{p}f^{pr}(| B |)+\frac{1}{q}g^{qr}(| C^* |) \end{bmatrix}\hat{k}(\lambda_1,\lambda_2),\hat{k}(\lambda_1,\lambda_2)\right\rangle\right).$$

Now, applying the definition of Berezin number and Lemma 2.3(a), we have

$$\mathbf{ber}^r(T)\le \max\left\{\mathbf{ber}\left(\frac{1}{p}f^{pr}(| C |)+\frac{1}{q}g^{qr}(| B^* |)\right),\mathbf{ber}\left(\frac{1}{p}f^{pr}(| B |)+\frac{1}{q}g^{qr}(| C^* |)\right)\right\}.$$

■

Inequality (3.1) induces the following inequality.

Corollary 3.2: *Let* $T=\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]\in\mathbb{B}(\mathscr{H}(\Omega_1)\oplus\mathscr{H}(\Omega_2))$. *Then*

$$\boldsymbol{ber}^r(T)\le \tfrac{1}{2}\max\{\boldsymbol{ber}(| C |^{2r\alpha}+| B^* |^{2r(1-\alpha)}),\boldsymbol{ber}(| B |^{2r\alpha}+| C^* |^{2r(1-\alpha)})\}$$

for any $r\ge 1$ *and* $0\le\alpha\le 1$.

Proof: Letting $f(t)=t^{\alpha}$, $g(t)=t^{1-\alpha}$ and $p=q=2$ in inequality (3.1), we get the desired inequality. ■

Theorem 3.3: *Let* $T_i=\left[\begin{smallmatrix} 0 & B_i \\ C_i & 0 \end{smallmatrix}\right]\in\mathbb{B}(\mathscr{H}_2(\Omega)\oplus\mathscr{H}_1(\Omega))$ *for any* $i=1,2,\ldots,n$. *Then*

$$\boldsymbol{ber}_p^p(T_1,T_2,\ldots,T_n)\le\max\left\{\boldsymbol{ber}\left(\sum_{i=1}^n\alpha\,| C_i |^p+(1-\alpha)\,| B_i^* |^p\right),\right.$$
$$\left.\boldsymbol{ber}\left(\sum_{i=1}^n\alpha\,| B_i |^p+(1-\alpha)\,| C_i^* |^p\right)\right\} \quad (3.2)$$

for $0\le\alpha\le 1$ *and* $p\ge 2$.

Proof: For any $(\lambda_1,\lambda_2) \in \Omega_1 \times \Omega_2$, let $\hat{k}_{(\lambda_1,\lambda_2)} = [\begin{smallmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{smallmatrix}]$ be the normalized reproducing kernel in $\mathscr{H}(\Omega_1) \oplus \mathscr{H}(\Omega_2)$. Then

$$\sum_{i=1}^{n} \mid \langle T_i\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle \mid^p$$

$$= \sum_{i=1}^{n} (\mid \langle T_i\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle \mid^2)^{p/2}$$

$$\le \sum_{i=1}^{n} (\langle \mid T_i \mid^{2\alpha} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle \langle \mid T_i^* \mid^{2(1-\alpha)} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle)^{p/2}$$

(by Lemma 2.2(a))

$$\le \sum_{i=1}^{n} \langle \mid T_i \mid^{p\alpha} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle \langle \mid T_i^* \mid^{p(1-\alpha)} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle \quad \text{(by Lemma 2.4(a))}$$

$$\le \sum_{i=1}^{n} \langle \mid T_i \mid^{p} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle^{\alpha} \langle \mid T_i^* \mid^{p} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle^{1-\alpha} \quad \text{(by Lemma 2.4(b))}$$

$$\le \sum_{i=1}^{n} (\alpha\langle \mid T_i \mid^{p} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle + (1-\alpha)\langle \mid T_i^* \mid^{p} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle)$$

(by Lemma 2.1(a))

$$= \sum_{i=1}^{n} \left(\alpha \left\langle \begin{bmatrix} \mid C_i \mid^p & 0 \\ 0 & \mid B_i \mid^p \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right.$$

$$\left. +(1-\alpha) \left\langle \begin{bmatrix} \mid B_i^* \mid^p & 0 \\ 0 & \mid C_i^* \mid^p \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right)$$

$$= \sum_{i=1}^{n} \left\langle \begin{bmatrix} \alpha \mid C_i \mid^p +(1-\alpha) \mid B_i^* \mid^p & 0 \\ 0 & \alpha \mid B_i \mid^p +(1-\alpha) \mid C_i^* \mid^p \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle$$

$$= \left\langle \begin{bmatrix} \sum_{i=1}^{n} \alpha \mid C_i \mid^p +(1-\alpha) \mid B_i^* \mid^p & 0 \\ 0 & \sum_{i=1}^{n} \alpha \mid B_i \mid^p +(1-\alpha) \mid C_i^* \mid^p \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle.$$

Therefore

$$\sum_{i=1}^{n} \mid \langle T_i\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle \mid^p$$

$$\leq \mathbf{ber}\left\langle \begin{bmatrix} \sum_{i=1}^{n} \alpha \mid C_i \mid^p + (1-\alpha) \mid B_i^* \mid^p & 0 \\ 0 & \sum_{i=1}^{n} \alpha \mid B_i \mid^p + (1-\alpha) \mid C_i^* \mid^p \end{bmatrix} \right.$$

$$\left. \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle.$$

By the definition of the Berezin number and Lemma 2.3(a), we get

$$\mathbf{ber}_p^p(T_1, T_2, \ldots, T_n)$$
$$\leq \max \left\{ \mathbf{ber}\left(\sum_{i=1}^{n} \alpha \mid C_i \mid^p + (1-\alpha) \mid B_i^* \mid^p \right), \mathbf{ber}\left(\sum_{i=1}^{n} \alpha \mid B_i \mid^p + (1-\alpha) \mid C_i^* \mid^p \right) \right\}.$$

■

Now, we would like to estimate the Berezin number for matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.

Proposition 3.4: *Let $T = \left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right] \in \mathbb{B}(\mathscr{H}_1(\Omega) \oplus \mathscr{H}_2(\Omega))$. Then*

$$\boldsymbol{ber}^r(T) \leq \frac{1}{2} \max\{\boldsymbol{ber}(|A|^r + |A^*|^r), \boldsymbol{ber}(|D|^r + |D^*|^r)\} \tag{3.3}$$

for $r \geq 1$.

Proof: Let $\hat{k}_{(\lambda_1,\lambda_2)} = \left[\begin{smallmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{smallmatrix}\right]$ be the normalized reproducing kernel of $\mathscr{H}(\Omega_1) \oplus \mathscr{H}(\Omega_2)$. Then

$$|\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \rangle|$$
$$\leq \langle |T|\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \rangle^{1/2} \langle |T^*|\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \rangle^{1/2}$$
$$\leq \frac{1}{2}\left\langle \begin{bmatrix} \mid A \mid & 0 \\ 0 & \mid D \mid \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle + \frac{1}{2}\left\langle \begin{bmatrix} \mid A^* \mid & 0 \\ 0 & \mid D^* \mid \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle$$
$$\leq \left(\frac{1}{2}\left\langle \begin{bmatrix} \mid A \mid & 0 \\ 0 & \mid D \mid \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle^r + \frac{1}{2}\left\langle \begin{bmatrix} \mid A^* \mid & 0 \\ 0 & \mid D^* \mid \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle^r \right)^{1/r}$$
$$\leq \left(\frac{1}{2}\left\langle \begin{bmatrix} \mid A \mid^r & 0 \\ 0 & \mid D \mid^r \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle + \frac{1}{2}\left\langle \begin{bmatrix} \mid A^* \mid^r & 0 \\ 0 & \mid D^* \mid^r \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right)^{1/r}$$
$$= \left(\left\langle \begin{bmatrix} \frac{1}{2}(\mid A \mid^r + |A^*|^r) & 0 \\ 0 & \frac{1}{2}(\mid D \mid^r + |D^*|^r) \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right)^{1/r}.$$

Thus

$$|\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle|^r \leq \left\langle \begin{bmatrix} \frac{1}{2}(|A|^r + |A^*|^r) & 0 \\ 0 & \frac{1}{2}(|D|^r + |D^*|^r) \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle.$$

Therefore

$$\mathbf{ber}^r(T) \leq \tfrac{1}{2}\max\{\mathbf{ber}(|A|^r + |A^*|^r), \mathbf{ber}(|D|^r + |D^*|^r)\}.$$

■

The following corollary deduces from inequalities (3.1) and (3.3) directly.

Corollary 3.5: *Let $T = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$ with $A, B, C, D \in \mathbb{B}(\mathscr{H}(\Omega))$. Then*

$$\boldsymbol{ber}(T) \leq \tfrac{1}{2}\max\{\boldsymbol{ber}(|C| + |B^*|), \boldsymbol{ber}(|B| + |C^*|)\} + \tfrac{1}{2}\max\{\boldsymbol{ber}(|A| + |A^*|), \boldsymbol{ber}(|D| + |D^*|)\}.$$

In particular,

$$\boldsymbol{ber}\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \leq \frac{1}{2}(\boldsymbol{ber}(|A| + |A^*|) + \boldsymbol{ber}(|B| + |B^*|)).$$

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