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Chebyshev type inequalities for Hilbert space operators

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ABSTRACT

We establish several operator extensions of the Chebyshev inequality. The main version deals with the Hadamard product of Hilbert space operators. More precisely, we prove that if $\mathscr{A}$ is a C^*-algebra, T is a compact Hausdorff space equipped with a Radon measure μ, $\alpha : T \to [0, +\infty)$ is a measurable function and $(A_t)_{t\in T}$, $(B_t)_{t\in T}$ are suitable continuous fields of operators in $\mathscr{A}$ having the synchronous Hadamard property, then

$$\int_T \alpha(s)d\mu(s) \int_T \alpha(t)(A_t \circ B_t)d\mu(t) \geq \left(\int_T \alpha(t)A_t d\mu(t)\right) \circ \left(\int_T \alpha(s)B_s d\mu(s)\right).$$

We apply states on C^*-algebras to obtain some versions related to synchronous functions. We also present some Chebyshev type inequalities involving the singular values of positive $n \times n$ matrices. Several applications are given as well.

1. Introduction and preliminaries

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ together with the operator norm $\|\cdot\|$. Let I stand for the identity operator. In the case when $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive (positive semidefinite for a matrix A) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$ and then we write $A \geq 0$. By a strictly positive operator (positive definite for a matrix) A, denoted by $A > 0$, we mean a positive invertible operator. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $B \geq A$ ($B > A$, resp.) if $B - A \geq 0$ ($B - A > 0$, resp.). For $A \in \mathbb{M}_n$, the singular values of A, denoted by $s_1(A), s_2(A), \cdots, s_n(A)$, are the eigenvalues of the positive matrix $|A| = (A^*A)^{\frac{1}{2}}$ enumerated as $s_1(A) \geq \cdots \geq s_n(A)$ with their multiplicities counted.

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The Gelfand map $f(t) \mapsto f(A)$ is an isometric $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a self-adjoint operator A and the C^*-algebra generated by I and A. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \geq g(A)$. Let f be a continuous real valued function on an interval J. The function f is called operator monotone (operator decreasing, resp.) if $A \leq B$ implies $f(A) \leq f(B)$ $(f(B) \leq f(A)$, resp.) for all $A, B \in \mathbb{B}_h^J(\mathscr{H})$, where $\mathbb{B}_h^J(\mathscr{H})$ is the set of all self-adjoint operators in $\mathbb{B}(\mathscr{H})$, whose spectra are contained in J; cf. [10].

Given an orthonormal basis $\{e_j\}$ of a Hilbert space $\mathscr{H}$, the Hadamard product $A \circ B$ of two operators $A, B \in \mathbb{B}(\mathscr{H})$ is defined by $\langle A \circ B e_i, e_j\rangle = \langle Ae_i, e_j\rangle\langle Be_i, e_j\rangle$. Clearly $A \circ B = B \circ A$. It is known that the Hadamard product can be presented by filtering the tensor product $A \otimes B$ through a positive linear map. In fact,

$$A \circ B = U^*(A \otimes B)U, \tag{1.1}$$

where $U : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is the isometry defined by $Ue_j = e_j \otimes e_j$; see [6]. It follows from (1.1) that if $A \geq 0$ and $B \geq 0$, then

$$A \circ B \geq 0. \tag{1.2}$$

For matrices, one easily observes [14] that the Hadamard product of $A = (a_{ij})$ and $B = (b_{ij})$ is $A \circ B = (a_{ij}b_{ij})$, a principal submatrix of the tensor product $A \otimes B = (a_{ij}B)_{1 \leq i,j \leq n}$. From now on when we deal with the Hadamard product of operators, we explicitly assume that an orthonormal basis is fixed.

The axiomatic theory of operator means has been developed by Kubo and Ando [8]. An operator mean is a binary operation σ defined on the set of strictly positive operators, if the following conditions hold:

(i) $A \leq C$, $B \leq D$ imply $A \,\sigma\, B \leq C \,\sigma\, D$;
(ii) $A_n \downarrow A$, $B_n \downarrow B$ imply $A_n \,\sigma\, B_n \downarrow A \,\sigma\, B$, where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \cdots$ and $A_n \to A$ as $n \to \infty$ in the strong operator topology;
(iii) $T^*(A \,\sigma\, B)T \leq (T^*AT) \,\sigma\, (T^*BT)$ $(T \in \mathbb{B}(\mathscr{H}))$;
(iv) $I \,\sigma\, I = I$.

There exists an affine order isomorphism between the class of operator means and the class of positive operator monotone functions f defined on $(0, \infty)$ with $f(1) = 1$ via $f(t)I = I \,\sigma\, (tI)$ $(t > 0)$. In addition, $A \,\sigma\, B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$ for all strictly positive operators A, B. The operator monotone function f is called the representing function of σ. Using a limit argument by $A_\varepsilon = A + \varepsilon I$, one can extend the definition of $A \,\sigma\, B$ to positive operators. The operator means corresponding to the operator monotone functions $f_{\sharp_\mu}(t) = t^\mu$ and $f_!(t) = \frac{2t}{1+t}$ on $[0, \infty)$ are the operator weighted geometric mean $A \sharp_\mu B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\mu A^{\frac{1}{2}}$ and the operator harmonic mean $A \,!\, B = 2(A^{-1} + B^{-1})^{-1}$, respectively.

Let us consider the real sequences $a = (a_1, \cdots, a_n)$, $b = (b_1, \cdots, b_n)$ and a non-negative sequence $w = (w_1, \cdots, w_n)$. Then the weighed Chebyshev function is defined by

$$T(w; a, b) := \sum_{j=1}^n w_j \sum_{j=1}^n w_j a_j b_j - \sum_{j=1}^n w_j a_j \sum_{j=1}^n w_j b_j.$$

In 1882, Chebyshev [3] proved that if a and b are monotone in the same sense, then $T(w; a, b) \geq 0$. Some integral generalizations of this inequality were given by Barza, Persson and Soria [1]. The Chebyshev inequality is a complement of the Grüss inequality; see [11] and the references therein.

A related notion is synchronicity. Two continuous functions $f, g : J \to \mathbb{R}$ are called synchronous on an interval J, if

$$\big(f(t) - f(s)\big)\big(g(t) - g(s)\big) \geq 0$$

for all $s, t \in J$. It is obvious that, if f, g are monotonic and have the same monotonicity, then they are synchronic. Dragomir [4] generalized the Chebyshev inequality for convex functions on a real normed space and applied his results to show that if $p_1, \cdots, p_n$ is a sequence of nonnegative numbers with $\sum_{j=1}^{n} p_j = 1$ and two sequences $(v_1, \cdots, v_n)$ and $(u_1, \cdots, u_n)$ in a real inner product space are synchronous, namely, $\langle v_k - v_j, u_k - u_j\rangle \geq 0$ for all $j, k = 1, \cdots, n$, then $\sum_{k=1}^{n} p_k\langle v_k, u_k\rangle \geq \langle \sum_{k=1}^{n} p_k v_k, \sum_{k=1}^{n} p_k u_k\rangle$. He also presented some Chebyshev inequalities for self-adjoint operators acting on a Hilbert space in [5].

In this paper we provide several operator extensions of the Chebyshev inequality. In the second section, we present our main results dealing with the Hadamard product of Hilbert space operators and weighted operator geometric means. The key notion is the so-called synchronous Hadamard property. More Chebyshev type inequalities regarding operator means are presented in Section 3. In Section 4, we apply states on C^*-algebras to obtain some versions related to synchronous functions. We present some Chebyshev type inequalities involving the singular values of positive $n \times n$ matrices in the last section.

2. Chebyshev inequality dealing with Hadamard product

This section is devoted to presentation of some operator Chebyshev inequalities dealing with the Hadamard product. The key notion is the so-called synchronous Hadamard property.

Let $\mathscr{A}$ be a C^*-algebra of operators acting on a Hilbert space and let T be a compact Hausdorff space equipped with a Radon measure μ. A field $(A_t)_{t\in T}$ of operators in $\mathscr{A}$ is called a continuous field of operators if the function $t \mapsto A_t$ is norm continuous on T and the function $t \mapsto \|A_t\|$ is integrable. Then one can form the Bochner integral $\int_T A_t d\mu(t)$, which is the unique element in $\mathscr{A}$ such that

$$\varphi\Big(\int_T A_t d\mu(t)\Big) = \int_T \varphi(A_t) d\mu(t)$$

for every linear functional φ in the norm dual $\mathscr{A}^*$ of $\mathscr{A}$. By [13, p. 78], since $t \mapsto A_t$ is a continuous function from T to $\mathscr{A}$, for every operator $A_t \in \mathscr{A}$ we can consider an element of the form

$$I_\lambda(A_t) = \sum_{k=1}^{n} A_t(s_k)\mu(E_k),$$

where the E_k's form a partition of T into disjoint Borel subsets, and

$$s_k \in E_k \subseteq \big\{t \in T : \big\|A_t - A_t(s_k)\big\| \leq \varepsilon\big\} \quad (1 \leq k \leq n),$$

with $\lambda = \{E_1, \cdots, E_n, \varepsilon\}$. Then $(I_\lambda(A_t))_{\lambda\in\Lambda}$ is a uniformly convergent net to $\int_T A_t d\mu(t)$. Let $\mathcal{C}(T, \mathscr{A})$ denote the set of continuous functions on T with values in $\mathscr{A}$. It is easy to see that the set $\mathcal{C}(T, \mathscr{A})$ is a C^*-algebra under the pointwise operations and the norm $\|(A_t)\| = \sup_{t\in T} \|A_t\|$; cf. [7]. Now since tensor product of two operators is norm continuous, for any operator $B \in \mathscr{A}$ we have

$$\int_T (A_t \otimes B) d\mu(t) = \Big(\int_T A_t d\mu(t)\Big) \otimes B.$$

Also, for any operator $C \in \mathscr{A}$

$$\int_T (C^* A_t C) d\mu(t) = C^* \Big(\int_T A_t d\mu(t)\Big) C.$$

Therefore

$$\int_T (A_t \circ B) d\mu(t) = \int_T V^*(A_t \otimes B)V d\mu(t) = V^* \int_T (A_t \otimes B) d\mu(t) V$$
$$= V^*\Big(\int_T A_t d\mu(t) \otimes B\Big)V = \int_T A_t d\mu(t) \circ B \quad (A_t, B \in \mathscr{A}). \tag{2.1}$$

Let us give our key definition.

Definition 2.1. Two fields $(A_t)_{t\in T}$ and $(B_t)_{t\in T}$ are said to have the synchronous Hadamard property if

$$(A_t - A_s) \circ (B_t - B_s) \geq 0$$

for all $s, t \in T$.

The first result reads as follows.

Theorem 2.2. *Let $\mathscr{A}$ be a C^*-algebra, T be a compact Hausdorff space equipped with a Radon measure μ, let $(A_t)_{t\in T}$ and $(B_t)_{t\in T}$ be fields in $\mathcal{C}(T, \mathscr{A})$ with the synchronous Hadamard property and let $\alpha : T \to [0, +\infty)$ be a measurable function. Then*

$$\int_T \alpha(s) d\mu(s) \int_T \alpha(t)(A_t \circ B_t) d\mu(t) \geq \Big(\int_T \alpha(t) A_t d\mu(t)\Big) \circ \Big(\int_T \alpha(s) B_s d\mu(s)\Big). \tag{2.2}$$

Proof. We have

$$\int_T \alpha(s) d\mu(s) \int_T \alpha(t)(A_t \circ B_t) d\mu(t) - \Big(\int_T \alpha(t) A_t d\mu(t)\Big) \circ \Big(\int_T \alpha(s) B_s d\mu(s)\Big)$$
$$= \int_T \int_T \alpha(s)\alpha(t)(A_t \circ B_t) d\mu(t) d\mu(s) - \int_T \Big(\int_T \alpha(t) A_t d\mu(t)\Big) \circ \alpha(s) B_s d\mu(s) \quad \big(\text{by } (2.1)\big)$$
$$= \int_T \int_T \alpha(s)\alpha(t)(A_t \circ B_t) d\mu(t) d\mu(s) - \int_T \int_T \alpha(t)\alpha(s)(A_t \circ B_s) d\mu(t) d\mu(s) \quad \big(\text{by } (2.1)\big)$$
$$= \int_T \int_T \big(\alpha(s)\alpha(t)(A_t \circ B_t) - \alpha(t)\alpha(s)(A_t \circ B_s)\big) d\mu(t) d\mu(s)$$
$$= \frac{1}{2} \int_T \int_T \big[\alpha(s)\alpha(t)(A_t \circ B_t) - \alpha(t)\alpha(s)(A_t \circ B_s) - \alpha(s)\alpha(t)(A_s \circ B_t) + \alpha(t)\alpha(s)(A_s \circ B_s)\big] d\mu(t) d\mu(s)$$
$$= \frac{1}{2} \int_T \int_T \big[\alpha(s)\alpha(t)(A_t - A_s) \circ (B_t - B_s)\big] d\mu(t) d\mu(s)$$
$$\geq 0 \quad \big(\text{since the fields } (A_t) \text{ and } (B_t) \text{ have the synchronous Hadamard property}\big). \qquad \square$$

In the discrete case $T = \{1, \cdots, n\}$, set $\alpha(i) = \omega_i \geq 0$ $(1 \leq i \leq n)$. Then Theorem 2.2 forces the following result.

Corollary 2.3. *(See [9, Theorem 2.1].) Let $A_1 \geq \cdots \geq A_n$, $B_1 \geq \cdots \geq B_n$ be self-adjoint operators and $\omega_1, \cdots, \omega_n$ be positive numbers. Then*

$$\sum_{j=1}^{n} \omega_j \sum_{j=1}^{n} \omega_j (A_j \circ B_j) \geq \left(\sum_{j=1}^{n} \omega_j A_j \right) \circ \left(\sum_{j=1}^{n} \omega_j B_j \right).$$

3. More Chebyshev type inequalities regarding operator means

Recall that a continuous function $f : J \to \mathbb{R}$ is super-multiplicative if $f(xy) \geq f(x)f(y)$, for all $x, y \in J$. In the next result we need the notion of increasing field. Let T be a compact Hausdorff space as well as a totally ordered set under an order $\preceq$. We say (A_t) is an increasing (decreasing, resp.) field, whenever $t \preceq s$ implies that $A_t \leq A_s$ ($A_t \geq A_s$, resp.). In this section we frequently employ some known relationships between the Hadamard product and operator means; cf. [12, Chapter 6].

Theorem 3.1. *Let $\mathscr{A}$ be a C^*-algebra, T be a compact Hausdorff space equipped with a Radon measure μ being also a totally ordered set, let $(A_t)_{t \in T}$, $(B_t)_{t \in T}$, $(C_t)_{t \in T}$, $(D_t)_{t \in T}$ be positive increasing fields in $\mathcal{C}(T, \mathscr{A})$, let $\alpha : T \to [0, +\infty)$ be a measurable function and σ be an operator mean with the super-multiplicative representing function. Then*

$$\int_T \alpha(s) d\mu(s) \int_T \alpha(t) \big((A_t \circ B_t) \,\sigma\, (C_t \circ D_t) \big) d\mu(t) \geq \left(\int_T \alpha(t)(A_t \,\sigma\, C_t) d\mu(t) \right) \circ \left(\int_T \alpha(s)(B_s \,\sigma\, D_s) d\mu(s) \right).$$

Proof. Let $s, t \in T$. Without loss of generality assume that $s \preceq t$. Then by the property (i) of the operator mean we have $0 \leq (A_t \,\sigma\, B_t) - (A_s \,\sigma\, B_s)$. Then

$$\begin{aligned}
&\int_T \alpha(s) d\mu(s) \int_T \big(\alpha(t)(A_t \circ B_t) \,\sigma\, (C_t \circ D_t) \big) d\mu(t) - \left(\int_T \alpha(t)(A_t \,\sigma\, C_t) d\mu(t) \right) \circ \left(\int_T \alpha(s)(B_s \,\sigma\, D_s) d\mu(s) \right) \\
&\quad = \int_T \int_T \alpha(s)\alpha(t) \big((A_t \circ B_t) \,\sigma\, (C_t \circ D_t) \big) d\mu(t) d\mu(s) \\
&\qquad - \int_T \int_T \alpha(t)\alpha(s) \big((A_t \,\sigma\, C_t) \circ (B_s \,\sigma\, D_s) \big) d\mu(t) d\mu(s) \quad \big(\text{by (2.1)}\big) \\
&\quad \geq \int_T \int_T \alpha(s)\alpha(t) \big((A_t \,\sigma\, C_t) \circ (B_t \,\sigma\, D_t) \big) d\mu(t) d\mu(s) \\
&\qquad - \int_T \int_T \alpha(t)\alpha(s) \big((A_t \,\sigma\, C_t) \circ (B_s \,\sigma\, D_s) \big) d\mu(t) d\mu(s) \quad \big(\text{by [12, Theorem 6.7]}\big) \\
&\quad = \frac{1}{2} \int_T \int_T \alpha(s)\alpha(t) \big[\big((A_t \,\sigma\, C_t) \circ (B_t \,\sigma\, D_t) \big) - \big((A_t \,\sigma\, C_t) \circ (B_s \,\sigma\, D_s) \big) \\
&\qquad + \big((A_s \,\sigma\, C_s) \circ (B_s \,\sigma\, D_s) \big) - \big((A_s \,\sigma\, C_s) \circ (B_t \,\sigma\, D_t) \big) \big] d\mu(t) d\mu(s) \\
&\quad = \frac{1}{2} \int_T \int_T \alpha(s)\alpha(t) \big[(A_t \,\sigma\, C_t) - (A_s \,\sigma\, C_s) \big] \circ \big[(B_t \,\sigma\, D_t) - (B_s \,\sigma\, D_s) \big] d\mu(t) d\mu(s) \\
&\quad \geq 0 \quad \big(\text{by (1.2)}\big). \qquad \square
\end{aligned}$$

A discrete version of the theorem above is the following result obtained by taking $T = \{1, \cdots, n\}$.

Corollary 3.2. *Let $A_{i+1} \geq A_i \geq 0$, $B_{i+1} \geq B_i \geq 0$, $C_{i+1} \geq C_i \geq 0$, $D_{i+1} \geq D_i \geq 0$ $(1 \leq i \leq n-1)$, $\omega_1, \cdots, \omega_n$ be positive numbers and σ be an operator mean with the super-multiplicative representing function. Then*

$$\sum_{j=1}^{n} \omega_j \sum_{j=1}^{n} \omega_j \left[(A_j \circ B_j)\, \sigma\, (C_j \circ D_j)\right] \geq \left(\sum_{j=1}^{n} \omega_j (A_j\, \sigma\, C_j)\right) \circ \left(\sum_{j=1}^{n} \omega_j (B_j\, \sigma\, D_j)\right).$$

Theorem 3.3. *Let $\mathscr{A}$ be a C^*-algebra, T be a compact Hausdorff space equipped with a Radon measure μ being also a totally ordered set, let $(A_t)_{t \in T}$, $(B_t)_{t \in T}$ be positive increasing fields in $\mathcal{C}(T, \mathscr{A})$ and let $\alpha : T \to [0, +\infty)$ be a measurable function. Then*

$$\int_T \alpha(s) d\mu(s) \int_T \alpha(t)(A_t \circ B_t) d\mu(t) \geq \left(\int_T \alpha(t)(A_t \sharp_\mu B_t) d\mu(t)\right) \circ \left(\int_T \alpha(s)(A_s \sharp_{1-\mu} B_s) d\mu(s)\right)$$

for all $\mu \in [0, 1]$.

Proof. Let $s, t \in T$. Without loss of generality assume that $s \preceq t$. Then by the property (i) of the operator mean, we have $0 \leq (A_t \sharp_\mu B_t) - (A_s \sharp_\mu B_s)$ and $0 \leq (A_t \sharp_{1-\mu} B_t) - (A_s \sharp_{1-\mu} B_s)$. Then

$$\begin{aligned}
&\int_T \alpha(s) d\mu(s) \int_T \alpha(t)(A_t \circ B_t) d\mu(t) - \left(\int_T \alpha(t)(A_t \sharp_\mu B_t) d\mu(t)\right) \circ \left(\int_T \alpha(s)(A_s \sharp_{1-\mu} B_s) d\mu(s)\right) \\
&= \int_T \int_T \alpha(s)\alpha(t)(A_t \circ B_t) d\mu(t) d\mu(s) - \int_T \int_T \alpha(t)\alpha(s)\big((A_t \sharp_\mu B_t) \circ (A_s \sharp_{1-\mu} B_s)\big) d\mu(t) d\mu(s) \quad \big(\text{by (2.1)}\big) \\
&\geq \int_T \int_T \alpha(s)\alpha(t)\big((A_t \sharp_\mu B_t) \circ (A_t \sharp_{1-\mu} B_t)\big) d\mu(t) d\mu(s) \\
&\quad - \int_T \int_T \alpha(t)\alpha(s)\big((A_t \sharp_\mu B_t) \circ (A_s \sharp_{1-\mu} B_s)\big) d\mu(t) d\mu(s) \quad \big(\text{by [12, Theorem 6.6]}\big) \\
&= \frac{1}{2} \int_T \int_T \big[\alpha(s)\alpha(t)\big((A_t \sharp_\mu B_t) \circ (A_t \sharp_{1-\mu} B_t)\big) - \alpha(t)\alpha(s)\big((A_t \sharp_\mu B_t) \circ (A_s \sharp_{1-\mu} B_s)\big) \\
&\quad + \alpha(t)\alpha(s)\big((A_s \sharp_\mu B_s) \circ (A_s \sharp_{1-\mu} B_s)\big) - \alpha(s)\alpha(t)\big((A_s \sharp_\mu B_s) \circ (A_t \sharp_{1-\mu} B_t)\big)\big] d\mu(t) d\mu(s) \\
&= \frac{1}{2} \int_T \int_T \alpha(s)\alpha(t)\big[(A_t \sharp_\mu B_t) - (A_s \sharp_\mu B_s)\big] \circ \big[(A_t \sharp_{1-\mu} B_t) - (A_s \sharp_{1-\mu} B_s)\big] d\mu(t) d\mu(s) \\
&\geq 0 \quad \big(\text{by (1.2)}\big). \qquad \square
\end{aligned}$$

In the discrete case $T = \{1, \cdots, n\}$, setting $\alpha(i) = \omega_i \geq 0$ $(1 \leq i \leq n)$ in Theorem 3.3 we reach the next assertion.

Corollary 3.4. *Let $A_n \geq \cdots \geq A_1 \geq 0$, $B_n \geq \cdots \geq B_1 \geq 0$ and $\omega_1, \cdots, \omega_n$ be positive numbers. Then*

$$\sum_{j=1}^{n} \omega_j \sum_{j=1}^{n} \omega_j (A_j \circ B_j) \geq \left(\sum_{j=1}^{n} \omega_j (A_j \sharp_\mu B_j)\right) \circ \left(\sum_{j=1}^{n} \omega_j (A_j \sharp_{1-\mu} B_j)\right)$$

for all $\mu \in [0, 1]$.

Proposition 3.5. *Let $f : [0, \infty) \to \mathbb{R}$ be a super-multiplicative and operator monotone function, $A_1 \geq \cdots \geq A_n \geq 0$, $B_1 \geq \cdots \geq B_n \geq 0$ and $\omega_1, \cdots, \omega_n$ be positive numbers. Then*

$$\sum_{j=1}^{n} \omega_j \sum_{j=1}^{n} \omega_j f(A_j \circ B_j) \geq \left(\sum_{j=1}^{n} \omega_j f(A_j)\right) \circ \left(\sum_{j=1}^{n} \omega_j f(B_j)\right).$$

Proof.

$$\begin{aligned}
&\sum_{j=1}^{n} \omega_j \sum_{j=1}^{n} \omega_j f(A_j \circ B_j) - \left(\sum_{j=1}^{n} \omega_j f(A_j)\right) \circ \left(\sum_{j=1}^{n} \omega_j f(B_j)\right) \\
&\quad \geq \sum_{j=1}^{n} \omega_j \sum_{j=1}^{n} \omega_j \big(f(A_j) \circ f(B_j)\big) - \left(\sum_{j=1}^{n} \omega_j f(A_j)\right) \circ \left(\sum_{j=1}^{n} \omega_j f(B_j)\right) \quad \text{(by [12, Theorem 6.3])} \\
&\quad = \sum_{i,j=1}^{n} \big[\omega_i \omega_j \big(f(A_j) \circ f(B_j)\big) - \omega_i \omega_j \big(f(A_i) \circ f(B_j)\big)\big] \\
&\quad = \frac{1}{2} \sum_{i,j=1}^{n} \omega_i \omega_j \big[\big(f(A_j) \circ f(B_j)\big) - \big(f(A_i) \circ f(B_j)\big) + \big(f(A_i) \circ f(B_i)\big) - \big(f(A_j) \circ f(B_i)\big)\big] \\
&\quad = \frac{1}{2} \sum_{i,j=1}^{n} \omega_i \omega_j \big[\big(f(A_j) - f(A_i)\big) \circ \big(f(B_j) - f(B_i)\big)\big] \geq 0 \quad \text{(by the operator monotonicity of } f\text{)}. \qquad \square
\end{aligned}$$

Example 3.6. Let $A_1 \geq \cdots \geq A_n \geq 0$, $B_1 \geq \cdots \geq B_n \geq 0$ and $\omega_1, \cdots, \omega_n$ be positive numbers. Then

$$\sum_{j=1}^{n} \omega_j \sum_{j=1}^{n} \omega_j (A_j \circ B_j)^p \geq \left(\sum_{j=1}^{n} \omega_j A_j^p\right) \circ \left(\sum_{j=1}^{n} \omega_j B_j^p\right)$$

for each $p \in [0, 1]$.

In the finite dimensional case we get the following.

Proposition 3.7. *Let $A_1 \geq \cdots \geq A_k \geq 0$, $B_1 \geq \cdots \geq B_k \geq 0$ be $n \times n$ matrices and $\omega_1, \cdots, \omega_k$ be positive numbers. Then*

$$\left(\sum_{j=1}^{k} \omega_j\right)^n \det\left(\sum_{j=1}^{k} \omega_j (A_j \circ B_j)\right) \geq \left(\sum_{j=1}^{k} \omega_j^n \det(A_j)\right)\left(\sum_{j=1}^{k} \omega_j^n \det(B_j)\right).$$

Proof.

$$\begin{aligned}
\left(\sum_{j=1}^{k} \omega_j\right)^n \det\left(\sum_{j=1}^{k} \omega_j (A_j \circ B_j)\right) &= \det\left(\sum_{j=1}^{k} \omega_j \sum_{j=1}^{k} \omega_j (A_j \circ B_j)\right) \\
&\geq \det\left(\left(\sum_{j=1}^{k} \omega_j A_j\right) \circ \left(\sum_{j=1}^{k} \omega_j B_j\right)\right) \quad \text{(by Corollary 2.3)} \\
&\geq \det\left(\sum_{j=1}^{k} \omega_j A_j\right) \det\left(\sum_{j=1}^{k} \omega_j B_j\right) \quad \text{(by [16, Theorem 7.27])} \\
&\geq \left(\sum_{j=1}^{k} \omega_j^n \det(A_j)\right)\left(\sum_{j=1}^{k} \omega_j^n \det(B_j)\right) \quad \text{(by [16, Theorem 7.7])}. \qquad \square
\end{aligned}$$

Proposition 3.8. *Let $A_1 \geq \cdots \geq A_k > 0$, $B_k \geq \cdots \geq B_1 \geq 0$ be $n \times n$ matrices and $\omega_1, \cdots, \omega_k$ be positive numbers. Then*

$$\left(\sum_{j=1}^{k} \omega_j\right)\left(\sum_{j=1}^{k} \omega_j \operatorname{tr}\left(A_j^{-1} B_j\right)\right) \geq \left(\sum_{j=1}^{k} \omega_j \operatorname{tr}(A_j)^{-1}\right)\left(\sum_{j=1}^{k} \omega_j \operatorname{tr}(B_j)\right).$$

Proof.

$$\begin{aligned}\left(\sum_{j=1}^{k} \omega_j\right)\left(\sum_{j=1}^{k} \omega_j \operatorname{tr}\left(A_j^{-1} B_j\right)\right) &\geq \left(\sum_{j=1}^{k} \omega_j\right)\left(\sum_{j=1}^{k} \omega_j \operatorname{tr}(A_j)^{-1} \operatorname{tr}(B_j)\right) \quad \text{(by [16, p. 224])} \\ &\geq \left(\sum_{j=1}^{k} \omega_j \operatorname{tr}(A_j)^{-1}\right)\left(\sum_{j=1}^{k} \omega_j \operatorname{tr}(B_j)\right) \quad \text{(by the Chebyshev inequality).}\end{aligned}$$

□

4. Chebyshev inequality for synchronous functions involving states

In this section, we apply the continuous functional calculus to synchronous functions and present some Chebyshev type inequalities involving states on C^*-algebras. Our main result of this section reads as follows.

Theorem 4.1. *Let $\mathscr{A}$ be a unital C^*-algebra, τ_1, τ_2 be states on $\mathscr{A}$ and $f, g : J \to \mathbb{R}$ be synchronous functions. Then*

$$\tau_1\big(f(A)g(A)\big) + \tau_2\big(f(B)g(B)\big) \geq \tau_1\big(f(A)\big)\tau_2\big(g(B)\big) + \tau_2\big(f(B)\big)\tau_1\big(g(A)\big) \tag{4.1}$$

for all $A, B \in \mathbb{B}_h^J(\mathscr{H})$.

Proof. For the synchronous functions f, g and for each $s, t \in J$

$$f(t)g(t) + f(s)g(s) - f(t)g(s) - f(s)g(t) \geq 0.$$

Fix $s \in J$. By the functional calculus for the operator A we have

$$f(A)g(A) + f(s)g(s) - f(A)g(s) - f(s)g(A) \geq 0,$$

whence

$$\tau_1\big(f(A)g(A)\big) + f(s)g(s) - \tau_1\big(f(A)\big)g(s) - f(s)\tau_1\big(g(A)\big) \geq 0.$$

Now for the operator B

$$\tau_1\big(f(A)g(A)\big) + f(B)g(B) - \tau_1\big(f(A)\big)g(B) - f(B)\tau_1\big(g(A)\big) \geq 0.$$

For the state τ_2 we have

$$\tau_1\big(f(A)g(A)\big) + \tau_2\big(f(B)g(B)\big) \geq \tau_1\big(f(A)\big)\tau_2\big(g(B)\big) + \tau_2\big(f(B)\big)\tau_1\big(g(A)\big). \qquad \square$$

Example 4.2.

(i) Let τ be a state on $\mathbb{B}(\mathscr{H})$ and $p, q > 0$. Since $f(t) = t^p$ and $g(t) = t^q$ are synchronous

$$\tau\big(A^{p+q}\big) + \tau\big(B^{p+q}\big) \geq \tau\big(A^p\big)\tau\big(B^q\big) + \tau\big(B^p\big)\tau\big(A^q\big) \quad (A, B \geq 0).$$

In a similar fashion, for self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$,

$$\tau\big(e^{\alpha A+\beta A}\big) + \tau\big(e^{\alpha B+\beta B}\big) \geq \tau\big(e^{\alpha A}\big)\tau\big(e^{\beta B}\big) + \tau\big(e^{\beta B}\big)\tau\big(e^{\alpha A}\big) \quad (\alpha, \beta \geq 0).$$

(ii) Let A, B be positive matrices, C be a positive definite matrix with $\operatorname{tr}(C) = \alpha$ and $p, q \geq 0$. Utilizing $\tau(A) = \frac{1}{\alpha}\operatorname{tr}(A \circ C)$ in (i) we have

$$\operatorname{tr}\big(A^{p+q} \circ C + B^{p+q} \circ C\big) \geq \frac{1}{\alpha}\big(\operatorname{tr}\big(A^p \circ C\big)\operatorname{tr}\big(B^q \circ C\big) + \operatorname{tr}\big(A^q \circ C\big)\operatorname{tr}\big(B^p \circ C\big)\big).$$

(iii) Let $f, g : J \to \mathbb{R}$ be synchronous functions. Then for $n \times n$ matrices A, B with spectra in J

$$\operatorname{tr}\big(f(A)g(A) + f(B)g(B)\big) \geq \frac{1}{n}\big(\operatorname{tr}\big(f(A)\big)\operatorname{tr}\big(g(B)\big) + \operatorname{tr}\big(g(A)\big)\operatorname{tr}\big(f(B)\big)\big).$$

Using Theorem 4.1 we obtain the next two corollaries.

Corollary 4.3. *Let $\mathscr{A}$ be a unital C^*-algebra, τ be a state on $\mathscr{A}$ and $f, g : J \to \mathbb{R}$ be synchronous functions. Then*

$$\tau\big(f(A)g(A)\big) \geq \tau\big(f(A)\big)\tau\big(g(A)\big)$$

for all operator $A \in \mathbb{B}_h^J(\mathscr{H})$.

Proof. Put $B = A$ in inequality (4.1) to get the result. □

Corollary 4.4. *(See [5, Theorem 1].) Let $f, g : J \to \mathbb{R}$ be synchronous functions. Then*

$$\big\langle f(A)g(A)x, x\big\rangle + \big\langle f(B)f(B)y, y\big\rangle \geq \big\langle f(A)x, x\big\rangle\big\langle g(B)y, y\big\rangle + \big\langle f(B)y, y\big\rangle\big\langle g(A)x, x\big\rangle$$

for all operators $A, B \in \mathbb{B}_h^J(\mathscr{H})$ and all unit vectors $x, y \in \mathscr{H}$.

Proof. Apply Theorem 4.1 to the states τ_1, τ_2 defined by $\tau_1(A) = \langle Ax, x\rangle$, $\tau_2(A) = \langle Ay, y\rangle$ $(A \in \mathbb{B}(\mathscr{H}))$ for fixed unit vectors $x, y \in \mathscr{H}$. □

Using the same strategy as in the proof of [15, Lemma 2.1] we get the next theorem.

Theorem 4.5. *Let τ be a state on a unital C^*-algebra $\mathscr{A}$ and $f : J \to [0, +\infty)$, $g : J \to \mathbb{R}$ be continuous functions such that f is decreasing and g is operator decreasing on a compact interval J. Then*

$$\tau\big(f(A)g(A)\big) \geq \tau\big(f(B)\big)\tau\big(g(A)\big)$$

for all $A, B \in \mathbb{B}_h^J(\mathscr{H})$ with $A \leq B$.

Proof. Put $\alpha = \inf_{x\in J} g(x)$ and $\beta = \sup_{x\in J} g(x)$. Then $\alpha \leq g(x) \leq \beta$ $(x \in J)$. So $\alpha I \geq g(B) \geq \beta I$, whence $\alpha \geq \tau(g(B)) \geq \beta$. Therefore, there exists a number $t_0 \in J$ such that $\tau(g(B)) = g(t_0)$.

Then if $x \in J$, $x \geq t_0$, then $g(x) \leq \tau(g(B))$, $f(x) \leq f(t_0)$, and if $x \in J$, $x \leq t_0$, then $g(x) \geq \tau(g(B))$, $f(x) \geq f(t_0)$. Hence

$$\big(f(x) - f(t_0)\big)\big(g(x) - \tau\big(g(B)\big)\big) \geq 0$$

for all $x \in J$. Thus

$$f(x)\big(g(x) - \tau\big(g(B)\big)\big) \geq f(t_0)\big(g(x) - \tau\big(g(B)\big)\big)$$

for all $x \in J$. Hence

$$f(A)\big(g(A) - \tau\big(g(B)\big)\big) \geq f(t_0)\big(g(A) - \tau\big(g(B)\big)\big).$$

Now

$$\begin{aligned} \tau\big(f(A)g(A)\big) - \tau\big(g(B)\big)\tau\big(f(A)\big) &= \tau\big(f(A)\big(g(A) - \tau\big(g(B)\big)\big)\big) \\ &\geq \tau\big(f(t_0)\big(g(A) - \tau\big(g(B)\big)\big)\big) \\ &= f(t_0)\big(\tau\big(g(A)\big) - \tau\big(g(B)\big)\big) \\ &\geq 0 \quad \text{(since } g \text{ is operator decreasing).} \quad \square \end{aligned}$$

Remark 4.6. The assumption $A \leq B$ is necessary in Theorem 4.5, since if $\tau(A) = \frac{1}{2}\operatorname{tr}(A)$ on $\mathbb{M}_2$, $f(t) = g(t) = \frac{1}{t}$, $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then we observe that $A \nleq B$ and $\tau(A^{-2}) = \frac{13}{72} < \frac{5}{12} = \tau(A^{-1})\tau(B^{-1})$.

Corollary 4.7. *Suppose that $f : J \to [0, +\infty)$ and $g : J \to \mathbb{R}$ are continuous functions such that f is decreasing and g is operator decreasing. Then*

$$\big\langle f(A)g(A)x, x\big\rangle - \big\langle f(B)x, x\big\rangle\big\langle g(A)x, x\big\rangle \geq 0$$

for all operators $A, B \in \mathbb{B}_h^J(\mathscr{H})$ such that $A \leq B$ and for all unit vectors $x \in \mathscr{H}$.

Proof. Apply Theorem 4.5 to the state τ defined by $\tau(A) = \langle Ax, x\rangle$ $(A \in \mathbb{B}(\mathscr{H}))$ for a fixed unit vector $x \in H$. $\square$

Using the same strategy as in the proof of Theorem 4.1 we get the next result.

Theorem 4.8. *Let $\mathscr{A}$ be a unital C^*-algebra, τ_1, τ_2 be states on $\mathscr{A}$ and $f, g : \mathbb{R} \to \mathbb{R}$ be synchronous functions. Then*

$$\tau_2\big(f(A)g(A)\big) + f\big(\tau_1(B)\big)g\big(\tau_1(B)\big) \geq f\big(\tau_1(A)\big)\tau_2\big(g(B)\big) + \tau_1\big(f(B)\big)g\big(\tau_2(A)\big) \tag{4.2}$$

for all self-adjoint operators A, B.

We now immediately get the next corollaries.

Corollary 4.9. *Let $f, g : J \to \mathbb{R}$ be synchronous functions. Then*

$$\big\langle f(A)g(A)x, x\big\rangle + f\big(\langle By, y\rangle\big)g\big(\langle By, y\rangle\big) \geq f\big(\langle Ax, x\rangle\big)\big\langle g(B)y, y\big\rangle + \big\langle f(B)y, y\big\rangle g\big(\langle Ax, x\rangle\big)$$

for all operators $A, B \in \mathbb{B}_h^J(\mathscr{H})$ and all unit vectors $x, y \in \mathscr{H}$.

Corollary 4.10. *(See [5, Theorem 2].) Let $f, g : J \to \mathbb{R}$ are synchronous functions. Then*

$$\big\langle f(A)g(A)x, x\big\rangle - f\big(\langle Ax, x\rangle\big)g\big(\langle Ax, x\rangle\big) \geq \big[\big\langle f(A)x, x\big\rangle - f\big(\langle Ax, x\rangle\big)\big]\big[g\big(\langle Ax, x\rangle\big) - \big\langle g(A)x, x\big\rangle\big]$$

for all operator $A \in \mathbb{B}_h^J(\mathscr{H})$ and any unit vector $x \in \mathscr{H}$.

Corollary 4.11. *Let $\mathscr{A}$ be a unital C^*-algebra, τ be a state on $\mathscr{A}$ and $f, g : \mathbb{R} \to \mathbb{R}$ be synchronous functions. Then*

$$\tau\big(f(B)g(B)\big) - \tau\big(f(B)\big)\tau\big(g(B)\big) \geq \big(\tau\big(f(B)\big) - f\big(\tau(A)\big)\big)\big(g\big(\tau(A)\big) - \tau\big(g(B)\big)\big)$$

for all self-adjoint operators A, B.

Proof. By using inequality (4.2) we get

$$\begin{aligned}
&\tau\big(f(B)g(B)\big) - \tau\big(f(B)\big)\tau\big(g(B)\big) \\
&\quad \geq f\big(\tau(A)\big)\tau\big(g(B)\big) + \tau\big(f(B)\big)g\big(\tau(A)\big) - f\big(\tau(A)\big)g\big(\tau(A)\big) - \tau\big(f(B)\big)\tau\big(g(B)\big) \\
&\quad = \big(\tau\big(f(B)\big) - f\big(\tau(A)\big)\big)\big(g\big(\tau(A)\big) - \tau\big(g(B)\big)\big). \qquad \square
\end{aligned}$$

By using Corollary 4.11 and the Davis–Choi–Jensen inequality [12] we obtain the next result.

Corollary 4.12. *Let $\mathscr{A}$ be a unital C^*-algebra, τ be a state on $\mathscr{A}$ and $f, g : \mathbb{R} \to \mathbb{R}$ be synchronous such that one of them is convex while the other is concave on $\mathbb{R}$. Then*

$$\tau\big(f(A)g(A)\big) - \tau\big(f(A)\big)\tau\big(g(A)\big) \geq \big(\tau\big(f(A)\big) - f\big(\tau(A)\big)\big)\big(g\big(\tau(A)\big) - \tau\big(g(A)\big)\big) \geq 0$$

for all self-adjoint operator A.

In the next proposition we establish a version of the Aczél–Chebyshev type inequality.

Proposition 4.13. *Let $\mathscr{A}$ be a unital C^*-algebra, τ be a state on $\mathscr{A}$ and f, g be continuous functions such that $0 \leq f(x) \leq \alpha$ and $0 \leq g(x) \leq \beta$ for some non-negative real numbers α, β. Then*

$$\big(\alpha\beta - \tau\big(f(B)g(B)\big)\big) \geq \big(\alpha - \tau\big(f(B)\big)\big)\big(\beta - \tau\big(g(A)\big)\big) \tag{4.3}$$

for all positive operators $A, B \in \mathscr{A}$.

Proof. If $\alpha = 0$ or $\beta = 0$, inequality (4.3) is trivial. Now assume that $\alpha > 0$ and $\beta > 0$. Then (4.3) is equivalent to the inequality

$$\big(1 - \tau\big(f(B)g(B)\big)\big) \geq \big(1 - \tau\big(f(B)\big)\big)\big(1 - \tau\big(g(A)\big)\big),$$

with $0 \leq f(x) \leq 1$ and $0 \leq g(x) \leq 1$. Then we have

$$\big(1 - \tau\big(f(B)g(B)\big)\big) \geq \big(1 - \tau\big(f(B)\big)\big) \geq \big(1 - \tau\big(f(B)\big)\big)\big(1 - \tau\big(g(A)\big)\big) \geq 0. \qquad \square$$

5. Chebyshev type inequalities involving singular values

In this section we deal with some singular value versions of the Chebyshev inequality for positive $n \times n$ matrices. We need the following known result.

Lemma 5.1. *(See [2, Corollary III.2.2].) Let A, B be $n \times n$ Hermitian matrices. Then*

$$\lambda_j^{\downarrow}(A + B) \geq \lambda_n^{\downarrow}(A) + \lambda_j^{\downarrow}(B) \quad (1 \leq j \leq n).$$

Theorem 5.2. *Let $f, g : [0, +\infty) \to [0, +\infty)$ be synchronous functions. Then*

$$s_j\big(f(A)g(A)\big) + s_j\big(f(B)g(B)\big) \geq s_n\big(f(A)\big)s_n\big(g(B)\big) + \frac{1}{2}\Big(s_j\big(g(A)\big)s_j\big(f(B)\big) + s_j\big(g(B)\big)s_j\big(f(A)\big)\Big)$$

for all positive matrices $A, B \in \mathbb{M}_n$ and all $j = 1, 2, \cdots, n$.

Proof. For synchronous functions f, g we have

$$f(t)g(t) + f(s)g(s) \geq f(t)g(s) + f(s)g(t) \quad (s, t \geq 0).$$

If we fix $s \in [0, +\infty)$, then

$$f(A)g(A) + f(s)g(s)I \geq f(A)g(s) + f(s)g(A).$$

Hence

$$\begin{aligned} s_j\big(f(A)g(A)\big) + f(s)g(s) &= s_j\big(f(A)g(A) + f(s)g(s)\big) \\ &\geq s_j\big(f(A)g(s) + f(s)g(A)\big) \\ &\geq s_n\big(f(A)g(s)\big) + s_j\big(f(s)g(A)\big) \quad \big(\text{by Lemma 5.1}\big) \\ &= s_n\big(f(A)\big)g(s) + f(s)s_j\big(g(A)\big) \quad (1 \leq j \leq n). \end{aligned}$$

Using functional calculus for B we get

$$s_j\big(f(A)g(A)\big) + f(B)g(B) \geq s_n\big(f(A)\big)g(B) + s_j\big(g(A)\big)f(B) \quad (1 \leq j \leq n).$$

Thus

$$\begin{aligned} s_j\big(f(A)g(A)\big) + s_j\big(f(B)g(B)\big) &\geq s_j\Big(s_n\big(f(A)\big)g(B) + s_j\big(g(A)\big)f(B)\Big) \\ &\geq s_n\Big(s_n\big(f(A)g(B)\big)\Big) + s_j\Big(s_j\big(g(A)\big)f(B)\Big) \quad \big(\text{by Lemma 5.1}\big) \\ &= s_n\big(f(A)\big)s_n\big(g(B)\big) + s_j\big(g(A)\big)s_j\big(f(B)\big) \quad (1 \leq j \leq n). \end{aligned} \tag{5.1}$$

In inequality (5.1), if we interchange the roles of A and B, then we get

$$s_j\big(f(B)g(B)\big) + s_j\big(f(A)g(A)\big) \geq s_n\big(f(B)\big)s_n\big(g(A)\big) + s_j\big(g(B)\big)s_j\big(f(A)\big) \quad (1 \leq j \leq n). \tag{5.2}$$

By (5.1) and (5.2)

$$s_j\big(f(A)g(A)\big) + s_j\big(f(B)g(B)\big) \geq s_n\big(f(A)\big)s_n\big(g(B)\big) + \frac{1}{2}\Big(s_j\big(g(A)\big)s_j\big(f(B)\big) + s_j\big(g(B)\big)s_j\big(f(A)\big)\Big)$$

for all $1 \leq j \leq n$. □

In the following example we show that the constant $\frac{1}{2}$ is the best possible one.

Example 5.3. For arbitrary synchronous functions $f, g : [0, +\infty) \to [0, +\infty)$, let us put $A = B = I_{n \times n}$. Then $s_j(f(A)g(B)) = s_j(f(B)g(B)) = f(1)g(1)$ and $s_j(f(B)g(A)) = s_j(g(B)f(A)) = f(1)g(1)$ $(1 \leq j \leq n)$. Thus

$$s_j\big(f(A)g(A)\big) + s_j\big(f(B)g(B)\big) = s_n\big(f(A)\big)s_n\big(g(B)\big) + \frac{1}{2}\Big(s_j\big(g(A)\big)s_j\big(f(B)\big) + s_j\big(g(B)\big)s_j\big(f(A)\big)\Big)$$

for all $j = 1, 2, \cdots, n$.

Using the same strategy as in the proof of Theorem 5.2 we get the next result.

Theorem 5.4. *Let $f, g : [0, +\infty) \to [0, +\infty)$ be synchronous functions. Then*

$$f\big(s_j(A)\big)g\big(s_j(A)\big) + s_j\big(f(B)g(B)\big) \geq f\big(s_j(A)\big)s_n\big(g(B)\big) + s_j\big(f(B)\big)g\big(s_j(A)\big)$$

for all positive matrices $A, B \in \mathbb{M}_n$ and for all $j = 1, 2, \cdots, n$.

Example 5.5. Let A, B be positive $n \times n$ matrices and $p, q > 0$. Then

$$s_j(A)^p s_j(A)^q + s_j\big(B^{p+q}\big) \geq s_j(A)^p s_n\big(B^q\big) + s_j\big(B^p\big)s_j(A)^q \quad (1 \leq j \leq n).$$

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