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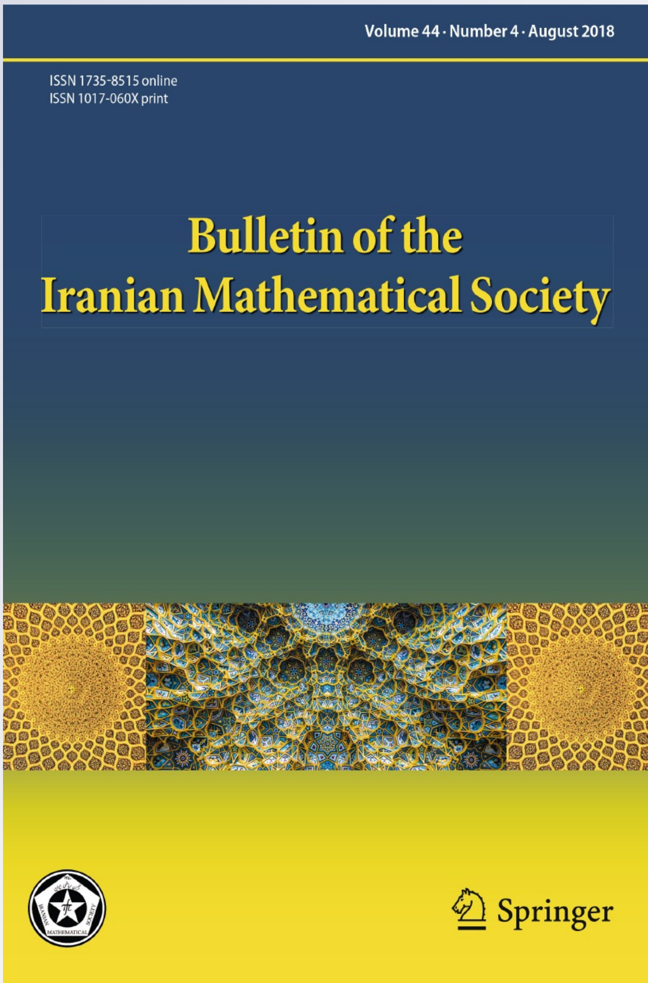

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ORIGINAL PAPER

Some Extensions of the Young and Heinz Inequalities for Matrices

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Abstract

In this paper, we present some extensions of the Young and Heinz inequalities for the Hilbert–Schmidt norm as well as any unitarily invariant norm. Furthermore, we give some inequalities dealing with matrices. More precisely, for two positive semidefinite matrices A and B we show that

$$\left\|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\right\|_2^2 \leq \left\|AX+XB\right\|_2^2-2r\left\|AX-XB\right\|_2^2 \\ -r_0\left(\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-AX\right\|_2^2+\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-XB\right\|_2^2\right),$$

where X is an arbitrary $n\times n$ matrix, $0<\nu\leq\frac{1}{2}$, $r=\min\{\nu,1-\nu\}$ and $r_0=\min\{2r,1-2r\}$.

Keywords Convex function · Heinz inequality · Hilbert–Schmidt norm · Positive semidefinite matrix · Unitarily invariant norm · Young inequality

Mathematics Subject Classification Primary 15A60; Secondary 47A30 · 26A51 · 65F35

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1 Introduction

Let $\mathcal{M}_n$ be the C^*-algebra of all $n \times n$ complex matrices and $\langle \cdot, \cdot \rangle$ be the standard scalar product in $\mathbb{C}^n$. A capital letter means an $n \times n$ matrix in $\mathcal{M}_n$. For Hermitian matrices A, B, we write $A \geq 0$ if A is positive semidefinite, $A > 0$ if A is positive definite, and $A \geq B$ if $A - B \geq 0$. A norm $|||\,.\,|||$ on $\mathcal{M}_n$ is called unitarily invariant norm if $|||UAV||| = |||A|||$ for all $A \in \mathcal{M}_n$ and all unitary matrices $U, V \in \mathcal{M}_n$. The Hilbert–Schmidt norm is defined by $\|A\|_2 = \left(\sum_{j=1}^{n} s_j^2(A)\right)^{1/2}$, where $s(A) = (s_1(A), \ldots, s_n(A))$ denotes the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$, arranged in the decreasing order with their multiplicities counted. This norm is unitarily invariant. It is known that if $A = [a_{ij}] \in \mathcal{M}_n$, then $\|A\|_2 = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}$. The trace norm of A can be expressed as $\operatorname{tr}(|A|) = ||A||_1 = \sum_{j=1}^{n} s_j(A)$.

The classical Young's inequality says that for positive real numbers a, b and $0 \leq \nu \leq 1$, we have $a^{\nu} b^{1-\nu} \leq \nu a + (1-\nu) b$. When $\nu = \frac{1}{2}$, Young's inequality is the arithmetic–geometric mean inequality, $\sqrt{ab} \leq \frac{a+b}{2}$.

Zhao and Wu in [13], refined Young's inequality in the following form

$$a^{1-\nu} b^{\nu} + S_1(\nu) + r(\sqrt{a} - \sqrt{b})^2 \leq (1-\nu)a + \nu b, \tag{1.1}$$

where

$$S_1(\nu) = \left((-1)^{r_0} 2\nu + (-1)^{r_0+1} \left[\frac{r_0+1}{2}\right]\right) \left(\sqrt[4]{b^{2-k} a^k} - \sqrt[4]{a^{k+1} b^{1-k}}\right)^2,$$

$0 < \nu \leq 1, r = \min\{\nu, 1-\nu\}, r_0 = [4\nu]$ and $k = [2\nu]$. Here $[x]$ is the greatest integer less than or equal to x. Also, they proved a reverse of (1.1) as follows

$$(1-\nu)a + \nu b \leq a^{1-\nu} b^{\nu} + R(\sqrt{a} - \sqrt{b})^2 - S_1(\nu), \tag{1.2}$$

where $0 < \nu \leq 1$ and $R = \max\{\nu, 1-\nu\}$. They showed if $a, b > 0$ and $0 < \nu < 1$, then

$$(a^{1-\nu} b^{\nu})^2 + r^2(a-b)^2 + S_1(\nu) \leq ((1-\nu)a + \nu b)^2, \tag{1.3}$$

and

$$((1-\nu)a + \nu b)^2 \leq (a^{1-\nu} b^{\nu})^2 + (1-\nu)^2(a-b)^2 - S_1(\nu), \tag{1.4}$$

where $r = \min\{\nu, 1-\nu\}$. Applying inequalities (1.3) and (1.4) we have the following inequalities:

If $0 < \nu \leq \frac{1}{2}$, then

$$\begin{aligned}&\nu^2(a^2+b^2) - (2\nu^2 ab + 2r_0 a\sqrt{ab} - r_0(ab+a^2))\\ &\quad\leq ((1-\nu)a+\nu b)^2 - (a^{1-\nu}b^{\nu})^2\\ &\quad\leq (1-\nu)^2(a^2+b^2) - (2(1-\nu)^2 ab + r_0 b\sqrt{ab} - r_0(ab+b^2)).\end{aligned}$$

If $\frac{1}{2} < \nu < 1$, then

$$\begin{aligned}&(1-\nu)^2(a^2+b^2) - (2(1-\nu)^2 ab + 2r_0 b\sqrt{ab} - r_0(ab+b^2))\\ &\quad\leq ((1-\nu)a+\nu b)^2 - (a^{1-\nu}b^{\nu})^2\\ &\quad\leq \nu^2(a^2+b^2) - (2\nu^2 ab + 2r_0 a\sqrt{ab} - r_0(ab+a^2)),\end{aligned}$$

where $r = \min\{\nu, 1-\nu\}$ and $r_0 = \min\{2r, 1-2r\}$.

The Heinz means are defined as $H_\nu(a,b) = \frac{a^{1-\nu}b^{\nu}+a^{\nu}b^{1-\nu}}{2}$ for $a, b > 0$ and $0 \leq \nu \leq 1$. These interesting means interpolate between the geometric and arithmetic means. In fact, the Heinz inequalities assert that $\sqrt{ab} \leq H_\nu(a,b) \leq \frac{a+b}{2}$, where $a, b > 0$ and $0 \leq \nu \leq 1$.

A matrix version of Young's inequality [2] says that if $A, B \in \mathcal{M}_n(\mathbb{C})$ are positive semidefinite and $0 \leq \nu \leq 1$, then

$$s_j(A^{1-\nu}B^{\nu}) \leq s_j((1-\nu)A + \nu B) \tag{1.5}$$

for $j = 1, 2, \ldots, n$. It follows from (1.5) that if $A, B \in \mathcal{M}_n$ are positive semidefinite and $0 \leq \nu \leq 1$, then a trace version of Young's inequality holds

$$\operatorname{tr}|A^{1-\nu}B^{\nu}| \leq \operatorname{tr}((1-\nu)A + \nu B). \tag{1.6}$$

A determinant version of Young's inequality says that [8]

$$\det\left(A^{1-\nu}B^{\nu}\right) \leq \det\left((1-\nu)A + \nu B\right). \tag{1.7}$$

In [11], it is shown the Young inequality for arbitrary unitarily invariant norms as follows

$$|||A^{1-\nu}XB^{\nu}||| \leq (1-\nu)|||AX||| + \nu|||XB||| \tag{1.8}$$

in which A, B are positive semidefinite $n \times n$ and $0 < \nu \leq 1$. Some mathematicians proved several refinements of the Young and Heinz inequalities for matrices; see [3–5,7,10] and references therein. Sababheh [12] showed that for any $A, B, X \in \mathcal{M}_n$ such that A and B are positive semidefinite, the following relation holds

$$\begin{aligned}&|||A^{1-\nu}XB^{\nu}|||+\nu(|||AX|||+|||XB|||)\\&\quad-\left(2\nu\sqrt{|||AX||||||XB|||}-r_0(\sqrt{|||AX|||}+\sqrt[4]{|||AX||||||XB|||})^2\right)\\&\quad\le(1-\nu)|||AX|||+\nu|||XB|||,\end{aligned}\tag{1.9}$$

where $0<\nu\le\frac{1}{2}$, $r=\min\{\nu,1-\nu\}$ and $r_0=\min\{2r,1-2r\}$.

Based on the refined and reversed Young inequalities (1.1) and (1.2), Zhao and Wu [13], proved that if $A,B,X\in\mathcal{M}_n$ such that A and B are two positive semidefinite matrices, then

(i) If $0<\nu\le\frac{1}{2}$,

$$\begin{aligned}&r^2\left\|AX-XB\right\|_2^2+r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-AX\right\|_2^2\\&\quad\le\left\|(1-\nu)AX+\nu XB\right\|_2^2-\left\|A^{1-\nu}XB^{\nu}\right\|_2^2\\&\quad\le R^2\left\|AX-XB\right\|_2^2-r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-XB\right\|_2^2,\end{aligned}\tag{1.10}$$

(ii) if $\frac{1}{2}<\nu<1$,

$$\begin{aligned}&R^2\left\|AX-XB\right\|_2^2+r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-XB\right\|_2^2\\&\quad\le\left\|(1-\nu)AX+\nu XB\right\|_2^2-\left\|A^{1-\nu}XB^{\nu}\right\|_2^2\\&\quad\le r^2\left\|AX-XB\right\|_2^2-r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-AX\right\|_2^2,\end{aligned}\tag{1.11}$$

where $r=\min\{\nu,1-\nu\}$, $R=\max\{\nu,1-\nu\}$ and $r_0=\min\{2r,1-2r\}$.

In this paper, we generalized some extensions of the Young and Heinz inequalities for the Hilbert–Schmidt norm as well as any unitarily invariant norm. Also, we give some inequalities dealing with matrices. Furthermore, we refine inequalities (1.6)–(1.8).

2 Main Results

For our purpose we need the following lemma.

Lemma 2.1 [1, Theorem 2] *Let ϕ be a strictly increasing convex function defined on an interval I. If x, y, z and w are points in I such that $z-w\le x-y$, where $w\le z\le x$ and $y\le x$, then*

$$\phi(z)-\phi(w)\le\phi(x)-\phi(y).$$

Theorem 2.2 *Let $\phi:[0,\infty)\to\mathbb{R}$ be a strictly increasing convex function. If $a,b>0$, then*

(i) *For* $0 < \nu \le \frac{1}{2}$,

$$\begin{aligned}&\phi(\nu(a+b)) - \phi\left(2\nu\sqrt{ab} + 2r_0\sqrt{a}\sqrt[4]{ab} - r_0(\sqrt{ab}+a)\right)\\&\quad\le \phi\left((1-\nu)a+\nu b\right) - \phi\left(a^{1-\nu}b^{\nu}\right)\\&\quad\le \phi\left((1-\nu)(a+b)\right) - \phi\left(2(1-\nu)\sqrt{ab} + 2r_0\sqrt{b}\sqrt[4]{ab} - r_0(\sqrt{ab}+b)\right),\end{aligned} \tag{2.1}$$

(ii) *for* $\frac{1}{2} < \nu < 1$,

$$\begin{aligned}&\phi\left((1-\nu)(a+b)\right) - \phi\left(2(1-\nu)\sqrt{ab} + 2r_0\sqrt{b}\sqrt[4]{ab} - r_0(\sqrt{ab}+b)\right)\\&\quad\le \phi\left((1-\nu)a+\nu b\right) - \phi\left(a^{1-\nu}b^{\nu}\right)\\&\quad\le \phi\left(\nu(a+b)\right) - \phi\left(2\nu\sqrt{ab} + 2r_0\sqrt{a}\sqrt[4]{ab} - r_0(\sqrt{ab}+a)\right),\end{aligned} \tag{2.2}$$

where $r = \min\{\nu, 1-\nu\}$ *and* $r_0 = \min\{2r, 1-2r\}$.

Proof Let $0 < \nu \le \frac{1}{2}$. If we put $x = (1-\nu)a + \nu b$, $y = a^{1-\nu}b^{\nu}$, $z = \nu(a+b)$, $w = 2r_0\sqrt{a}\sqrt[4]{ab} + 2\nu\sqrt{ab} - r_0(\sqrt{ab}+a)$, $z' = (1-\nu)(a+b)$ and $w' = 2(1-\nu)\sqrt{ab} + 2r_0\sqrt[4]{ab}\sqrt{b} - r_0(\sqrt{ab}+b)$, then $y \le x$ and $x \le z'$. It follows from

$$\begin{aligned}&2r_0\sqrt{a}\sqrt[4]{ab} + 2\nu\sqrt{ab} - r_0(\sqrt{ab}+a)\\&\quad\le r_0(a+\sqrt{ab}) + \nu(a+b) - r_0(\sqrt{ab}+a)\\&\qquad\text{(by the arithmetic–geometric mean)}\\&\quad= \nu(a+b)\\&\quad\le (1-\nu)a + \nu b\end{aligned}$$

and

$$\begin{aligned}&2(1-\nu)\sqrt{ab} + 2r_0\sqrt[4]{ab}\sqrt{b} - r_0(\sqrt{ab}+b)\\&\quad\le (1-\nu)(a+b) + r_0(b+\sqrt{ab}) - r_0(\sqrt{ab}+a)\\&\qquad\text{(by the arithmetic–geometric mean)}\\&\quad= (1-\nu)(a+b),\end{aligned}$$

where $w \le z \le x$, $w' \le z'$. Using inequalities (1.1) and (1.2) we have

$$\begin{aligned}&\nu(a+b) - \left(2\nu\sqrt{ab} + 2r_0\sqrt{a}\sqrt[4]{ab} - r_0(\sqrt{ab}+a)\right)\\&\quad\le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu}\\&\quad\le (1-\nu)(a+b) - \left(2(1-\nu)\sqrt{ab} + 2r_0\sqrt{b}\sqrt[4]{ab} - r_0(\sqrt{ab}+b)\right).\end{aligned} \tag{2.3}$$

Hence

$$z - w \leq x - y \leq z' - w'.$$

Applying Lemma 2.1 we reach inequality (2.1). Now, if $\frac{1}{2} < \nu < 1$, then

$$\begin{aligned}
&(1-\nu)(a+b) - \left(2(1-\nu)\sqrt{ab} + 2r_0\sqrt{b}\sqrt[4]{ab} - r_0(\sqrt{ab}+b)\right)\\
&\leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu}\\
&\leq \nu(a+b) - \left(2\nu\sqrt{ab} + 2r_0\sqrt{a}\sqrt[4]{ab} - r_0(\sqrt{ab}+a)\right). \qquad (2.4)
\end{aligned}$$

In a similar fashion, we have inequality (2.2). □

By taking $\phi(x) = x^m$ $(m \geq 1)$, we have the next result.

Corollary 2.3 *Let* $a, b > 0$ *and* $m \geq 1$. *Then*

(i) *If* $0 < \nu \leq \frac{1}{2}$, *then*

$$\begin{aligned}
&\left(\nu(a+b)\right)^m - \left(2\nu\sqrt{ab} + 2r_0\sqrt{a}\sqrt[4]{ab} - r_0(\sqrt{ab}+a)\right)^m\\
&\leq ((1-\nu)a + \nu b)^m - (a^{1-\nu}b^{\nu})^m\\
&\leq ((1-\nu)(a+b))^m - \left(2(1-\nu)\sqrt{ab} + 2r_0\sqrt{b}\sqrt[4]{ab} - r_0(\sqrt{ab}+b)\right)^m;
\end{aligned}$$

(ii) *if* $\frac{1}{2} < \nu < 1$, *then*

$$\begin{aligned}
&((1-\nu)(a+b))^m - \left(2(1-\nu)\sqrt{ab} + 2r_0\sqrt{b}\sqrt[4]{ab} - r_0(\sqrt{ab}+b)\right)^m\\
&\leq ((1-\nu)a + \nu b)^m - (a^{1-\nu}b^{\nu})^m\\
&\leq (\nu(a+b))^m - \left(2\nu\sqrt{ab} + 2r_0\sqrt{a}\sqrt[4]{ab} - r_0(\sqrt{ab}+a)\right)^m,
\end{aligned}$$

where $r = \min\{\nu, 1-\nu\}$ *and* $r_0 = \min\{2r, 1-2r\}$.

In the following result, we show a refinement of the Heinz inequality.

Corollary 2.4 *Let* $\phi : [0, \infty) \to \mathbb{R}$ *be a strictly increasing convex function. If* $a, b > 0$, *then*

$$\begin{aligned}
&\phi(r(a+b)) - \phi(2r\sqrt{ab} + r_0\sqrt[4]{ab}(\sqrt{a}+\sqrt{b}) - \frac{r_0}{2}(\sqrt{a}+\sqrt{b})^2)\\
&\leq \phi\left(\frac{a+b}{2}\right) - \phi(H_\nu(a,b))\\
&\leq \phi(R(a+b)) - \phi\left(2R\sqrt{ab} + r_0\sqrt[4]{ab}(\sqrt{a}+\sqrt{b}) - \frac{r_0}{2}(\sqrt{a}+\sqrt{b})^2\right)
\end{aligned}$$

for $0 \leq \nu \leq 1$, $R = \max\{\nu, 1-\nu\}$, $r = \min\{\nu, 1-\nu\}$ *and* $r_0 = \min\{2r, 1-2r\}$.

Proof Let $0 \leq \nu \leq 1$. By interchanging a with b in inequalities (2.3) and (2.4), respectively, then we get

$$\begin{aligned} & r(a+b) - \left(2r\sqrt{ab} + r_0\sqrt[4]{ab}(\sqrt{a}+\sqrt{b}) - \frac{r_0}{2}(\sqrt{a}+\sqrt{b})^2\right) \\ & \leq \frac{a+b}{2} - H_\nu(a,b) \\ & \leq R(a+b) - \left(2R\sqrt{ab} + r_0\sqrt[4]{ab}(\sqrt{a}+\sqrt{b}) - \frac{r_0}{2}(\sqrt{a}+\sqrt{b})^2\right). \end{aligned} \tag{2.5}$$

Now, we put $x = \frac{a+b}{2}$, $y = H_\nu(a,b)$, $z = r(a+b)$, $w = 2r\sqrt{ab}+r_0\sqrt[4]{ab}(\sqrt{a}+\sqrt{b})-\frac{r_0}{2}(\sqrt{a}+\sqrt{b})^2$, $z' = R(a+b)$ and $w' = 2R\sqrt{ab}+r_0\sqrt[4]{ab}(\sqrt{a}+\sqrt{b})-\frac{r_0}{2}(\sqrt{a}+\sqrt{b})^2$. Using the arithmetic–geometric mean and (2.5) we have $y \leq x$, $w \leq z \leq x$, $w' \leq z'$, $y \leq x \leq z'$ and

$$z - w \leq x - y \leq z' - w'.$$

Applying Lemma 2.1 we get the desired result. □

Example 2.5 If we take $\phi(x) = x^m$ $(m \geq 1)$ in Corollary 2.4, then for positive numbers a and b we reach the inequality

$$\begin{aligned} & (r(a+b))^m - (2r\sqrt{ab} + r_0\sqrt[4]{ab}(\sqrt{a}+\sqrt{b}) - \frac{r_0}{2}(\sqrt{a}+\sqrt{b})^2)^m \\ & \leq \left(\frac{a+b}{2}\right)^m - (H_\nu(a,b))^m \\ & \leq (R(a+b))^m - (2R\sqrt{ab} + r_0\sqrt[4]{ab}(\sqrt{a}+\sqrt{b}) - \frac{r_0}{2}(\sqrt{a}+\sqrt{b})^2)^m, \end{aligned}$$

where $0 \leq \nu \leq 1$, $R = \max\{\nu, 1-\nu\}$, $r = \min\{\nu, 1-\nu\}$ and $r_0 = \min\{2r, 1-2r\}$.

3 Some Applications

In this section, we apply numerical inequalities that we achieved in Sect. 2 for Hilbert space operators. First, we improve the inequalities (1.6), (1.7) and (1.8). To achieve this, we need the following lemmas.

Lemma 3.1 *Let $A, B \in \mathcal{M}_n$. Then*

$$\sum_{j=1}^{n} s_j(AB) \leq \sum_{j=1}^{n} s_j(A)s_j(B).$$

The next lemma is a Heinz–Kato type inequality for unitarily invariant norms that known in [8].

Lemma 3.2 *Let $A, B, X \in \mathcal{M}_n$ such that A and B are positive semidefinite. If $0 \le \nu \le 1$, then*

$$|||A^{1-\nu} X B^{\nu}||| \le |||AX|||^{1-\nu} |||XB|||^{\nu}.$$

In particular,

$$\mathrm{tr}|A^{1-\nu} B^{\nu}| \le (\mathrm{tr} A)^{1-\nu} (\mathrm{tr} B)^{\nu}.$$

The third lemma is the Minkowski inequality for determinants that known in [9].

Lemma 3.3 *Let $A, B \in \mathcal{M}_n$ be positive definite. Then*

$$\det(A+B)^{\frac{1}{n}} \ge \det A^{\frac{1}{n}} + \det B^{\frac{1}{n}}.$$

In the next result we show an extension of inequality (1.9).

Theorem 3.4 *Let $A, B \in \mathcal{M}_n$ be positive definite. If $0 < \nu \le \frac{1}{2}$, then*

$$\begin{aligned}
&\left(\mathrm{tr}|A^{1-\nu} B^{\nu}|\right)^m + \nu^m \left(\mathrm{tr} A + \mathrm{tr} B\right)^m \\
&\quad - \left(2\nu(\mathrm{tr}(A)\mathrm{tr}(B))^{\frac{1}{2}} - r_0((\mathrm{tr}(A)\mathrm{tr}(B))^{\frac{1}{4}} - (\mathrm{tr}(A))^{\frac{1}{2}})^2\right)^m \\
&\le \left(\mathrm{tr}((1-\nu)A + \nu B)\right)^m
\end{aligned} \tag{3.1}$$

and if $\frac{1}{2} \le \nu \le 1$, then

$$\begin{aligned}
&\left(\mathrm{tr}|A^{1-\nu} B^{\nu}|\right)^m + (1-\nu)^m \left(\mathrm{tr} A + \mathrm{tr} B\right)^m \\
&\quad - \left(2(1-\nu)(\mathrm{tr}(A)\mathrm{tr}(B))^{\frac{1}{2}} - r_0((\mathrm{tr}(A)\mathrm{tr}(B))^{\frac{1}{4}} - (\mathrm{tr}(B))^{\frac{1}{2}})^2\right)^m \\
&\le \left(\mathrm{tr}((1-\nu)A + \nu B)\right)^m,
\end{aligned} \tag{3.2}$$

where $m = 1, 2, \ldots$, $r = \min\{\nu, 1-\nu\}$ and $r_0 = \min\{2r, 1-2r\}$.

Proof Let $0 < \nu \le \frac{1}{2}$. Then

$$\begin{aligned}
&\left(\mathrm{tr}|A^{1-\nu} B^{\nu}|\right)^m + \nu^m \left(\mathrm{tr} A + \mathrm{tr} B\right)^m \\
&\quad - \left(2\nu(\mathrm{tr}(A)\mathrm{tr}(B))^{\frac{1}{2}} - r_0((\mathrm{tr}(A)\mathrm{tr}(B))^{\frac{1}{4}} - (\mathrm{tr}(A))^{\frac{1}{2}})^2\right)^m \\
&\le \left((\mathrm{tr}(A))^{1-\nu} (\mathrm{tr}(B))^{\nu}\right)^m + \nu^m \left(\mathrm{tr} A + \mathrm{tr} B\right)^m \\
&\quad - \left(2\nu(\mathrm{tr}(A)\mathrm{tr}(B))^{\frac{1}{2}} - r_0((\mathrm{tr}(A)\mathrm{tr}(B))^{\frac{1}{4}} - (\mathrm{tr}(A))^{\frac{1}{2}})^2\right)^m \\
&\qquad\qquad \text{(by Lemma 3.2)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big((1-\nu)\mathrm{tr}(A) + \nu \mathrm{tr}(B)\Big)^m \qquad \text{(by Corollary 2.3)}\\
&= \Big(\mathrm{tr}((1-\nu)A + \nu B)\Big)^m.
\end{aligned}
$$

Thus, we get inequality (3.1). Using Corollary 2.3, Lemma 3.2 and with a same argument in the proof of (3.1), we have (3.2) for $\frac{1}{2} \leq \nu \leq 1$. □

Theorem 3.5 *Let $A, B \in \mathcal{M}_n$ be positive definite and $0 < \nu \leq \frac{1}{2}$. Then*

$$
\begin{aligned}
&\det(A^{1-\nu}B^{\nu})^m + \nu^{mn}\Big(\det A + \det B\Big)^m\\
&\quad - \Big(2\nu(\det(A)\det(B))^{\frac{1}{2}} - r_0((\det(A)\det(B))^{\frac{1}{4}} - (\det(A))^{\frac{1}{2}})^2\Big)^m\\
&\leq \det((1-\nu)A + \nu B)^m,
\end{aligned}
$$

holds for $m = 1, 2, \ldots$ and $r_0 = \min\{2\nu, 1-2\nu\}$.

Proof

$$
\begin{aligned}
\det((1-\nu)A + \nu B)^m &= \Big(\det((1-\nu)A + \nu B)^{\frac{1}{n}}\Big)^{mn}\\
&\geq \Big(\det((1-\nu)A)^{\frac{1}{n}} + \det(\nu B)^{\frac{1}{n}}\Big)^{mn} \qquad \text{(by Lemma 3.3)}\\
&= \Big((1-\nu)\det A^{\frac{1}{n}} + \nu \det B^{\frac{1}{n}}\Big)^{mn}\\
&\geq \Big((\det A^{\frac{1}{n}})^{1-\nu}(\det B^{\frac{1}{n}})^{\nu}\Big)^{mn} + \nu^{mn}\Big(\det A + \det B\Big)^m\\
&\quad - \Big(2\nu(\det(A)\det(B))^{\frac{1}{2}} - r_0((\det(A)\det(B))^{\frac{1}{4}} - (\det(A))^{\frac{1}{2}})^2\Big)^m\\
&\qquad \text{(by Corollary 2.3)}\\
&= \det(\mathrm{A}^{1-\nu}\mathrm{B}^{\nu})^m + \nu^{mn}\Big(\det A + \det B\Big)^m\\
&\quad - \Big(2\nu(\det(A)\det(B))^{\frac{1}{2}} - r_0((\det(A)\det(B))^{\frac{1}{4}} - (\det(A))^{\frac{1}{2}})^2\Big)^m.
\end{aligned}
$$

□

Theorem 3.6 *Let $A, B \in \mathcal{M}_n$ be positive definite. Then*

$$
\begin{aligned}
&\Big((1-\nu)|||AX||| + \nu|||XB|||\Big)^m\\
&\geq |||A^{1-\nu}XB^{\nu}|||^m + \nu^m\Big(|||AX||| + |||XB|||\Big)^m\\
&\quad - \Big(2\nu(|||AX||||||XB|||)^{\frac{1}{2}} - r_0((|||AX||||||XB|||)^{\frac{1}{4}} - (|||AX|||)^{\frac{1}{2}})^2\Big)^m,
\end{aligned}
$$

where $m = 1, 2, \ldots$, $0 < \nu \leq \frac{1}{2}$ and $r_0 = \min\{2\nu, 1-2\nu\}$.

Proof Applying Lemma 3.3 and Corollary 2.3 we have

$$\begin{aligned}
&|||A^{1-\nu}XB^{\nu}|||^m + \nu^m\Big(|||AX||| + |||XB|||\Big)^m \\
&\quad - \Big(2\nu(|||AX||||||XB|||)^{\frac{1}{2}} - r_0((|||AX||||||XB|||)^{\frac{1}{4}} - (|||AX|||)^{\frac{1}{2}})^2\Big)^m \\
&\leq \Big(|||AX|||^{1-\nu}|||XB|||^{\nu}\Big)^m + \nu^m\Big(|||AX||| + |||XB|||\Big)^m \\
&\quad - \Big(2\nu(|||AX||||||XB|||)^{\frac{1}{2}} - r_0((|||AX||||||XB|||)^{\frac{1}{4}} - (|||AX|||)^{\frac{1}{2}})^2\Big)^m \\
&\qquad\qquad \text{(by Lemma 3.3)} \\
&\leq \Big((1-\nu)|||AX||| + \nu|||XB|||\Big)^m \qquad \text{(by Corollary 2.3)}.
\end{aligned}$$

□

Remark 3.7 If $\frac{1}{2} \leq \nu \leq 1$, then similarly, we can prove the following inequalities

$$\begin{aligned}
&\det((1-\nu)A + \nu B)^m \geq \det(\mathrm{A}^{1-\nu}\mathrm{B}^{\nu})^m + (1-\nu)^{mn}\Big(\det A + \det B\Big)^m \\
&\quad - \Big(2(1-\nu)(\det(A)\det(\mathrm{B}))^{\frac{1}{2}} - r_0((\det(\mathrm{A})\det(B))^{\frac{1}{4}} - (\det(\mathrm{B}))^{\frac{1}{2}})^2\Big)^m,
\end{aligned}$$

and

$$\begin{aligned}
&\Big((1-\nu)|||AX||| + \nu|||XB|||\Big)^m \\
&\quad \geq |||A^{1-\nu}XB^{\nu}|||^m + (1-\nu)^m\Big(|||AX||| + |||XB|||\Big)^m \\
&\quad - \Big(2(1-\nu)(|||AX||||||XB|||)^{\frac{1}{2}} - r_0((|||AX||||||XB|||)^{\frac{1}{4}} - (|||XB|||)^{\frac{1}{2}})^2\Big)^m,
\end{aligned}$$

for all positive definite matrices $A, B \in \mathcal{M}_n, m = 1, 2, \ldots$ and $r_0 = \min\{2-2\nu, 2\nu-1\}$.

In [6], the authors showed that

$$2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \leq |||AX + XB|||,$$

where A, B are positive definite matrices and X is an arbitrary matrix. Using this inequality, inequalities (1.10) and (1.11), we have the next result.

Proposition 3.8 *Let $A, B, X \in \mathcal{M}_n$ such that A, B are positive semidefinite. Then* (i) *If* $0 < \nu \leq \frac{1}{2}$, *then*

$$\begin{aligned}
&r^2\left\|AX + XB\right\|_2^2 - 4\left(r^2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + r_0\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2 - r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} + AX\right\|_2^2\right) \\
&\quad \leq \left\|(1-\nu)AX + \nu XB\right\|_2^2 - \left\|A^{1-\nu}XB^{\nu}\right\|_2^2
\end{aligned}$$

$$\leq R^2\left\|AX+XB\right\|_2^2-\left(4\left(R^2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right)\right.$$
$$\left.-r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}+XB\right\|_2^2\right); \tag{3.3}$$

(ii) *if* $\frac{1}{2}<\nu<1$, *then*

$$R^2\left\|AX+XB\right\|_2^2-\left(4\left(R^2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+r_0\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right)-r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-XB\right\|_2^2\right)$$
$$\leq\left\|(1-\nu)AX+\nu XB\right\|_2^2-\left\|A^{1-\nu}XB^{\nu}\right\|_2^2$$
$$\leq r^2\left\|AX+XB\right\|_2^2-\left(4\left(r^2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+r_0\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\right)\right.$$
$$\left.-r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}+AX\right\|_2^2\right),$$

where $r=\min\{\nu,1-\nu\}$, $R=\max\{\nu,1-\nu\}$ *and* $r_0=\min\{2r,1-2r\}$.

Proof Let $0<\nu\leq\frac{1}{2}$. Applying

$$\left\|AX-XB\right\|_2^2=\left\|AX+XB\right\|_2^2-4\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2,$$
$$\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-AX\right\|_2^2=\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}+AX\right\|_2^2-4\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2.$$

and inequality (1.10), we get the first inequality. For $\frac{1}{2}<\nu\leq 1$, we can prove the second form of inequalities in a similar fashion. □

Applying Lemma 2.1 and inequality (3.3), we have the following theorem.

Theorem 3.9 *Let* $A, B, X\in\mathcal{M}_n$ *such that* A *and* B *are positive semidefinite. If* $\phi:[0,\infty)\to\mathbb{R}$ *is a strictly increasing convex function and* $0<\nu\leq\frac{1}{2}$, *then*

$$\phi\left(r^2\left\|AX+XB\right\|_2^2\right)-\phi\left(4\left(r^2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+r_0\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\right)\right.$$
$$\left.-r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}+AX\right\|_2^2\right)$$
$$\leq\phi\left(\left\|(1-\nu)AX+\nu XB\right\|_2^2\right)-\phi\left(\left\|A^{1-\nu}XB^{\nu}\right\|_2^2\right)$$
$$\leq\phi\left(R^2\left\|AX+XB\right\|_2^2\right)-\phi\left(4\left(R^2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right)\right.$$
$$\left.-r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}+XB\right\|_2^2\right),$$

where $r=\min\{\nu,1-\nu\}$, $R=\max\{\nu,1-\nu\}$ *and* $r_0=\min\{2r,1-2r\}$.

Remark 3.10 Note that for $\frac{1}{2} < \nu < 1$, we can get the similar inequality.

Example 3.11 If $\phi(x) = x^{\frac{m}{2}}$ $(m \geq 2)$, then using Theorem 3.9 we have

$$\begin{aligned} r^m \left\| AX + XB \right\|_2^m &- \left(4r^2 \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2 + r_0 \left\| A^{\frac{3}{4}} X B^{\frac{1}{4}} \right\|_2^2 - r_0 \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} + AX \right\|_2^2 \right)^{\frac{m}{2}} \\ &\leq \left\| (1-\nu) AX + \nu XB \right\|_2^m - \left\| A^{1-\nu} X B^{\nu} \right\|_2^m \\ &\leq R^m \left\| AX + XB \right\|_2^m - \left(4\left(R^2 \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2 + \left\| A^{\frac{1}{4}} X B^{\frac{3}{4}} \right\|_2^2 \right) \right. \\ &\quad \left. - r_0 \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} + XB \right\|_2^2 \right)^{\frac{m}{2}}. \end{aligned}$$

Replacing a and b by their squares in inequality (1.1), for $0 < \nu \leq \frac{1}{2}$, we have

$$(a^{1-\nu} b^{\nu})^2 + r_0(\sqrt{ab} - a)^2 + r(a-b)^2 \leq (1-\nu)a^2 + \nu b^2. \tag{3.4}$$

Now, applying (3.4), we have the following lemma.

Lemma 3.12 *If $a, b \geq 0$ and $0 \leq \nu \leq 1$, then*

$$(a^{1-\nu} b^{\nu} + a^{\nu} b^{1-\nu})^2 + 2r(a-b)^2 + r_0[(\sqrt{ab} - a)^2 + (\sqrt{ab} - b)^2] \leq (a+b)^2,$$

where $r = \min\{\nu, 1-\nu\}$ and $r_0 = \min\{2r, 1-2r\}$.

Proof We have

$$\begin{aligned} &(a+b)^2 - (a^{1-\nu} b^{\nu} + a^{\nu} b^{1-\nu})^2 \\ &= a^2 + b^2 - a^{2\nu} b^{2(1-\nu)} - a^{2(1-\nu)} b^{2\nu} \\ &= (1-\nu)a^2 + \nu b^2 - a^{2(1-\nu)} b^{2\nu} + \nu a^2 + (1-\nu) b^2 - a^{2\nu} b^{2(1-\nu)} \\ &\geq r(a-b)^2 + r_0(\sqrt{ab} - a)^2 + r(a-b)^2 + r_0(\sqrt{ab} - b)^2 \\ &= 2r(a-b)^2 + r_0[(\sqrt{ab} - a)^2 + (\sqrt{ab} - b)^2]. \end{aligned}$$

It follows the desired result. □

Now, applying Theorem 3.12, we improve the Heinz inequality for the Hilbert–Schmidt norm as follows:

Theorem 3.13 *Let $A, B, X \in \mathcal{M}_n$ such that A and B are positive semidefinite. If $0 < \nu \leq \frac{1}{2}$, then*

$$\begin{aligned} \left\| A^{\nu} X B^{1-\nu} + A^{1-\nu} X B^{\nu} \right\|_2^2 &\leq \left\| AX + XB \right\|_2^2 - 2r \left\| AX - XB \right\|_2^2 \\ &\quad - r_0 \left(\left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} - AX \right\|_2^2 + \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} - XB \right\|_2^2 \right), \end{aligned}$$

where $r = \min\{\nu, 1-\nu\}$ and $r_0 = \min\{2r, 1-2r\}$.

Proof Since $A, B \geq 0$, it follows that there are unitary matrices $U, V \in \mathcal{M}_n$ such that $A = UDU^*$ and $B = VEV^*$, where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $E = \text{diag}(\mu_1, \ldots, \mu_n)$ and $\lambda_i, \mu_i \geq 0$ $(i = 1, 2, \ldots, n)$. If $Y = U^*XV = [y_{ij}]$, then

$$(A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu) = U((\lambda_i^{1-\nu}\mu_j^\nu + \lambda_i^\nu\mu_j^{1-\nu})y_{ij})U^*,$$
$$AX + XB = U[(\lambda_i + \mu_j) \circ Y]V^*, \qquad AX - XB = U[(\lambda_i - \mu_j) \circ Y]V^*,$$
$$A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX = U[((\lambda_i\mu_j)^{\frac{1}{2}} - \lambda_i) \circ Y]V^*$$
$$A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB = U[((\lambda_i\mu_j)^{\frac{1}{2}} - \mu_j) \circ Y]V^*,$$

whence

$$\begin{aligned}\left\|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\right\|_2^2 &= \Big(\sum_{i,j=1}^n (\lambda_i^{1-\nu}\mu_j^\nu + \lambda_i^\nu\mu_j^{1-\nu})^2|y_{ij}|^2\Big)\\ &\leq \sum_{i,j=1}^n (\lambda_i + \mu_j)^2|y_{ij}|^2 - 2r\sum_{i,j=1}^n (\lambda_i - \mu_j)^2|y_{ij}|^2\\ &\quad - r_0\sum\Big[(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}} - \lambda_i)^2 + (\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}} - \mu_j)^2\Big]|y_{ij}|^2\\ &\qquad\text{(by Theorem 3.12)}\\ &= \left\|AX + XB\right\|_2^2 - 2r\left\|AX - XB\right\|_2^2\\ &\quad - r_0\Big(\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX\right\|_2^2 + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB\right\|_2^2\Big).\end{aligned}$$

□

Applying the triangle inequality and Lemma 3.2 we have the following result.

Proposition 3.14 *Let $A, B, X \in \mathcal{M}_n$ such that A and B are positive semidefinite. If $0 < \nu \leq \frac{1}{2}$, then*

$$\begin{aligned}&|||A^{1-\nu}XB^\nu + A^\nu XB^{1-\nu}|||\\ &\quad\leq (1-2\nu)(|||AX||| + |||XB|||) - \Big(2(2\nu\sqrt{|||AX||||||XB|||})\\ &\quad - r_0\Big((\sqrt{|||AX|||} + \sqrt[4]{|||AX||||||XB|||})^2 + (\sqrt{|||XB|||} + \sqrt[4]{|||AX||||||XB|||})^2\Big)\Big)\end{aligned}$$

Proof We have

$$\begin{aligned}&|||A^{1-\nu}XB^\nu + A^\nu XB^{1-\nu}|||\\ &\quad\leq |||A^{1-\nu}XB^\nu||| + |||A^\nu XB^{1-\nu}|||\\ &\quad\leq |||AX|||^{1-\nu}|||XB|||^\nu + |||AX|||^\nu|||XB|||^{1-\nu}\\ &\quad\leq |||AX||| + |||XB||| - 2\nu(|||AX||| + |||XB|||)\end{aligned}$$

$$- \Big(2(2\nu\sqrt{|||AX||||||XB|||}) - r_0\Big((\sqrt{|||AX|||} + \sqrt[4]{|||AX||||||XB|||})^2$$
$$+ (\sqrt{|||XB|||} + \sqrt[4]{|||AX||||||XB|||})^2\Big)\Big).$$

□

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