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Research Article

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Further refinements of generalized numerical radius inequalities for Hilbert space operators

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Abstract: In this paper, we show some refinements of generalized numerical radius inequalities involving the Young and Heinz inequality. In particular, we present

$$w_p^p(A_1^*T_1B_1,\dots,A_n^*T_nB_n)\le\frac{n^{1-\frac{1}{r}}}{2^{\frac{1}{r}}}\bigg\|\sum_{i=1}^n[B_i^*f^2(|T_i|)B_i]^{rp}+[A_i^*g^2(|T_i^*|)A_i]^{rp}\bigg\|^{\frac{1}{r}}-\inf_{\|x\|=1}\eta(x),$$

where $T_i,A_i,B_i\in\mathbb{B}(\mathscr{H})$ $(1\le i\le n)$, f and g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ for all $t\in[0,\infty)$, $p,r\ge 1$, $N\in\mathbb{N}$, and

$$\eta(x)=\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^N\Big(\sqrt[2^j]{\langle(A_i^*g^2(|T_i^*|)A_i)^px,x\rangle^{2^{j-1}-k_j}\langle(B_i^*f^2(|T_i|)B_i)^px,x\rangle^{k_j}}-\sqrt[2^j]{\langle(B_i^*f^2(|T_i|)B_i)^px,x\rangle^{k_j+1}\langle(A_i^*g^2(|T_i^*|)A_i)^px,x\rangle^{2^{j-1}-k_j-1}}\Big)^2.$$

Keywords: Euclidean operator radius, Heinz means, numerical radius, positive operator, Young inequality

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1 Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. In the case where $\dim\mathscr{H}=n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n\times n$ matrices with entries in the complex field. The numerical radius of $T\in\mathbb{B}(\mathscr{H})$ is defined by

$$w(T):=\sup\{|\langle Tx,x\rangle|:x\in\mathscr{H},\|x\|=1\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $T\in\mathbb{B}(\mathscr{H})$, $\frac{1}{2}\|T\|\le w(T)\le\|T\|$, see [4]. The quantity $w(T)$ is useful in studying perturbation, convergence and approximation problems as well as interactive methods, etc. For more information, see [1, 3, 5, 6, 11, 17, 18].

The classical Young inequality says that if $0\le\nu\le1$, then $a^\nu b^{1-\nu}\le\nu a+(1-\nu)b$ $(a,b>0)$. During the last decades several generalizations, reverses, refinements and applications of the Young inequality in various settings have been given; see [2, 10] and the references therein. A refinement of the scalar Young inequality is presented in [10] as follows:

$$a^\nu b^{1-\nu}\le\nu a+(1-\nu)b-r_0(a^{\frac{1}{2}}-b^{\frac{1}{2}})^2,\quad r_0=\min\{\nu,1-\nu\}.\tag{1.1}$$

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Recently, Sababheh and Choi obtained, in [14], a refinement of the Young inequality, that is,

$$a^{\nu}b^{1-\nu} \le \nu a + (1-\nu)b - S_N(\nu), \tag{1.2}$$

in which

$$S_N(\nu) := \sum_{j=1}^{N}\Big((-1)^{r_j}2^{j-1}\nu + (-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big)\Big(\sqrt[2^j]{b^{2^{j-1}-k_j}a^{k_j}} - \sqrt[2^j]{a^{k_j+1}b^{2^{j-1}-k_j-1}}\Big)^2,$$

where $N \in \mathbb{N}$, $r_j = [2^j\nu]$ and $k_j = [2^{j-1}\nu]$. Here $[x]$ is the greatest integer less than or equal to x. When $N = 1$, inequality (1.2) reduces to (1.1).

It follows from $\nu a + (1-\nu)b \le (\nu a^r + (1-\nu)b^r)^{\frac{1}{r}}$ $(r \ge 1)$ and inequality (1.1) that

$$a^{\nu}b^{1-\nu} \le (\nu a^r + (1-\nu)b^r)^{\frac{1}{r}} - S_N(\nu).$$

In particular, for $\nu = \frac{1}{2}$, we get

$$a^{\frac{1}{2}}b^{\frac{1}{2}} \le \Big(\frac{1}{2}\Big)^{\frac{1}{r}}(a^r + b^r)^{\frac{1}{r}} - \frac{1}{2}\sum_{j=1}^{N}\Big(\sqrt[2^j]{b^{2^{j-1}-k_j}a^{k_j}} - \sqrt[2^j]{a^{k_j+1}b^{2^{j-1}-k_j-1}}\Big)^2.$$

If $N = 1$, then we reach [10, inequality (2.1)] as follows:

$$a^{\frac{1}{2}}b^{\frac{1}{2}} \le \Big(\frac{1}{2}\Big)^{\frac{1}{r}}(a^r + b^r)^{\frac{1}{r}} - \frac{1}{2}(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2.$$

Let $T_i \in \mathbb{B}(\mathscr{H})$ $(1 \le i \le n)$. The Euclidean operator radius of $T_1, \dots, T_n$ is defined in [13] by

$$w_e(T_1, \dots, T_n) := \sup_{\|x\|=1}\Big(\sum_{i=1}^{n}|\langle T_i x, x\rangle|^2\Big)^{\frac{1}{2}}.$$

In [12], the functional w_p of operators $T_1, \dots, T_n$ for $p \ge 1$ is defined by

$$w_p(T_1, \dots, T_n) := \sup_{\|x\|=1}\Big(\sum_{i=1}^{n}|\langle T_i x, x\rangle|^p\Big)^{\frac{1}{p}}.$$

Let $T_1, \dots, T_n \in \mathbb{B}(\mathscr{H})$. Recently, Sheikhhosseini et al. showed in [16] that

$$w_p^p(A_1^*T_1B_1, \dots, A_n^*T_nB_n) \le \frac{n^{1-\frac{1}{r}}}{2^{\frac{1}{r}}}\Big\|\sum_{i=1}^{n}[B_i^*f^2(|T_i|)B_i]^{rp} + [A_i^*g^2(|T_i^*|)A_i]^{rp})\Big\|^{\frac{1}{r}} - \inf_{\|x\|=1}\zeta(x), \tag{1.3}$$

where

$$\zeta(x) = \frac{1}{2}\sum_{i=1}^{n}(\langle[B_i^*f^2(|T_i|)B_i]^p x, x\rangle^{\frac{1}{2}} - \langle[A_i^*f^2(|T_i^*|)A_i]^p x, x\rangle^{\frac{1}{2}})^2.$$

They also presented the following inequality:

$$w_p(T_1, \dots, T_n) \le \frac{1}{2}\Big[\sum_{i=1}^{n}\Big(\||T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)}\| - 2\inf_{\|x\|=1}\zeta_i(x)\Big)^p\Big]^{\frac{1}{p}}, \tag{1.4}$$

in which, $0 \le \alpha \le 1$, $p \ge 1$ and $\zeta_i(x) = \frac{1}{2}(\langle|T_i|^{2\alpha}x, x\rangle^{\frac{1}{2}} - \langle|T_i^*|^{2(1-\alpha)}x, x\rangle^{\frac{1}{2}})^2$.

In the same paper, they showed

$$w_p^p(T_1, \dots, T_n) \le \frac{1}{2}\Big\|\sum_{i=1}^{n}(|T_i|^{2\alpha p} + |T_i^*|^{2(1-\alpha)p})\Big\| - \inf_{\|x\|=1}\zeta(x), \tag{1.5}$$

where $\zeta(x) = \frac{1}{2}\sum_{i=1}^{n}(\langle|T_i|^{2\alpha p}x, x\rangle^{\frac{1}{2}} - \langle|T_i^*|^{2(1-\alpha)p}x, x\rangle^{\frac{1}{2}})^2$.

Moreover, they established the inequalities

$$w_p^p(T_1, \dots, T_n) \le \Big\|\sum_{i=1}^{n}\alpha|T_i|^p + (1-\alpha)|T_i^*|^p\Big\| - \inf_{\|x\|=1}\zeta(x), \tag{1.6}$$

and

$$w_p^r(|T_1|,\dots,|T_n|)w_q^r(|T_1^*|,\dots,|T_n^*|) \le \frac{r}{p}\Big\|\sum_{i=1}^n |T_i|^p\Big\| + \frac{r}{q}\Big\|\sum_{i=1}^n |T_i^*|^q\Big\| - \inf_{\|x\|=\|y\|=1}\delta(x,y), \tag{1.7}$$

where $\zeta(x) = \min\{\alpha, 1-\alpha\}\sum_{i=1}^n(\langle |T_i|^p x, x\rangle^{\frac{1}{2}} - \langle |T_i^*|^p x, x\rangle^{\frac{1}{2}})^2$ and

$$\delta(x,y) = \frac{r}{p}\Big(\sqrt[2]{\sum_{i=1}^n \langle |T_i|x,y\rangle^p} - \sqrt[2]{\sum_{i=1}^n \langle |T_i^*|x,y\rangle^q}\Big)^2.$$

Assume that $X \in \mathbb{B}(\mathscr{H})$. The mixed Heinz means are defined by

$$H_\alpha(A,B) = \frac{A^\alpha X B^{1-\alpha} + A^{1-\alpha} X B^\alpha}{2},$$

in which, $0 \le \alpha \le 1$ and $A, B \ge 0$, see [8]. In [15], Sattari et al. showed that

$$w^r(A^\alpha X B^{1-\alpha}) \le \|X\|^r \|\alpha A^r + (1-\alpha)B^r\|, \tag{1.8}$$

where $A, B, X \in \mathbb{B}(\mathscr{H})$, with A, B positive, $r \ge 2$ and $0 \le \alpha \le 1$.

Using inequality (1.8), they presented an upper bound for the Heinz means of matrices as follows:

$$w^r(H_\alpha(A,B)) \le \|X\|^r \Big\|\frac{A^r + B^r}{2}\Big\|. \tag{1.9}$$

In the present paper, we refine inequalities (1.3)–(1.9). We also find an upper bound for the functional w_p.

2 Main results

To prove our numerical radius inequalities, we need several known lemmas. The first lemma is a simple result of the classical Jensen, Young and a generalized mixed Cauchy–Schwarz inequalities [7, 9].

Lemma 2.1. *Let $a, b \ge 0$, $0 \le \nu \le 1$ and $r \ne 0$. Then the following hold:*

(a) $a^\nu b^{1-\nu} \le \nu a + (1-\nu)b \le (\nu a^r + (1-\nu)b^r)^{\frac{1}{r}}$ *for* $r \ge 1$.

(b) *If $T \in \mathbb{B}(\mathscr{H})$ and $x, y \in \mathscr{H}$ are any vectors, then*

$$|\langle Tx, y\rangle|^2 \le \langle |T|^{2\nu}x, x\rangle\langle |T^*|^{2(1-\nu)}y, y\rangle.$$

(c) *If f, g are nonnegative continuous functions on $[0,\infty)$ which satisfy the relation $f(t)g(t) = t$ $(t \in [0,\infty))$, then*

$$|\langle Tx, y\rangle| \le \|f(|T|)x\|\|g(|T^*|)x\|$$

for all $x, y \in \mathscr{H}$.

Lemma 2.2 (McCarty inequality [9]). *Let $T \in \mathbb{B}(\mathscr{H})$, $T \ge 0$, and let $x \in \mathscr{H}$ be a unit vector. Then*

(a) $\langle Tx, x\rangle^r \le \langle T^r x, x\rangle$ *for* $r \ge 1$,

(b) $\langle T^r x, x\rangle \le \langle Tx, x\rangle^r$ *for* $0 < r \le 1$.

Now, by using inequality (1.2), we get the first result.

Theorem 2.3. *Let $A, B, X \in \mathbb{B}(\mathscr{H})$, with A, B positive, $r \ge 2$ and $0 \le \nu \le 1$. Then*

$$w^r(A^\nu X B^{1-\nu}) \le \|X\|^r\Big[\|\nu A^r + (1-\nu)B^r\| - \inf_{\|x\|=1}\eta(x)\Big], \tag{2.1}$$

where

$$\eta(x) = \sum_{j=1}^N\Big((-1)^{r_j}2^{j-1}\nu + (-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big) \times \Big(\sqrt[2^j]{\langle B^r x, x\rangle^{2^{j-1}-k_j}\langle A^r x, x\rangle^{k_j}} - \sqrt[2^j]{\langle A^r x, x\rangle^{k_j+1}\langle B^r x, x\rangle^{2^{j-1}-k_j-1}}\Big)^2.$$

Proof. Let $x \in \mathscr{H}$ be a unit vector. Then, by Lemma 2.2 and inequality (1.2),

$$\begin{aligned}
|\langle A^{\nu}XB^{1-\nu}x, x\rangle|^{r} &= |\langle XB^{1-\nu}x, A^{\nu}x\rangle|^{r} \\
&\leq \|X\|^{r}\|B^{1-\nu}x\|^{r}\|A^{\nu}x\|^{r} \\
&= \|X\|^{r}\langle B^{2(1-\nu)}x, x\rangle^{\frac{r}{2}}\langle A^{2\nu}x, x\rangle^{\frac{r}{2}} \\
&\leq \|X\|^{r}\langle A^{r}x, x\rangle^{\nu}\langle B^{r}x, x\rangle^{1-\nu} \\
&\leq \|X\|^{r}[\nu\langle A^{r}x, x\rangle + (1-\nu)\langle B^{r}x, x\rangle] - \|X\|^{r}\sum_{j=1}^{N}\Big((-1)^{r_j}2^{j-1}\nu + (-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big) \\
&\quad \times \Big(\sqrt[2^j]{\langle B^{r}x, x\rangle^{2^{j-1}-k_j}\langle A^{r}x, x\rangle^{k_j}} - \sqrt[2^j]{\langle A^{r}x, x\rangle^{k_j+1}\langle B^{r}x, x\rangle^{2^{j-1}-k_j-1}}\Big)^{2}.
\end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$, with $\|x\| = 1$, in the above inequality we deduce the desired inequality. □

Remark 2.4. Let $N = 1$ in inequality (2.1). Then

$$w^{r}(A^{\nu}XB^{1-\nu}) \leq \|X\|^{r}\Big[\|\nu A^{r} + (1-\nu)B^{r}\| - \inf_{\|x\|=1}\eta(x)\Big], \tag{2.2}$$

in which, $\eta(x) = r_0(\langle A^{r}x, x\rangle^{\frac{1}{2}} - \langle B^{r}x, x\rangle^{\frac{1}{2}})^{2}$ and $r_0 = \min\{\nu, 1-\nu\}$. Hence, inequality (2.2) is a refinement of inequality (1.8).

Using Theorem 2.3 we can find an upper bound for Heinz means of matrices that is a refinement of (1.9).

Theorem 2.5. *Suppose $A, B, X \in \mathbb{B}(\mathscr{H})$, with A, B positive. Then*

$$w^{r}(H_{\nu}(A, B)) \leq \|X\|^{r}\Big[\Big\|\frac{A^{r}+B^{r}}{2}\Big\| - \frac{1}{2}\inf\zeta(x)\Big],$$

where $r \geq 2$, $0 \leq \nu \leq 1$, $n \in \mathbb{N}$, and

$$\begin{aligned}
\zeta(x) = \sum_{j=1}^{N}&\Big((-1)^{r_j}2^{j-1} + (-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big) \\
&\times \Big(\sqrt[2^j]{\langle B^{r}x, x\rangle^{2^{j-1}-k_j}\langle A^{r}x, x\rangle^{k_j}} - \sqrt[2^j]{\langle A^{r}x, x\rangle^{k_j+1}\langle B^{r}x, x\rangle^{2^{j-1}-k_j-1}}\Big)^{2}.
\end{aligned}$$

Proof. For a unit vector $x \in \mathscr{H}$, we have

$$\begin{aligned}
\Big|\Big\langle\frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2}x, x\Big\rangle\Big|^{r} &\leq \Big(\frac{|\langle A^{\nu}XB^{1-\nu}x, x\rangle| + |\langle A^{1-\nu}XB^{\nu}x, x\rangle|}{2}\Big)^{r} \\
&\leq \frac{|\langle A^{\nu}XB^{1-\nu}x, x\rangle|^{r} + |\langle A^{1-\nu}XB^{\nu}x, x\rangle|^{r}}{2} \\
&\leq \frac{\|X\|^{r}}{2}[\langle \nu A^{r} + (1-\nu)B^{r}x, x\rangle] - \frac{\|X\|^{r}}{2}\sum_{j=1}^{N}\Big((-1)^{r_j}2^{j-1}\nu + (-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big) \\
&\quad \times \Big(\sqrt[2^j]{b^{2^{j-1}-k_j}a^{k_j}} - \sqrt[2^j]{a^{k_j+1}b^{2^{j-1}-k_j-1}}\Big)^{2} \\
&\quad + \frac{\|X\|^{r}}{2}[\langle(1-\nu)A^{r} + \nu B^{r}x, x\rangle] - \frac{\|X\|^{r}}{2}\sum_{j=1}^{N}\Big((-1)^{r_j}2^{j-1}\nu + (-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big) \\
&\quad \times \Big(\sqrt[2^j]{\langle B^{r}x, x\rangle^{2^{j-1}-k_j}\langle A^{r}x, x\rangle^{k_j}} - \sqrt[2^j]{\langle A^{r}x, x\rangle^{k_j+1}\langle B^{r}x, x\rangle^{2^{j-1}-k_j-1}}\Big)^{2} \\
&= \|X\|^{r}\Big[\Big\langle\frac{A^{r}+B^{r}}{2}x, x\Big\rangle\Big] - \frac{\|X\|^{r}}{2}\sum_{j=1}^{N}\Big((-1)^{r_j}2^{j-1} + (-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big) \\
&\quad \times \Big(\sqrt[2^j]{\langle B^{r}x, x\rangle^{2^{j-1}-k_j}\langle A^{r}x, x\rangle^{k_j}} - \sqrt[2^j]{\langle A^{r}x, x\rangle^{k_j+1}\langle B^{r}x, x\rangle^{2^{j-1}-k_j-1}}\Big)^{2}.
\end{aligned}$$

If we take the supremum over $x \in \mathscr{H}$, with $\|x\| = 1$, then we deduce the desired inequality. □

In the next theorem we show a refinement of inequality (1.3).

Theorem 2.6. *Let $T_i, A_i, B_i \in \mathbb{B}(\mathscr{H})$ $(1 \le i \le n)$ and let f and g be nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then*

$$w_p^p(A_1^*T_1B_1,\ldots,A_n^*T_nB_n) \le \frac{n^{1-\frac{1}{r}}}{2^{\frac{1}{r}}}\Big\|\sum_{i=1}^{n}[B_i^*f^2(|T_i|)B_i]^{rp} + [A_i^*g^2(|T_i^*|)A_i]^{rp}\Big\|^{\frac{1}{r}} - \inf_{\|x\|=1}\eta(x),$$

where $p, r \ge 1$, $N \in \mathbb{N}$, and

$$\eta(x) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{N}\Big(\sqrt[2^j]{\langle (A_i^*g^2(|T_i^*|)A_i)^p x, x\rangle^{2^{j-1}-k_j}\langle (B_i^*f^2(|T_i|)B_i)^p x, x\rangle^{k_j}} - \sqrt[2^j]{\langle (B_i^*f^2(|T_i|)B_i)^p x, x\rangle^{k_j+1}\langle (A_i^*g^2(|T_i^*|)A_i)^p x, x\rangle^{2^{j-1}-k_j-1}}\Big)^2.$$

Proof. Let $x \in \mathscr{H}$ be any unit vector. Then, by Lemma (2.1) (c), Lemma (2.2) (a) and (1.2),

$$\begin{aligned}
\sum_{i=1}^{n}|\langle A_i^*T_iB_ix, x\rangle|^p &= \sum_{i=1}^{n}|\langle T_iB_ix, A_ix\rangle|^p \\
&\le \sum_{i=1}^{n}\|f(|T_i|)B_ix\|^p\|g(|T_i^*|)A_ix\|^p \\
&= \sum_{i=1}^{n}\langle f(|T_i|)B_ix, f(|T_i|)B_ix\rangle^{\frac{p}{2}}\langle g(|T_i^*|)A_ix, g(|T_i^*|)A_ix\rangle^{\frac{p}{2}} \\
&= \sum_{i=1}^{n}\langle B_i^*f^2(|T_i|)B_ix, x\rangle^{\frac{p}{2}}\langle A_i^*g^2(|T_i^*|)A_ix, x\rangle^{\frac{p}{2}} \\
&\le \sum_{i=1}^{n}\langle (B_i^*f^2(|T_i|)B_i)^p x, x\rangle^{\frac{1}{2}}\langle (A_i^*g^2(|T_i^*|)A_i)^p x, x\rangle^{\frac{1}{2}} \\
&\le \sum_{i=1}^{n}\Big[\Big(\frac{1}{2}\langle (B_i^*f^2(|T_i|)B_i)^{pr}x, x\rangle + \frac{1}{2}\langle (A_i^*g^2(|T_i^*|)A_i)^{pr}x, x\rangle\Big)^{\frac{1}{r}}\Big] \\
&\quad - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{N}\Big(\sqrt[2^j]{\langle (A_i^*g^2(|T_i^*|)A_i)^p x, x\rangle^{2^{j-1}-k_j}\langle (B_i^*f^2(|T_i|)B_i)^p x, x\rangle^{k_j}} \\
&\quad - \sqrt[2^j]{\langle (B_i^*f^2(|T_i|)B_i)^p x, x\rangle^{k_j+1}\langle (A_i^*g^2(|T_i^*|)A_i)^p x, x\rangle^{2^{j-1}-k_j-1}}\Big)^2 \\
&\le \frac{n^{1-\frac{1}{r}}}{2^{\frac{1}{r}}}\Big\langle\Big(\sum_{i=1}^{n}([B_i^*f^2(|T_i|)B_i]^{rp} + [A_i^*g^2(|T_i^*|)A_i]^{rp})\Big)x, x\Big\rangle^{\frac{1}{r}} \\
&\quad - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{N}\Big(\sqrt[2^j]{\langle (A_i^*g^2(|T_i^*|)A_i)^p x, x\rangle^{2^{j-1}-k_j}\langle (B_i^*f^2(|T_i|)B_i)^p x, x\rangle^{k_j}} \\
&\quad - \sqrt[2^j]{\langle (B_i^*f^2(|T_i|)B_i)^p x, x\rangle^{k_j+1}\langle (A_i^*g^2(|T_i^*|)A_i)^p x, x\rangle^{2^{j-1}-k_j-1}}\Big)^2.
\end{aligned}$$

By taking the supremum on the unit vector x in $\mathscr{H}$ we reach the desired inequality. □

Corollary 2.7. *Let $A_i, B_i \in \mathbb{B}(\mathscr{H})$ $(1 \le i \le n)$. Then for $r, p \ge 1$ we have*

$$w_p^p(A_1^*B_1,\ldots,A_n^*B_n) \le \frac{n^{1-\frac{1}{r}}}{2^{\frac{1}{r}}}\Big\|\sum_{i=1}^{n}(|B_i|^{2rp} + |A_i|^{2rp})\Big\|^{\frac{1}{r}} - \inf_{\|x\|=1}\eta(x),$$

where

$$\eta(x) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{N}\Big(\sqrt[2^j]{\langle |A_i|^{2p}x, x\rangle^{2^{j-1}-k_j}\langle |B_i|^{2p}x, x\rangle^{k_j}} - \sqrt[2^j]{\langle |B_i|^{2p}x, x\rangle^{k_j+1}\langle |A_i|^{2p}x, x\rangle^{2^{j-1}-k_j-1}}\Big)^2.$$

Proof. Choosing $f(t) = g(t) = t^{\frac{1}{2}}$ and $T_i = I$ for $i = 1, 2, \ldots, n$ in Theorem 2.6, we get the desired result. □

Corollary 2.8. *Let $T_i \in \mathbb{B}(\mathscr{H})$ $(1 \le i \le n)$, and let f and g be nonnegative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$ for all $t \in [0,\infty)$ and $r, p \ge 1$. Then*

$$w_p^p(T_1,\dots,T_n) \le \frac{n^{1-\frac{1}{r}}}{2^{\frac{1}{r}}}\Big\| \sum_{i=1}^n (|f^{2rp}(|T_i|) + g^{2rp}(|T_i^*|)) \Big\|^{\frac{1}{r}} - \inf_{\|x\|=1} \eta(x), \tag{2.3}$$

where

$$\eta(x) = \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^N\Big(\sqrt[2^j]{\langle g^{2p}(|T_i^*|)x,x\rangle^{2^{j-1}-k_j}\langle f^{2p}(|T_i|)x,x\rangle^{k_j}} - \sqrt[2^j]{\langle f^{2p}(|T_i|)x,x\rangle^{k_j+1}\langle g^{2p}(|T_i^*|)x,x\rangle^{2^{j-1}-k_j-1}}\Big)^2.$$

In particular,

$$w_p^p(T_1,\dots,T_n) \le \frac{1}{2}\Big\| \sum_{i=1}^n (|T_i|^{2\alpha p} + |T_i^*|^{2(1-\alpha)p}) \Big\| - \inf_{\|x\|=1} \eta(x), \tag{2.4}$$

where $0 \le \alpha \le 1$ and

$$\eta(x) = \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^N\Big(\sqrt[2^j]{\langle |T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j}\langle |T_i|^{2\alpha p}x,x\rangle^{k_j}} - \sqrt[2^j]{\langle |T_i|^{2\alpha p}x,x\rangle^{k_j+1}\langle |T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j-1}}\Big)^2.$$

Proof. Selecting $A_i = B_i = I$ for $i = 1, 2, \dots, n$ in Theorem 2.6, we get the first result. Letting $f(t) = t^\alpha$, $g(t) = t^{1-\alpha}$, $r = 1$ and $B_i = A_i = I$ for $i = 1, 2, \dots, n$ in inequality (2.3), we reach the second inequality. □

Remark 2.9. Note that inequality (2.4) is a refinement of inequality (1.5), since

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^n(\langle |T_i|^{2\alpha p}x,x\rangle^{\frac{1}{2}} - \langle |T_i^*|^{2(1-\alpha)p}x,x\rangle^{\frac{1}{2}})^2 \le \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^N\Big(&\sqrt[2^j]{\langle |T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j}\langle |T_i|^{2\alpha p}x,x\rangle^{k_j}} \\ &- \sqrt[2^j]{\langle |T_i|^{2\alpha p}x,x\rangle^{k_j+1}\langle |T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j-1}}\Big)^2.\end{aligned}$$

Now, by letting $n = 2$, $N = 1$, $T_1 = B$ and $T_2 = C$ in Theorem 2.6, we obtain the following consequence.

Corollary 2.10. *Let $B, C \in \mathbb{B}(\mathscr{H})$. Then, for all $p \ge 1$ and $0 \le \alpha \le 1$,*

$$w_p^p(B,C) \le \frac{1}{2}\||B|^{2\alpha p} + |B^*|^{2(1-\alpha)p} + |C|^{2\alpha p} + |C^*|^{2(1-\alpha)p}\| - \inf_{\|x\|=1}\eta(x),$$

where

$$\eta(x) = \frac{1}{2}[(\langle |B|^{2\alpha p}x,x\rangle^{\frac{1}{2}} - \langle |B^*|^{2(1-\alpha)p}x,x\rangle^{\frac{1}{2}})^2 + (\langle |C|^{2\alpha p}x,x\rangle^{\frac{1}{2}} - \langle |C^*|^{2(1-\alpha)p}x,x\rangle^{\frac{1}{2}})^2].$$

Theorem 2.11. *Let $T_i \in \mathbb{B}(\mathscr{H})$ $(1 \le i \le n)$. Then*

$$w_p(T_1,\dots,T_n) \le \frac{1}{2}\Big[\sum_{i=1}^n\Big(\||T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)}\| - 2\inf_{\|x\|=1}\eta_i(x)\Big)^p\Big]^{\frac{1}{p}}, \tag{2.5}$$

where $p \ge 1$, $0 \le \alpha \le 1$ and

$$\eta_i(x) = \frac{1}{2}\sum_{j=1}^N\Big(\sqrt[2^j]{\langle |T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j}\langle |T_i|^{2\alpha p}x,x\rangle^{k_j}} - \sqrt[2^j]{\langle |T_i|^{2\alpha p}x,x\rangle^{k_j+1}\langle |T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j-1}}\Big)^2.$$

Proof. By using of Lemma 2.1 and inequality (1.2), for any unit vector $x \in \mathscr{H}$, we have

$$\begin{aligned}\sum_{i=1}^n|\langle T_ix,x\rangle|^p &\le \sum_{i=1}^n(\langle |T_i|^{2\alpha}x,x\rangle^{\frac{1}{2}}\langle |T_i^*|^{2(1-\alpha)}x,x\rangle^{\frac{1}{2}})^p \\ &\le \frac{1}{2^p}\sum_{i=1}^n\Big[\langle |T_i|^{2\alpha}x,x\rangle + \langle |T_i^*|^{2(1-\alpha)}x,x\rangle - \sum_{j=1}^N\Big(\sqrt[2^j]{\langle |T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j}\langle |T_i|^{2\alpha p}x,x\rangle^{k_j}} \\ &\qquad - \sqrt[2^j]{\langle |T_i|^{2\alpha p}x,x\rangle^{k_j+1}\langle |T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j-1}}\Big)^2\Big]^p \\ &= \frac{1}{2^p}\sum_{i=1}^n\Big[\langle |T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)}x,x\rangle - \sum_{j=1}^N\Big(\sqrt[2^j]{\langle |T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j}\langle |T_i|^{2\alpha p}x,x\rangle^{k_j}} \\ &\qquad - \sqrt[2^j]{\langle |T_i|^{2\alpha p}x,x\rangle^{k_j+1}\langle |T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j-1}}\Big)^2\Big]^p.\end{aligned}$$

Thus,

$$\Big(\sum_{i=1}^{n}|\langle T_i x, x\rangle|^p\Big)^{\frac{1}{p}} \le \frac{1}{2}\Big[\sum_{i=1}^{n}\Big(\langle|T_i|^{2\alpha}+|T_i^*|^{2(1-\alpha)}x,x\rangle - \sum_{j=1}^{N}\Big(\sqrt[2^j]{\langle|T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j}\langle|T_i|^{2\alpha p}x,x\rangle^{k_j}} - \sqrt[2^j]{\langle|T_i|^{2\alpha p}x,x\rangle^{k_j+1}\langle|T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j-1}}\Big)^2\Big)^p\Big]^{\frac{1}{p}}$$

$$= \frac{1}{2}\Big[\sum_{i=1}^{n}(\langle|T_i|^{2\alpha}+|T_i^*|^{2(1-\alpha)}x,x\rangle - 2\eta_i(x))^p\Big]^{\frac{1}{p}}.$$

Now, by taking the supremum over all unit vectors $x \in \mathscr{H}$, we get the desired result. □

Remark 2.12. If $N = 1$ in inequality (2.5), then we reach inequality (1.4). It follows from

$$\frac{1}{2}(\langle|T_i|^{2\alpha}x,x\rangle^{\frac{1}{2}} - \langle|T_i^*|^{2(1-\alpha)}x,x\rangle^{\frac{1}{2}})^2 \le \frac{1}{2}\sum_{j=1}^{N}\Big(\sqrt[2^j]{\langle|T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j}\langle|T_i|^{2\alpha p}x,x\rangle^{k_j}} - \sqrt[2^j]{\langle|T_i|^{2\alpha p}x,x\rangle^{k_j+1}\langle|T_i^*|^{2(1-\alpha)p}x,x\rangle^{2^{j-1}-k_j-1}}\Big)^2$$

that inequality (2.5) is a refinement of inequality (1.4).

Theorem 2.13. *Let $T_i \in \mathbb{B}(\mathscr{H})$ $(1 \le i \le n)$. Then, for $0 \le \alpha \le 1$ and $p \ge 2$,*

$$w_p^p(T_1,\dots,T_n) \le \Big\|\sum_{i=1}^{n}(\alpha|T_i|^p + (1-\alpha)|T_i^*|^p)\Big\| - \inf_{\|x\|=1}\eta(x), \tag{2.6}$$

where

$$\eta(x) = \sum_{i=1}^{n}\Big(\sum_{j=1}^{N}\Big((-1)^{r_j}2^{j-1}\alpha + (-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big) \times \Big(\sqrt[2^j]{\langle|T_i^*|^p x,x\rangle^{2^{j-1}-k_j}\langle|T_i|^p x,x\rangle^{k_j}} - \sqrt[2^j]{\langle|T_i|^p x,x\rangle^{k_j+1}\langle|T_i^*|^p x,x\rangle^{2^{j-1}-k_j-1}}\Big)^2\Big)$$

Proof. For every unit vector $x \in \mathscr{H}$, by Lemma 2.1 (b), Lemma 2.2 (b) and (1.2), we have

$$\begin{aligned}\sum_{i=1}^{n}|\langle T_i x,x\rangle|^p &= \sum_{i=1}^{n}(|\langle T_i x,x\rangle|^2)^{\frac{p}{2}}\\ &\le \sum_{i=1}^{n}(\langle|T_i|^{2\alpha}x,x\rangle\langle|T_i^*|^{2(1-\alpha)}x,x\rangle)^{\frac{p}{2}}\\ &\le \sum_{i=1}^{n}(\langle|T_i|px,x\rangle^{\alpha}\langle|T_i^*|^p x,x\rangle^{1-\alpha})\\ &\le \sum_{i=1}^{n}(\alpha\langle|T_i|^p x,x\rangle + (1-\alpha)\langle|T_i^*|^p x,x\rangle) - \sum_{i=1}^{n}\Big(\sum_{j=1}^{N}\Big((-1)^{r_j}2^{j-1}\alpha + (-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big)\\ &\qquad \times \Big(\sqrt[2^j]{\langle|T_i^*|^p x,x\rangle^{2^{j-1}-k_j}}\langle|T_i|^p x,x\rangle^{k_j} - \sqrt[2^j]{\langle|T_i|^p x,x\rangle^{k_j+1}\langle|T_i^*|^p x,x\rangle^{2^{j-1}-k_j-1}}\Big)^2\Big)\\ &\le \sum_{i=1}^{n}\langle(\alpha|T_i|^p + (1-\alpha)|T_i^*|^p)x,x\rangle - \inf_{\|x\|=1}\eta(x)\\ &= \Big\langle\sum_{i=1}^{n}(\alpha|T_i|^p + (1-\alpha)|T_i^*|^p)x,x\Big\rangle - \inf_{\|x\|=1}\eta(x).\end{aligned}$$

Now by taking the supremum over a unit vector $x \in \mathscr{H}$ we get the desired result. □

Remark 2.14. If we put $N = 1$ in inequality (2.6), then we get inequality (1.6). Hence, inequality (2.6) is a refinement of (1.6).

In [12, Remark 3.10], Moslehian et al. showed

$$w_p^p(B, C) \le \frac{1}{2}\| |B|^p + |B^*|^p + |C|^p + |C^*|^p \|, \tag{2.7}$$

in which, $B, C \in \mathbb{B}(\mathscr{H})$ and $p \ge 2$. In the following result we show a refinement of (2.7).

Corollary 2.15. *Let $B, C \in \mathbb{B}(\mathscr{H})$. Then, for $p \ge 2$,*

$$w_p^p(B, C) \le \frac{1}{2}\| |B|^p + |B^*|^p + |C|^p + |C^*|^p \| - \inf_{\|x\|=1} \eta(x), \tag{2.8}$$

where $\eta(x) = \frac{1}{2}((\langle |B|^p x, x\rangle^{\frac{1}{2}} - \langle |B^*|^p x, x\rangle^{\frac{1}{2}})^2 - (\langle |C|^p x, x\rangle^{\frac{1}{2}} - \langle |C^*|^p x, x\rangle^{\frac{1}{2}})^2)$.

In particular, if $A \in \mathbb{B}(\mathscr{H})$, then

$$w^2(A) \le \frac{1}{2}\|A^*A + AA^*\|.$$

Proof. If we take $N = 1$, $n = 2$, $T_1 = B$, $T_2 = C$, and $\alpha = \frac{1}{2}$ in Theorem 2.13, we get the first inequality.

In this particular case, let $A = B + iC$ be the Cartesian decomposition of A. Then $A^*A + AA^* = 2(B^2 + C^2)$ and $\inf_{\|x\|=1} \eta(x) = 0$. Thus, for $p = 2$, inequality (2.8) can be written as

$$w_2^2(B, C) \le \|B^2 + C^2\| = \frac{1}{2}\|A^*A + AA^*\|.$$

The desired inequality follows by noting that

$$w_2^2(B, C) = \sup_{\|x\|=1}\{|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\} = \sup_{\|x\|=1} |\langle Ax, x\rangle|^2 = w^2(A). \qquad \square$$

Theorem 2.16. *Let $T_i \in \mathbb{B}(\mathscr{H})$ $(1 \le i \le n)$, $r \ge 1$, and $p \ge q \ge 1$, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then*

$$w_p^r(|T_1|, \dots, |T_n|) w_q^r(|T_1^*|, \dots, |T_n^*|) \le \frac{r}{p}\Big\| \sum_{i=1}^n |T_i|^p \Big\| + \frac{r}{q}\Big\| \sum_{i=1}^n |T_i^*|^q \Big\| - \inf_{\|x\|=\|y\|=1} \lambda(x, y), \tag{2.9}$$

where

$$\lambda(x, y) = \sum_{j=1}^N \Big((-1)^{r_j} 2^{j-1}\Big(\frac{r}{p}\Big) + (-1)^{r_j+1}\Big[\frac{r_j + 1}{2}\Big]\Big)\Bigg(\sqrt[2^j]{\Big(\sum_{i=1}^n \langle |T_i^*|x, y\rangle^q\Big)^{2^{j-1}-k_j} \Big(\sum_{i=1}^n \langle |T_i|x, y\rangle^p\Big)^{k_j}}$$
$$- \sqrt[2^j]{\Big(\sum_{i=1}^n \langle |T_i|x, y\rangle^p\Big)^{k_j+1} \Big(\sum_{i=1}^n \langle |T_i^*|x, y\rangle^q\Big)^{2^{j-1}-k_j-1}}\Bigg)^2.$$

Proof. Let $x, y \in \mathbb{B}(\mathscr{H})$ be unit vectors. Applying inequality (1.2) and Lemma (2.2) (a), we get

$$\Bigg(\Big(\sum_{i=1}^n \langle |T_i|x, y\rangle^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^n \langle |T_i^*|x, y\rangle^q\Big)^{\frac{1}{q}}\Bigg)^r$$
$$\le \frac{r}{p}\sum_{i=1}^n \langle |T_i|x, y\rangle^p + \frac{r}{q}\sum_{i=1}^n \langle |T_i^*|x, y\rangle^q - \sum_{j=1}^N \Big((-1)^{r_j} 2^{j-1}\Big(\frac{r}{p}\Big) + (-1)^{r_j+1}\Big[\frac{r_j + 1}{2}\Big]\Big)$$
$$\times \Bigg(\sqrt[2^j]{\Big(\sum_{i=1}^n \langle |T_i^*|x, y\rangle^q\Big)^{2^{j-1}-k_j} \Big(\sum_{i=1}^n \langle |T_i|x, y\rangle^p\Big)^{k_j}} - \sqrt[2^j]{\Big(\sum_{i=1}^n \langle |T_i|x, y\rangle^p\Big)^{k_j+1} \Big(\sum_{i=1}^n \langle |T_i^*|x, y\rangle^q\Big)^{2^{j-1}-k_j-1}}\Bigg)^2$$
$$\le \frac{r}{p}\sum_{i=1}^n \langle |T_i|^p x, y\rangle + \frac{r}{q}\sum_{i=1}^n \langle |T_i^*|^q x, y\rangle - \sum_{j=1}^N \Big((-1)^{r_j} 2^{j-1}\Big(\frac{r}{p}\Big) + (-1)^{r_j+1}\Big[\frac{r_j + 1}{2}\Big]\Big)$$
$$\times \Bigg(\sqrt[2^j]{\Big(\sum_{i=1}^n \langle |T_i^*|x, y\rangle^q\Big)^{2^{j-1}-k_j} \Big(\sum_{i=1}^n \langle |T_i|x, y\rangle^p\Big)^{k_j}} - \sqrt[2^j]{\Big(\sum_{i=1}^n \langle |T_i|x, y\rangle^p\Big)^{k_j+1} \Big(\sum_{i=1}^n \langle |T_i^*|x, y\rangle^q\Big)^{2^{j-1}-k_j-1}}\Bigg)^2$$
$$= \frac{r}{p}\Big\langle\Big(\sum_{i=1}^n |T_i|^p\Big)x, y\Big\rangle + \frac{r}{q}\Big\langle\Big(\sum_{i=1}^n |T_i^*|^q\Big)x, y\Big\rangle - \sum_{j=1}^N \Big((-1)^{r_j} 2^{j-1}\Big(\frac{r}{p}\Big) + (-1)^{r_j+1}\Big[\frac{r_j + 1}{2}\Big]\Big)$$
$$\times \Bigg(\sqrt[2^j]{\Big(\sum_{i=1}^n \langle |T_i^*|x, y\rangle^q\Big)^{2^{j-1}-k_j} \Big(\sum_{i=1}^n \langle |T_i|x, y\rangle^p\Big)^{k_j}} - \sqrt[2^j]{\Big(\sum_{i=1}^n \langle |T_i|x, y\rangle^p\Big)^{k_j+1} \Big(\sum_{i=1}^n \langle |T_i^*|x, y\rangle^q\Big)^{2^{j-1}-k_j-1}}\Bigg)^2.$$

By taking the supremum on $x, y \in \mathscr{H}$ with $\|x\| = \|y\| = 1$, we get the desired inequality. $\square$

Remark 2.17. If we put $N = 1$ in inequality (2.9), then we come to inequality (1.7). It follows from

$$\frac{r}{p}\Big(\sqrt[2]{\sum_{i=1}^{n}\langle|T_i|x,y\rangle^p} - \sqrt[2]{\sum_{i=1}^{n}\langle|T_i^*|x,y\rangle^q}\Big)^2$$
$$\leq \sum_{j=1}^{N}\Big((-1)^{r_j}2^{j-1}\big(\frac{r}{p}\big) + (-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big)\Big(\sqrt[2^j]{\Big(\sum_{i=1}^{n}\langle|T_i^*|x,y\rangle^q\Big)^{2^{j-1}-k_j}\Big(\sum_{i=1}^{n}\langle|T_i|x,y\rangle^p\Big)^{k_j}}$$
$$- \sqrt[2^j]{\Big(\sum_{i=1}^{n}\langle|T_i|x,y\rangle^p\Big)^{k_j+1}\Big(\sum_{i=1}^{n}\langle|T_i^*|x,y\rangle^q\Big)^{2^{j-1}-k_j-1}}\Big)^2$$

that inequality (2.9) is a refinement of (1.7).

Corollary 2.18. *Let $T_i \in \mathbb{B}(\mathscr{H})$ $(1 \leq i \leq n)$. Then*

$$w_e(|T_1|,\dots,|T_n|)w_e(|T_1^*|,\dots,|T_n^*|) \leq \frac{1}{2}\Big(\Big\|\sum_{i=1}^{n}T_i^*T_i\Big\| + \Big\|\sum_{i=1}^{n}T_iT_i^*\Big\|\Big) - \inf_{\|x\|=\|y\|=1}\lambda(x,y),$$

where

$$\lambda(x,y) = \sum_{j=1}^{N}\Big((-1)^{r_j}2^{j-1}\big(\frac{1}{2}\big) + (-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big)$$
$$\times\Big(\sqrt[2^j]{\Big(\sum_{i=1}^{n}\langle|T_i^*|x,y\rangle\Big)^{2(2^{j-1}-k_j)}\Big(\sum_{i=1}^{n}\langle|T_i|x,y\rangle\Big)^{2k_j}}$$
$$- \sqrt[2^j]{\Big(\sum_{i=1}^{n}\langle|T_i|x,y\rangle\Big)^{2(k_j+1)}\Big(\sum_{i=1}^{n}\langle|T_i^*|x,y\rangle\Big)^{2(2^{j-1}-k_j-1)}}\Big)^2.$$

Proof. The result is obtained by letting $p = q = 2$ and $r = 1$ in inequality (2.9). □

Corollary 2.19. *Let $T_1,\dots,T_n \in \mathbb{B}(\mathscr{H})$ be positive operators. Then*

$$w_e(T_1,\dots,Tn) \leq \Big\|\sum_{i=1}^{n}T_i^2\Big\|^{\frac{1}{2}}.$$

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