



Generalizations of the Aluthge Transform of Operators

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Abstract. Let A be an operator with the polar decomposition $A = U|A|$. The Aluthge transform of the operator A , denoted by \tilde{A} , is defined as $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. In this paper, first we generalize the definition of Aluthge transform for non-negative continuous functions f, g such that $f(x)g(x) = x$ ($x \geq 0$). Then, by using this definition, we get some numerical radius inequalities. Among other inequalities, it is shown that if A is bounded linear operator on a complex Hilbert space \mathbb{H} , then

$$h(w(A)) \leq \frac{1}{4} \left\| h(g^2(|A|)) + h(f^2(|A|)) \right\| + \frac{1}{2} h(w(\tilde{A}_{f,g})),$$

where f, g are non-negative continuous functions such that $f(x)g(x) = x$ ($x \geq 0$), h is a non-negative and non-decreasing convex function on $[0, \infty)$ and $\tilde{A}_{f,g} = f(|A|)Ug(|A|)$.

1. Introduction and preliminaries

2. Introduction

Let $\mathbb{B}(\mathbb{H})$ denotes the C^* -algebra of all bounded linear operators on a complex Hilbert space \mathbb{H} with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. In the case when $\dim \mathbb{H} = n$, we identify $\mathbb{B}(\mathbb{H})$ with the matrix algebra \mathbb{M}_n of all $n \times n$ matrices with entries in the complex field. For an operator $A \in \mathbb{B}(\mathbb{H})$, let $A = U|A|$ (U is a partial isometry with $\ker U = \text{range}|A|^\perp$) be the polar decomposition of A . The Aluthge transform of the operator A , denoted by \tilde{A} , is defined as $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. In [7, 21], a more general notion called t -Aluthge transform has been introduced which has later been studied. This is defined for any $0 < t \leq 1$ by $\tilde{A}_t = |A|^t U |A|^{1-t}$. Clearly, for $t = \frac{1}{2}$ we obtain the usual Aluthge transform. For the case $t = 1$, the operator $\tilde{A}_1 = |A|U$ is called the Duggal transform of $A \in \mathbb{B}(\mathbb{H})$. For $A \in \mathbb{B}(\mathbb{H})$, we generalize the Aluthge transform of the operator A to the form

$$\tilde{A}_{f,g} = f(|A|)Ug(|A|),$$

in which f, g are non-negative continuous functions such that $f(x)g(x) = x$ ($x \geq 0$). The numerical radius of $A \in \mathbb{B}(\mathbb{H})$ is defined by

$$w(A) := \sup\{|\langle Ax, x \rangle| : x \in \mathbb{H}, \|x\| = 1\}.$$

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It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $A \in \mathbb{B}(\mathbb{H})$, $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$; see [8]. Let $r(\cdot)$ denote the spectral radius. It is well known that for every operator $A \in \mathbb{B}(\mathbb{H})$, we have $r(A) \leq w(A)$. An important inequality for $w(A)$ is the power inequality stating that $w(A^n) \leq w(A)^n$ ($n = 1, 2, \dots$). For further information about the numerical radius we refer the reader to [10–12] and references therein. The quantity $w(A)$ is useful in studying perturbation, convergence and approximation problems as well as integrative methods, etc. For more information see [3, 6, 9, 13–15, 17].

Let $A, B, C, D \in \mathbb{B}(\mathbb{H})$. The operator matrices $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ and $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ are called the diagonal and off-diagonal parts of the operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, respectively.

In [16], it has been shown that if A is an operator in $\mathbb{B}(\mathbb{H})$, then

$$w(A) \leq \frac{1}{2} (\|A\| + \|A^2\|^{\frac{1}{2}}). \tag{1}$$

Several refinements and generalizations of inequality (1) have been given; see [1, 4, 5, 21–24]. Yamazaki [22] showed that for $A \in \mathbb{B}(\mathbb{H})$ and $t \in [0, 1]$ we have

$$w(A) \leq \frac{1}{2} (\|A\| + w(\tilde{A}_t)). \tag{2}$$

Davidson and Power [7] proved that if A and B are positive operators in $\mathbb{B}(\mathbb{H})$, then

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \|AB\|^{\frac{1}{2}}. \tag{3}$$

Inequality (3) has been generalized in [2, 20] and improved in [18, 19]. In [20], the author extended this inequality to the form

$$\|A + B^*\| \leq \max\{\|A\|, \|B\|\} + \frac{1}{2} (\| |A|^t |B^*|^{1-t} \| + \| |A^*|^{1-t} |B|^t \|), \tag{4}$$

in which $A, B \in \mathbb{B}(\mathbb{H})$ and $t \in [0, 1]$.

In this paper, by applying the generalized Aluthge transform of operators, we establish some inequalities involving the numerical radius. In particular, we extend inequalities (2) and (4) for two non-negative continuous functions. We also show some upper bounds for the numerical radius of 2×2 operator matrices.

3. main results

To prove our numerical radius inequalities, we need several known lemmas.

Lemma 3.1. [1, Theorem 2.2] Let $X, Y, S, T \in \mathbb{B}(\mathbb{H})$. Then

$$r(XY + ST) \leq \frac{1}{2} (w(YX) + w(TS)) + \frac{1}{2} \sqrt{(w(YX) - w(TS))^2 + 4\|YS\|\|TX\|}.$$

Lemma 3.2. [16, 22] Let $A \in \mathbb{B}(\mathbb{H})$. Then

$$(a) \quad w(A) = \max_{\theta \in \mathbb{R}} \left\| \operatorname{Re} \left(e^{i\theta} A \right) \right\|.$$

$$(b) \quad w \left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \right) = \frac{1}{2} \|A\|.$$

Polarization identity: For all $x, y \in \mathbb{H}$, we have

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^3 \|x + i^k y\|^2 i^k.$$

Now, we are ready to present our first result. The following theorem shows a generalization of inequality (2).

Theorem 3.3. Let $A \in \mathbb{B}(\mathbb{H})$ and f, g be two non-negative continuous functions on $[0, \infty)$ such that $f(x)g(x) = x$ ($x \geq 0$). Then, for all non-negative and non-decreasing convex function h on $[0, \infty)$, we have

$$h(w(A)) \leq \frac{1}{4} \|h(g^2(|A|)) + h(f^2(|A|))\| + \frac{1}{2} h(w(\tilde{A}_{f,g})).$$

Proof. Let x be any unit vector. Then

$$\begin{aligned} \operatorname{Re} \langle e^{i\theta} Ax, x \rangle &= \operatorname{Re} \langle e^{i\theta} U|A|x, x \rangle \\ &= \operatorname{Re} \langle e^{i\theta} Ug(|A|)f(|A|x), x \rangle \\ &= \operatorname{Re} \langle e^{i\theta} f(|A|x), g(|A|)U^*x \rangle \\ &= \frac{1}{4} \left\| (e^{i\theta} f(|A|) + g(|A|)U^*)x \right\|^2 - \frac{1}{4} \left\| (e^{i\theta} f(|A|) - g(|A|)U^*)x \right\|^2 \\ &\hspace{15em} \text{(by polarization identity)} \\ &\leq \frac{1}{4} \left\| (e^{i\theta} f(|A|) + g(|A|)U^*)x \right\|^2 \\ &\leq \frac{1}{4} \left\| (e^{i\theta} f(|A|) + g(|A|)U^*) \right\|^2 \\ &= \frac{1}{4} \left\| (e^{i\theta} f(|A|) + g(|A|)U^*) (e^{-i\theta} f(|A|) + Ug(|A|)) \right\| \\ &= \frac{1}{4} \left\| g^2(|A|) + f^2(|A|) + e^{i\theta} \tilde{A}_{f,g} + e^{-i\theta} (\tilde{A}_{f,g})^* \right\| \\ &\leq \frac{1}{4} \|g^2(|A|) + f^2(|A|)\| + \frac{1}{4} \|e^{i\theta} \tilde{A}_{f,g} + e^{-i\theta} (\tilde{A}_{f,g})^*\| \\ &= \frac{1}{4} \|g^2(|A|) + f^2(|A|)\| + \frac{1}{2} \left\| \operatorname{Re} (e^{i\theta} \tilde{A}_{f,g}) \right\| \\ &\leq \frac{1}{4} \|g^2(|A|) + f^2(|A|)\| + \frac{1}{2} w(\tilde{A}_{f,g}). \end{aligned}$$

Now, taking the supremum over all unit vectors $x \in \mathbb{H}$ and applying Lemma 3.2 in the above inequality produces

$$w(A) \leq \frac{1}{4} \|g^2(|A|) + f^2(|A|)\| + \frac{1}{2} w(\tilde{A}_{f,g}).$$

Therefore,

$$\begin{aligned}
 h(w(A)) &\leq h\left(\frac{1}{4} \|g^2(|A|) + f^2(|A|)\| + \frac{1}{2} w(\tilde{A}_{f,g})\right) \\
 &= h\left(\frac{1}{2} \left\| \frac{g^2(|A|) + f^2(|A|)}{2} \right\| + \frac{1}{2} w(\tilde{A}_{f,g})\right) \\
 &\leq \frac{1}{2} h\left(\left\| \frac{g^2(|A|) + f^2(|A|)}{2} \right\|\right) + \frac{1}{2} h(w(\tilde{A}_{f,g})) \\
 &\hspace{15em} \text{(by the convexity of } h) \\
 &= \frac{1}{2} \left\| h\left(\frac{g^2(|A|) + f^2(|A|)}{2}\right) \right\| + \frac{1}{2} h(w(\tilde{A}_{f,g})) \\
 &\leq \frac{1}{4} \left\| h(g^2(|A|)) + h(f^2(|A|)) \right\| + \frac{1}{2} h(w(\tilde{A}_{f,g})) \\
 &\hspace{15em} \text{(by the convexity of } h).
 \end{aligned}$$

□

Theorem 3.3 includes some special cases as follows.

Corollary 3.4. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for all non-negative and non-decreasing convex function h on $[0, \infty)$ and all $t \in [0, 1]$, we have*

$$h(w(A)) \leq \frac{1}{4} \left\| h(|A|^{2t}) + h(|A|^{2(1-t)}) \right\| + \frac{1}{2} h(w(\tilde{A}_t)). \tag{5}$$

Corollary 3.5. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for all $t \in [0, 1]$ and $s \geq 1$, we have*

$$w^s(A) \leq \frac{1}{4} \left\| |A|^{2ts} + |A|^{2(1-t)s} \right\| + \frac{1}{2} w^s(\tilde{A}_t).$$

In particular,

$$w^s(A) \leq \frac{1}{2} \left(\|A\|^s + w^s(\tilde{A}) \right).$$

Proof. The first inequality follows from inequality (5) for the function $h(x) = x^s$ ($s \geq 1$). For the particular case, it is enough to put $t = \frac{1}{2}$. □

Theorem 3.3 gives the next result for the off-diagonal operator matrix $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$.

Theorem 3.6. *Let $A, B \in \mathbb{B}(\mathbb{H})$, f, g be two non-negative continuous functions on $[0, \infty)$ such that $f(x)g(x) = x$ ($x \geq 0$) and $s \geq 1$. Then*

$$\begin{aligned}
 w^s\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\leq \frac{1}{4} \max\left(\|g^{2s}(|A|) + f^{2s}(|A|)\|, \|g^{2s}(|B|) + f^{2s}(|B|)\|\right) \\
 &\quad + \frac{1}{4} \left(\|f(|B|)g(|A^*|)\|^s + \|f(|A|)g(|B^*|)\|^s \right).
 \end{aligned}$$

Proof. Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of A and B , respectively, and let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$.

It follows from the polar decomposition of $T = \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} \begin{bmatrix} |B| & 0 \\ 0 & |A| \end{bmatrix}$ that

$$\begin{aligned} \tilde{T}_{f,g} &= f(|T|) \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} g(|T|) \\ &= \begin{bmatrix} f(|B|) & 0 \\ 0 & f(|A|) \end{bmatrix} \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} \begin{bmatrix} g(|B|) & 0 \\ 0 & g(|A|) \end{bmatrix} \\ &= \begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ f(|A|)Vg(|B|) & 0 \end{bmatrix}. \end{aligned}$$

Using $|A^*|^2 = AA^* = U|A|^2U^*$ and $|B^*|^2 = BB^* = V|B|^2V^*$ we have $g(|A|) = U^*g(|A^*|)U$ and $g(|B|) = V^*g(|B^*|)V$ for every non-negative continuous function g on $[0, \infty)$. Therefore,

$$\begin{aligned} w(\tilde{T}_{f,g}) &= w\left(\begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ f(|A|)Vg(|B|) & 0 \end{bmatrix}\right) \\ &\leq w\left(\begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ 0 & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & 0 \\ f(|A|)Vg(|B|) & 0 \end{bmatrix}\right) \\ &= w\left(\begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ 0 & 0 \end{bmatrix}\right) + w\left(W^* \begin{bmatrix} 0 & f(|A|)Vg(|B|) \\ 0 & 0 \end{bmatrix} W\right) \\ &= w\left(\begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ 0 & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & f(|A|)Vg(|B|) \\ 0 & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}\|f(|B|)Ug(|A|)\| + \frac{1}{2}\|f(|A|)Vg(|B|)\| \\ &\quad \text{(by Lemma 3.2(b))} \\ &= \frac{1}{2}\|f(|B|)UU^*g(|A^*|)U\| + \frac{1}{2}\|f(|A|)VV^*g(|B^*|)V\| \\ &\leq \frac{1}{2}\|f(|B|)g(|A^*|)\| + \frac{1}{2}\|f(|A|)g(|B^*|)\|, \end{aligned} \tag{6}$$

where $W = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is unitary. Applying Theorem 3.3 and inequality (6), we have

$$\begin{aligned} w^s(T) &\leq \frac{1}{4}\|g^{2s}(|T|) + f^{2s}(|T|)\| + \frac{1}{2}(w^s(\tilde{T}_{f,g})) \\ &\leq \frac{1}{4}\max(\|g^{2s}(|A|) + f^{2s}(|A|)\|, \|g^{2s}(|B|) + f^{2s}(|B|)\|) \\ &\quad + \frac{1}{2}\left[\frac{1}{2}(\|f(|B|)g(|A^*|)\| + \|f(|A|)g(|B^*|)\|)\right]^s \\ &\leq \frac{1}{4}\max(\|g^{2s}(|A|) + f^{2s}(|A|)\|, \|g^{2s}(|B|) + f^{2s}(|B|)\|) \\ &\quad + \frac{1}{4}\|f(|B|)g(|A^*|)\|^s + \frac{1}{4}\|f(|A|)g(|B^*|)\|^s \\ &\quad \text{(by the convexity } h(x) = x^s\text{)}. \end{aligned}$$

□

Corollary 3.7. Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for all $t \in [0, 1]$ and $s \geq 1$, we have

$$w^{\frac{s}{2}}(AB) \leq \frac{1}{4} \max(\| |A|^{2ts} + |A|^{2(1-t)s} \|, \| |B|^{2ts} + |B|^{2(1-t)s} \|) + \frac{1}{4} (\| |A|^t |B^*|^{1-t} \|^s + \| |B|^t |A^*|^{1-t} \|^s).$$

Proof. Applying the power inequality of the numerical radius ($w(A^n) \leq w^n(A)$ [19]), we have

$$\begin{aligned} w^{\frac{s}{2}}(AB) &\leq \max(w^{\frac{s}{2}}(AB), w^{\frac{s}{2}}(BA)) \\ &= w^{\frac{s}{2}} \left(\begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix} \right) \\ &= w^{\frac{s}{2}} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^2 \right) \\ &\leq w^s \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \\ &\leq \frac{1}{4} \max(\| |A|^{2ts} + |A|^{2(1-t)s} \|, \| |B|^{2ts} + |B|^{2(1-t)s} \|) \\ &\quad + \frac{1}{4} (\| |A|^t |B^*|^{1-t} \|^s + \| |B|^t |A^*|^{1-t} \|^s) \\ &\quad \text{(by Theorem 3.6)}. \end{aligned}$$

□

Corollary 3.8. Let $A, B \in \mathbb{B}(\mathbb{H})$ be positive operators. Then, for all $t \in [0, 1]$ and $s \geq 1$, we have

$$\| |A^{\frac{1}{2}} B^{\frac{1}{2}} \|^s \leq \frac{1}{4} \max(\| |A|^{ts} + |A|^{(1-t)s} \|, \| |B|^{ts} + |B|^{(1-t)s} \|) + \frac{1}{4} (\| |A|^t |B|^{1-t} \|^s + \| |B|^t |A|^{1-t} \|^s).$$

Proof. Since the spectral radius of any operator is dominated by its numerical radius, then $r^{\frac{1}{2}}(AB) \leq w^{\frac{1}{2}}(AB)$. Applying a commutativity property of the spectral radius, we get

$$\begin{aligned} r^{\frac{s}{2}}(AB) &= r^{\frac{s}{2}}(A^{\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}}) \\ &= r^{\frac{s}{2}}(A^{\frac{1}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}}) \\ &= r^{\frac{s}{2}}(A^{\frac{1}{2}} B^{\frac{1}{2}} (A^{\frac{1}{2}} B^{\frac{1}{2}})^*) \\ &= \left\| |A^{\frac{1}{2}} B^{\frac{1}{2}} (A^{\frac{1}{2}} B^{\frac{1}{2}})^* \right\|^{\frac{s}{2}} \\ &= \left\| |A^{\frac{1}{2}} B^{\frac{1}{2}} \right\|^s. \end{aligned} \tag{7}$$

Now, the result follows from Corollary 3.7. □

An important special case of Theorem 3.6, which generalizes inequality (4) can be stated as follows.

Corollary 3.9. Let $A, B \in \mathbb{B}(\mathbb{H})$ and $s \geq 1$. Then

$$\| |A + B| \|^s \leq \frac{1}{2^{2-s}} \max(\| |A|^{2ts} + |A|^{2(1-t)s} \|, \| |B^*|^{2ts} + |B^*|^{2(1-t)s} \|) + \frac{1}{2^{2-s}} (\| |A|^t |B|^{1-t} \|^s + \| |B^*|^t |A^*|^{1-t} \|^s).$$

In particular, if A and B are normal, then

$$\|A + B\|^s \leq \frac{1}{2^{1-s}} \max(\|A\|^s, \|B\|^s) + \frac{1}{2^{1-s}} \|AB\|^{\frac{s}{2}}.$$

Proof. Applying Lemma 3.2 and Theorem 3.3, we have

$$\begin{aligned} \|A + B^*\|^s &= \|T + T^*\|^s \\ &\leq 2^s \max_{\theta \in \mathbb{R}} \left\| \operatorname{Re}(e^{i\theta} T) \right\|^s \\ &= 2^s w^s(T) \\ &\leq \frac{2^s}{4} \max(\| |A|^{2ts} + |A|^{2(1-t)s} \|, \| |B|^{2ts} + |B|^{2(1-t)s} \|) \\ &\quad + \frac{2^s}{4} (\| |A|^t |B^*|^{1-t} \|^s + \| |B|^t |A^*|^{1-t} \|^s) \\ &\hspace{10em} \text{(by Theorem 3.6),} \end{aligned}$$

where $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$, $f(x) = x^t$, and $g(x) = x^{1-t}$. Now, the desired result follows by replacing B by B^* . For the particular case $t = \frac{1}{2}$. If A and B are normal, then $|B^*| = |B|$ and $|A^*| = |A|$. Applying equality (7) for the operators $|A|^{\frac{1}{2}}$ and $|B|^{\frac{1}{2}}$, we have

$$\begin{aligned} \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|^s &= r^{\frac{s}{2}} (|A| |B|) \\ &\leq \| |A| |B| \|^{\frac{s}{2}} \\ &= \| U^* A B^* V \|^{\frac{s}{2}} \\ &= \| AB^* \|^{\frac{s}{2}}, \end{aligned}$$

where $A = U|A|$ and $B = V|B|$ are the polar decompositions of the operators A and B . This completes the proof of the corollary. \square

In the next result, we show another generalization of inequality (2).

Theorem 3.10. Let $A \in \mathbb{B}(\mathbb{H})$ and f, g, h be non-negative and non-decreasing continuous functions on $[0, \infty)$ such that $f(x)g(x) = x$ ($x \geq 0$) and h is convex. Then

$$h(w(A)) \leq \frac{1}{2} \left(h(w(\tilde{A}_{f,g})) + \|h(|A|)\| \right).$$

Proof. Let $A = U|A|$ be the polar decomposition of A . Then for every $\theta \in \mathbb{R}$, we have

$$\begin{aligned} \|\operatorname{Re}(e^{i\theta} A)\| &= r(\operatorname{Re}(e^{i\theta} A)) \\ &= \frac{1}{2} r(e^{i\theta} A + e^{-i\theta} A^*) \\ &= \frac{1}{2} r(e^{i\theta} U|A| + e^{-i\theta} |A|U^*) \\ &= \frac{1}{2} r(e^{i\theta} U g(|A|) f(|A|) + e^{-i\theta} f(|A|) g(|A|) U^*). \end{aligned} \tag{8}$$

Now, if we put $X = e^{i\theta}Ug(|A|)$, $Y = f(|A|)$, $S = e^{-i\theta}f(|A|)$ and $T = g(|A|)U^*$ in Lemma 3.1, then we get

$$\begin{aligned}
 & r(e^{i\theta}Ug(|A|)f(|A|) + e^{-i\theta}f(|A|)g(|A|)U^*) \\
 & \leq \frac{1}{2}(w(f(|A|)Ug(|A|)) + w(g(|A|)U^*f(|A|))) \\
 & \quad + \frac{1}{2}\sqrt{4\|e^{-i\theta}f(|A|)g(|A|)\| \|g(|A|)U^*e^{i\theta}Uf(|A|)\|} \\
 & \qquad \qquad \qquad \text{(by Lemma 3.1)} \\
 & \leq w(f(|A|)Ug(|A|)) + \sqrt{\|f(|A|)\| \|f(|A|)\| \|g(|A|)\| \|g(|A|)\|} \\
 & = w(f(|A|)Ug(|A|)) + \sqrt{f(\|A\|)g(\|A\|)g(\|A\|)f(\|A\|)} \\
 & = w(f(|A|)Ug(|A|)) + \sqrt{\|A\|\|A\|} \\
 & = w(\tilde{A}_{f,g}) + \|A\|. \tag{9}
 \end{aligned}$$

Note that, since $w(X) = w(X^*)$ ($X \in \mathbb{B}(\mathbb{H})$), in the first inequality we have

$$w(YX) - w(TS) = w(f(|A|)Ug(|A|)) - w(g(|A|)U^*f(|A|)) = 0.$$

Using inequalities (8), (9) and Lemma 3.2 we get

$$w(A) = \max_{\theta \in \mathbb{R}} \left\| \operatorname{Re}(e^{i\theta}A) \right\| \leq \frac{1}{2}(w(\tilde{A}_{f,g}) + \|A\|).$$

Hence

$$\begin{aligned}
 h(w(A)) & \leq h\left(\frac{1}{2}[w(\tilde{A}_{f,g}) + \|A\|]\right) \\
 & \qquad \qquad \qquad \text{(by the monotonicity of } h) \\
 & \leq \frac{1}{2}h(w(\tilde{A}_{f,g})) + \frac{1}{2}h(\|A\|) \\
 & \qquad \qquad \qquad \text{(by the convexity of } h) \\
 & = \frac{1}{2}h(w(\tilde{A}_{f,g})) + \frac{1}{2}\|h(|A|)\|,
 \end{aligned}$$

as required. \square

Remark 3.11. We can obtain Theorem 3.3 from Theorem 3.10, but we keep the proof for the readers. To see this, first note that by the hypotheses of Theorem 3.3 we have

$$\begin{aligned}
 h(|A|) & = h(g(|A|)f(|A|)) \\
 & \leq h\left(\frac{g^2(|A|) + f^2(|A|)}{2}\right) \quad \text{(by the arithmetic-geometric inequality)} \\
 & \leq \frac{1}{2}(h(g^2(|A|)) + h(f^2(|A|))) \quad \text{(by the convexity of } h). \tag{10}
 \end{aligned}$$

Hence, using Theorem 3.10 and inequality (10) we get

$$\begin{aligned}
 h(w(A)) & \leq \frac{1}{2}[h(w(\tilde{A}_{f,g})) + \|h(|A|)\|] \\
 & \leq \frac{1}{2}[h(w(\tilde{A}_{f,g})) + \frac{1}{2}\|h(g^2(|A|)) + h(f^2(|A|))\|] \\
 & = \frac{1}{2}h(w(\tilde{A}_{f,g})) + \frac{1}{4}\|h(g^2(|A|)) + h(f^2(|A|))\|.
 \end{aligned}$$

Remark 3.12. For the special case $f(x) = x^t$ and $g(x) = x^{1-t}$ ($t \in [0, 1]$), we obtain the inequality (2)

$$w(A) \leq \frac{1}{2} (w(\tilde{A}_t) + \|A\|),$$

where $A \in \mathbb{B}(\mathbb{H})$.

Let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. Using Theorem 3.10, we get the following result.

Corollary 3.13. Let $A, B \in \mathbb{B}(\mathbb{H})$ and f, g be two non-negative and non-decreasing continuous functions such that $f(x)g(x) = x$ ($x \geq 0$). Then

$$2w^s \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \max\{\|A\|^s, \|B\|^s\} + \frac{1}{2} (\|f(|B|)g(|A^*|)\|^s + \|f(|A|)g(|B^*|)\|^s),$$

where $s \geq 1$.

Proof. Using Theorem 3.10 and inequality (6), we have

$$\begin{aligned} 2w^s \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) &\leq \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\|^s + w^s(\tilde{T}_{f,g}) \\ &= \max\{\|A\|^s, \|B\|^s\} + \left(\frac{1}{2} [\|f(|B|)g(|A^*|)\| + \|f(|A|)g(|B^*|)\|] \right)^s \\ &\leq \max\{\|A\|^s, \|B\|^s\} + \frac{1}{2} (\|f(|B|)g(|A^*|)\|^s + \|f(|A|)g(|B^*|)\|^s) \end{aligned}$$

and the proof is complete. \square

Using similar arguments to the proof of Corollary 3.9, we get the following result.

Corollary 3.14. Let $A, B \in \mathbb{B}(\mathbb{H})$ and f, g be two non-negative and non-decreasing continuous functions on $[0, \infty)$ such that $f(x)g(x) = x$ ($x \geq 0$). Then

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \frac{1}{2} (\|f(|B|)g(|A|)\| + \|f(|A^*|)g(|B^*|)\|).$$

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