SOME BEREZIN NUMBER INEQUALITIES FOR OPERATOR MATRICES

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Abstract. The Berezin symbol $\widetilde{A}$ of an operator A acting on the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ over some (non-empty) set is defined by $\widetilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle$ $(\lambda \in \Omega)$, where $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ is the normalized reproducing kernel of $\mathcal{H}$. The Berezin number of operator A is defined by $\mathbf{ber}(A) = \sup\limits_{\lambda\in\Omega}\big|\tilde{A}(\lambda)\big| = \sup\limits_{\lambda\in\Omega}\big|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle\big|$. Moreover $\mathbf{ber}(A) \leqslant w(A)$ (numerical radius). In this paper, we present some Berezin number inequalities. Among other inequalities, it is shown that if $\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$, then

$$\mathbf{ber}(\mathbf{T}) \leqslant \frac{1}{2}\left(\mathbf{ber}(A) + \mathbf{ber}(D)\right) + \frac{1}{2}\sqrt{\left(\mathbf{ber}(A) - \mathbf{ber}(D)\right)^2 + (\|B\| + \|C\|)^2}.$$

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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$ with the identity I. In the case when $\dim\mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices having entries in the complex field. An operator $A \in \mathbb{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathcal{H}$, and then we write $A \geqslant 0$. Let $r(\,\cdot\,)$ denote the spectral radius. The numerical range and numerical radius of $A \in \mathbb{B}(\mathcal{H})$ are defined by

$$W(A) := \big\{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\big\} \quad \text{and} \quad w(A) := \sup\big\{|\lambda| : \lambda \in W(A)\big\},$$

respectively. It is well known that $w(\,\cdot\,)$ defines a norm on $\mathbb{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $A \in \mathbb{B}(\mathcal{H})$, $\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|$

(see [4, p. 9]). For further information about numerical radius we refer the reader to [1, 5, 16].
A functional Hilbert space is a Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on a (non-empty) set Ω, which has the property that point evaluations are continuous i.e., for each $\lambda \in \Omega$ the map $f \longrightarrow f(\lambda)$ is a continuous linear functional on $\mathcal{H}$. Then the Riesz representation theorem ensures that for each $\lambda \in \Omega$ there is a unique element k_λ of $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The collection $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of $\mathcal{H}$. If $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by [6, Problem 37]

$$k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z).$$

For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathcal{H}$. For a bounded linear operator A on $\mathcal{H}$, the function $\widetilde{A}$ defined on Ω by $\widetilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle$ is the Berezin symbol of A, which firstly have been introduced by Berezin [2, 3]. Berezin set and Berezin number of operator A are defined by (see [10])

$$\mathbf{Ber}(A) := \left\{\widetilde{A}(\lambda) : \lambda \in \Omega\right\} \quad \text{and} \quad \mathbf{ber}(A) := \sup\left\{|\widetilde{A}(\lambda)| : \lambda \in \Omega\right\},$$

respectively. It is clear that the Berezin symbol $\widetilde{A}$ is a bounded function on Ω whose values lies in the numerical range of the operator A and hence

$$\mathbf{Ber}(A) \subseteq W(A) \qquad \text{and} \qquad \mathbf{ber}(A) \leqslant w(A)$$

for all $A \in \mathbb{B}(\mathcal{H})$. Karaev [13] showed that if $\mathcal{H}^2$ is the Hardy space, then we take $A = \langle \cdot\, , z \rangle z$ in $\mathcal{H}^2$, an elementary calculation shows that $\widetilde{A}(\lambda) = |\lambda|^2(1 - |\lambda|^2)$, and thus

$$\mathbf{Ber}(A) = \left[0, \frac{1}{4}\right] \subsetneq [0, 1] = W(A) \qquad \text{and} \qquad \mathbf{ber}(A) = \frac{1}{4} \lneq 1 = w(A).$$

Moreover, Berezin number of an operator A satisfies the following properties:
(a) $\mathbf{ber}(\alpha A) = |\alpha| \mathbf{ber}(A)$ for all $\alpha \in \mathcal{C}$.
(b) $\mathbf{ber}(A + B) \leqslant \mathbf{ber}(A) + \mathbf{ber}(B)$.

Let $T_i \in \mathbb{B}(\mathcal{H}(\Omega))$ $(1 \leqslant i \leqslant n)$. Then we define the generalized Euclidean Berezin number of $T_1, \cdots, T_n$ as follows

$$\mathbf{ber}_p(T_1, \ldots, T_n) := \sup_{\lambda \in \Omega} \left(\sum_{i=1}^n \left| \left\langle T_i \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^p \right)^{\frac{1}{p}}.$$

The generalized Euclidean Berezin number $\mathbf{ber}_p$ ($p \geqslant 1$) has the following properties:
(a) $\mathbf{ber}_p(\alpha T_1, \ldots, \alpha T_n) = |\alpha|\, \mathbf{ber}_p(T_1, \ldots, T_n)$ for all $\alpha \in \mathcal{C}$;
(b) $\mathbf{ber}_p(T_1 + S_1, \ldots, T_n + S_n) \leqslant \mathbf{ber}_p(T_1, \ldots, T_n) + \mathbf{ber}_p(S_1, \ldots, S_n)$,
where $T_i, S_i \in \mathbb{B}(\mathcal{H}(\Omega))$ $(1 \leqslant i \leqslant n)$.
Namely, the Berezin symbol have been investigated in detail for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it has wide applications in various questions of analysis in the various questions of analysis and uniquely determines the operator (i.e., $\widetilde{A}(\lambda) = \widetilde{B}(\lambda)$ for all $\lambda \in \Omega$ implies $A = B$). For further information about Berezin symbol we refer the reader to [11, 12, 13, 15, 17] and references therein.
Let $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_n$ be Hilbert spaces, and consider the direct sum $\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j$. With respect to this decomposition, every operator $T \in \mathbb{B}(\mathcal{H})$ has an $n \times n$ operator matrix representation $T = [T_{ij}]$ with entries $T_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$, the space of all bounded linear operators from $\mathcal{H}_j$ to $\mathcal{H}_i$. Let $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_1)$, $B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$, $C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $D \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_2)$. The operator matrix $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ is called the diagonal part of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ is the off-diagonal part. Operator matrices provide a usual tool for studying Hilbert space operators, which have been extensively studied in the literature. J.C. Hou et al. [9] and A. Omer et al. [1] established useful estimates for the spectral radius, the numerical radius, and the operator norm of an $n \times n$ operator matrix $\mathbf{T} = [T_{ij}]$. In particular, they proved that

$$r(\mathbf{T}) \leqslant r\left([\|T_{ij}\|]\right), \quad w(\mathbf{T}) \leqslant w\left([\|T_{ij}\|]\right), \quad \|\mathbf{T}\| \leqslant \|[\|T_{ij}\|]\|$$

and $w(\mathbf{T}) \leqslant w\left([t_{ij}]\right)$, where $t_{ij} = \begin{cases} w(T_{ij}) & \text{if } i = j, \\ \|T_{ij}\| & \text{if } i \neq j. \end{cases}$

The Berezin number is named in honor of F. Berezin, who introduced this concept in [2]. In this paper, we establish some inequalities involving the Berezin number of operators. By using the some ideas of [1, 16] we give several upper bounds for the Berezin number and the generalized Euclidean Berezin number of Hilbert space operators.

2. Main results

Now we are in a position to present our first result.

Theorem 2.1. *Let $\mathbf{T} = [T_{ij}]$ be $n \times n$ operator matrix with $T_{ij} \in \mathbb{B}(\mathcal{H}(\Omega_j), \mathcal{H}(\Omega_i))$ $(1 \leqslant i, j \leqslant n)$. Then*

$$\mathbf{ber}(\mathbf{T}) \leqslant w\left([t_{ij}]\right),$$

where $t_{ij} = \begin{cases} \mathbf{ber}(T_{ij}) & \text{if } i = j, \\ \|T_{ij}\| & \text{if } i \neq j. \end{cases}$

Proof. Let $\mathcal{H} = \oplus_{i=1}^{n} \mathcal{H}(\Omega_i)$. For every $(\lambda_1, \ldots, \lambda_n) \in \Omega_1 \times \cdots \times \Omega_n$, let $\hat{\mathbf{k}}_{(\lambda_1,\ldots,\lambda_n)} = \begin{bmatrix} k_{\lambda_1} \\ \vdots \\ k_{\lambda_n} \end{bmatrix}$ be the normalized reproducing kernel of $\mathcal{H}$. Then

$$\begin{aligned}
\left|\widetilde{\mathbf{T}}(\lambda_1, \cdots, \lambda_n)\right| &= \left|\left\langle \mathbf{T}\hat{\mathbf{k}}_{(\lambda_1,\cdots,\lambda_n)}, \hat{\mathbf{k}}_{(\lambda_1,\cdots,\lambda_n)} \right\rangle\right| \\
&= \left| \sum_{i,j=1}^{n} \left\langle T_{ij} k_{\lambda_j}, k_{\lambda_i} \right\rangle \right| \\
&\leqslant \sum_{i,j=1}^{n} \left| \left\langle T_{ij} k_{\lambda_j}, k_{\lambda_i} \right\rangle \right| \\
&= \sum_{i=1}^{n} \left| \left\langle T_{ii} k_{\lambda_i}, k_{\lambda_i} \right\rangle \right| + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left| \left\langle T_{ij} k_{\lambda_j}, k_{\lambda_i} \right\rangle \right| \\
&\leqslant \sum_{i=1}^{n} \mathbf{ber}(T_{ii}) \|k_{\lambda_i}\|^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \|T_{ij}\| \|k_{\lambda_j}\| \|k_{\lambda_i}\| \\
&= \sum_{i,j=1}^{n} t_{ij} \|k_{\lambda_j}\| \|k_{\lambda_i}\| \\
&= \langle [t_{ij}] y, y \rangle \,,
\end{aligned}$$

where $y = \begin{bmatrix} \|k_{\lambda_1}\| \\ \vdots \\ \|k_{\lambda_n}\| \end{bmatrix}$. It follows from $\|y\| = 1$, that $\left|\widetilde{\mathbf{T}}(\lambda_1, \ldots, \lambda_n)\right| \leqslant w([t_{ij}])$. Hence

$$\mathbf{ber}(\mathbf{T}) = \sup_{(\lambda_1,\ldots,\lambda_n) \in \Omega_1 \times \cdots \times \Omega_n} \left|\widetilde{\mathbf{T}}(\lambda_1, \ldots, \lambda_n)\right| \leqslant w([t_{ij}])$$

as required. □

Corollary 2.1. *If* $\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$, *then*

$$\mathbf{ber}(\mathbf{T}) \leqslant \frac{1}{2} \left(\mathbf{ber}(A) + \mathbf{ber}(D)\right) + \frac{1}{2} \sqrt{\left(\mathbf{ber}(A) - \mathbf{ber}(D)\right)^2 + (\|B\| + \|C\|)^2}.$$

In particular, for $\mathbf{T} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$ *we have*

$$\mathbf{ber}(\mathbf{T}) \leqslant \max \left\{\mathbf{ber}(A), \mathbf{ber}(D)\right\}. \tag{2.1}$$

Proof. Using Theorem 2.1 we get the inequality

$$\begin{aligned}
&\mathbf{ber}\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\right) \\
&\leqslant w\left(\left[\begin{array}{cc} \mathbf{ber}A & \|B\| \\ \|C\| & \mathbf{ber}D \end{array}\right]\right) \\
&\leqslant r\left(\left[\begin{array}{cc} \mathbf{ber}A & \frac{\|B\|+\|C\|}{2} \\ \frac{\|B\|+\|C\|}{2} & \mathbf{ber}D \end{array}\right]\right) \qquad \text{(by [8, p. 44])} \\
&= \frac{1}{2}\left(\mathbf{ber}(A)+\mathbf{ber}(D)\right)+\frac{1}{2}\sqrt{\left(\mathbf{ber}(A)-\mathbf{ber}(D)\right)^2+(\|B\|+\|C\|)^2}.
\end{aligned}$$

In particular, if $\mathbf{T} = \left[\begin{array}{cc} A & 0 \\ 0 & D \end{array}\right] \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$, then

$$\begin{aligned}
\mathbf{ber}(\mathbf{T}) &\leqslant \frac{\mathbf{ber}(A)+\mathbf{ber}(D)+\sqrt{\left(\mathbf{ber}(A)-\mathbf{ber}(D)\right)^2}}{2} \\
&= \frac{\mathbf{ber}(A)+\mathbf{ber}(D)+|\mathbf{ber}(A)-\mathbf{ber}(D)|}{2} \\
&= \max\left\{\mathbf{ber}(A), \mathbf{ber}(D)\right\}.
\end{aligned}$$

□

We need the following lemmas for the next results. The next lemma follows from the spectral theorem for positive operators and Jensen inequality; see [14].

Lemma 2.1. *(The McCarty inequality) Let $T \in \mathbb{B}(\mathcal{H})$, $T \geqslant 0$ and $x \in \mathcal{H}$ such that $\|x\| \leqslant 1$. Then*

$$\langle Tx, x\rangle^r \leqslant \langle T^r x, x\rangle$$

for $r \geqslant 1$.

Proof. Let $r \geqslant 1$ and $x \in \mathcal{H}$ such that $\|x\| \leqslant 1$. Fix $u = \frac{x}{\|x\|}$. Using the McCarty inequality we have $\langle Tu, u\rangle^r \leqslant \langle T^r u, u\rangle$, whence

$$\begin{aligned}
\langle Tx, x\rangle^r &\leqslant \|x\|^{2r-2}\langle T^r x, x\rangle \\
&\leqslant \langle T^r x, x\rangle \qquad (\text{since } \|x\| \leqslant 1 \text{ and } 2r-2 \geqslant 0).
\end{aligned}$$

Hence, we get the desired result. □

Lemma 2.2. [14, Theorem 1] *Let $T \in \mathbb{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any vectors. If f, g are nonnegative continuous functions on $[0,\infty)$ which are satisfying the relation $f(t)g(t) = t\,(t \in [0,\infty))$, then*

$$|\langle Tx, y\rangle|^2 \leqslant \left\langle f^2(|T|)x, x\right\rangle \left\langle g^2(|T^*|)y, y\right\rangle,$$

in which $|T| = (T^*T)^{\frac{1}{2}}$.

Theorem 2.2. *Let $\mathbf{T} = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$, $r \geqslant 1$ and f, g be nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0,\infty))$. Then*

$$\mathbf{ber}^r(\mathbf{T}) \leqslant 2^{r-2}\mathbf{ber}^{\frac{1}{2}}\left(f^{2r}(|X|) + g^{2r}(|Y^*|)\right)\mathbf{ber}^{\frac{1}{2}}\left(f^{2r}(|Y|) + g^{2r}(|X^*|)\right).$$

Proof. For every $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$, let $\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel of $\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2)$ (i.e., $\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2 = 1$). Then

$$\begin{aligned}
&\left|\widetilde{\mathbf{T}}(\lambda_1,\lambda_2)\right|^r \\
&= \left|\left\langle \mathbf{T}\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}, \hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}\right\rangle\right|^r \\
&= |\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle + \langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|^r \\
&\leqslant (|\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle| + |\langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|)^r \qquad \text{(by the triangular inequality)} \\
&\leqslant \frac{2^r}{2}\left(|\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle|^r + |\langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|^r\right) \qquad \text{(by the convexity } f(t) = t^r\text{)} \\
&\leqslant \frac{2^r}{2}\Bigg(\left(\langle f^2(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}}\langle g^2(|X^*|)k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}}\right)^r \\
&\qquad + \left(\langle f^2(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}}\langle g^2(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}}\right)^r\Bigg) \qquad \text{(by Lemma 2.2)} \\
&\leqslant \frac{2^r}{2}\left(\langle f^{2r}(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}}\langle g^{2r}|X^*|k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}} + \langle f^{2r}(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}}\langle g^{2r}(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}}\right) \\
&\qquad\qquad \text{(by Lemma 2.1))} \\
&\leqslant \frac{2^r}{2}\left(\langle f^{2r}(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle + \langle g^{2r}(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle\right)^{\frac{1}{2}} \\
&\quad \times \left(\langle f^{2r}(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle + \langle g^{2r}(|X^*|)k_{\lambda_1}, k_{\lambda_1}\rangle\right)^{\frac{1}{2}} \quad \text{(by the Cauchy-Schwarz inequality)} \\
&= \frac{2^r}{2}\left\langle (f^{2r}(|X|) + g^{2r}(|Y^*|))k_{\lambda_2}, k_{\lambda_2}\right\rangle^{\frac{1}{2}}\left\langle (f^{2r}(|Y|) + g^{2r}(|X^*|))k_{\lambda_1}, k_{\lambda_1}\right\rangle^{\frac{1}{2}} \\
&\leqslant \frac{2^r}{2}\mathbf{ber}^{\frac{1}{2}}\left(f^{2r}(|X|) + g^{2r}(|Y^*|)\right)\mathbf{ber}^{\frac{1}{2}}\left(f^{2r}(|Y|) + g^{2r}(|X^*|)\right)\|k_{\lambda_1}\|\|k_{\lambda_2}\| \\
&\leqslant \frac{2^r}{2}\mathbf{ber}^{\frac{1}{2}}\left(f^{2r}(|X|) + g^{2r}(|Y^*|)\right)\mathbf{ber}^{\frac{1}{2}}\left(f^{2r}(|Y|) + g^{2r}(|X^*|)\right)\left(\frac{\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2}{2}\right) \\
&\qquad\qquad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{2^r}{4}\mathbf{ber}^{\frac{1}{2}}\left(f^{2r}(|X|) + g^{2r}(|Y^*|)\right)\mathbf{ber}^{\frac{1}{2}}\left(f^{2r}(|Y|) + g^{2r}(|X^*|)\right).
\end{aligned}$$

Now, taking the supremum over all $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$ we get the desired result. □

Theorem 2.2 includes a special case as follows.

Corollary 2.2. *Let* $\mathbf{T} = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, $0 \leqslant p \leqslant 1$ *and* $r \geqslant 1$. *Then*

$$\mathbf{ber}^r(\mathbf{T}) \leqslant 2^{r-2}\mathbf{ber}^{\frac{1}{2}}\left(|X|^{2rp} + |Y^*|^{2r(1-p)}\right)\mathbf{ber}^{\frac{1}{2}}\left(|Y|^{2rp} + |X^*|^{2r(1-p)}\right)$$

Proof. The result follows immediately from Theorem 2.2 for $f(t) = t^p$ and $g(t) = t^{1-p}$ $(0 \leqslant p \leqslant 1)$. □

Theorem 2.3. *Let* $A, B, X \in \mathbb{B}(\mathcal{H}(\Omega))$, $r \geqslant 1$ *and* f, g *be nonnegative continuous functions on* $[0,\infty)$ *satisfying the relation* $f(t)g(t) = t\,(t \in [0,\infty))$. *Then*

$$\mathbf{ber}^r(A^*XB) \leqslant \mathbf{ber}\left(\frac{1}{p}\left[B^*f^2(|X|)B\right]^{\frac{rp}{2}} + \frac{1}{q}\left[A^*g^2(|X^*|)A\right]^{\frac{rq}{2}}\right),$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $pr \geqslant qr \geqslant 2$.

Proof. For every $\lambda \in \Omega$, let $\hat{\mathbf{k}}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}(\Omega)$. Then

$$\begin{aligned}
\left|\langle A^*XB\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle\right|^r &= \left|\langle XB\hat{\mathbf{k}}_\lambda, A\hat{\mathbf{k}}_\lambda\rangle\right|^r \\
&\leqslant \left(\langle f^2(|X|)B\hat{\mathbf{k}}_\lambda, B\hat{\mathbf{k}}_\lambda\rangle\langle g^2(|X^*|)A\hat{\mathbf{k}}_\lambda, A\hat{\mathbf{k}}_\lambda\rangle\right)^{\frac{r}{2}} \\
&\qquad \text{(by Lemma 2.2)} \\
&\leqslant \frac{1}{p}\langle f^2(|X|)B\hat{\mathbf{k}}_\lambda, B\hat{\mathbf{k}}_\lambda\rangle^{\frac{rp}{2}} + \frac{1}{q}\langle g^2(|X^*|)A\hat{\mathbf{k}}_\lambda, A\hat{\mathbf{k}}_\lambda\rangle^{\frac{rq}{2}} \\
&\qquad \text{(by Young's inequaliy)} \\
&= \frac{1}{p}\langle B^*f^2(|X|)B\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle^{\frac{rp}{2}} + \frac{1}{q}\langle A^*g^2(|X^*|)A\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle^{\frac{rq}{2}} \\
&\leqslant \frac{1}{p}\langle\left[B^*f^2(|X|)B\right]^{\frac{rp}{2}}\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle + \frac{1}{q}\langle\left[A^*g^2(|X^*|)A\right]^{\frac{rq}{2}}\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle \\
&\qquad \text{(by Lemma 2.1)} \\
&= \left\langle\left(\frac{1}{p}\left[B^*f^2(|X|)B\right]^{\frac{rp}{2}} + \frac{1}{q}\left[A^*g^2(|X^*|)A\right]^{\frac{rq}{2}}\right)\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\right\rangle \\
&\leqslant \mathbf{ber}\left(\frac{1}{p}\left[B^*f^2(|X|)B\right]^{\frac{rp}{2}} + \frac{1}{q}\left[A^*g^2(|X^*|)A\right]^{\frac{rq}{2}}\right),
\end{aligned}$$

whence

$$\begin{aligned}
\mathbf{ber}^r(A^*XB) &= \sup_{\lambda\in\Omega}\left|\langle A^*XB\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle\right|^r \\
&\leqslant \mathbf{ber}\left(\frac{1}{p}\left[B^*f^2(|X|)B\right]^{\frac{rp}{2}} + \frac{1}{q}\left[A^*g^2(|X^*|)A\right]^{\frac{rq}{2}}\right).
\end{aligned}$$

□

Remark 2.1. *In Theorem 2.3, if we add the hypothesis of contraction for operators A and B (i.e., $A^*A \leqslant I$ and $B^*B \leqslant I$) then by using Lemma 2.1 and with a similar fashion in the proof in the Theorem 2.3 we get the inequality*

$$\mathbf{ber}^r(A^*XB) \leqslant \mathbf{ber}\left(\frac{1}{p}B^* f^{rp}(|X|)B + \frac{1}{q}A^* g^{rq}(|X^*|)A\right), \tag{2.2}$$

where $r \geqslant 1$, $\frac{1}{p}+\frac{1}{q}=1$ such that $pr \geqslant qr \geqslant 2$ and f, g be nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t\,(t \in [0,\infty))$.

The next result follows from Theorem 2.3 and inequality (2.2) for $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ $(0 \leqslant \alpha \leqslant 1)$.

Corollary 2.3. *Let $A, B, X \in \mathbb{B}(\mathcal{H}(\Omega))$, $r \geqslant 1$, $\frac{1}{p}+\frac{1}{q}=1$ such that $pr \geqslant qr \geqslant 2$ and $0 \leqslant \alpha \leqslant 1$. Then*

$$\mathbf{ber}^r(A^*XB) \leqslant \mathbf{ber}\left(\frac{1}{p}\left[B^*|X|^{2\alpha}B\right]^{\frac{rp}{2}} + \frac{1}{q}\left[A^*|X^*|^{2(1-\alpha)}A\right]^{\frac{rq}{2}}\right),$$

In particular, if A and B be contraction, then

$$\mathbf{ber}^r(A^*XB) \leqslant \mathbf{ber}\left(\frac{1}{p}B^*|X|^{rp\alpha}B + \frac{1}{q}A^*|X^*|^{rp(1-\alpha)}A\right).$$

Now, we need the following lemma for the next result.

Lemma 2.3. *Let $X, Y \in \mathbb{B}(\mathcal{H}(\Omega))$. If $\mathbf{ber}\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right) \leqslant \mathbf{ber}\left(\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}\right)$, then $\mathbf{ber}(X) \leqslant \mathbf{ber}(Y)$.*

Proof. For every $\lambda \in \Omega$, let $\hat{\mathbf{k}}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}(\Omega)$. Then

$$\begin{aligned}
\left|\left\langle X\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\right\rangle\right| &= \left|\left\langle \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\mathbf{k}}_\lambda \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{\mathbf{k}}_\lambda \\ 0 \end{bmatrix}\right\rangle\right| \\
&\leqslant \mathbf{ber}\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right) \qquad \text{(by the definition of } \mathbf{ber}) \\
&\leqslant \mathbf{ber}\left(\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\leqslant \mathbf{ber}\,(Y) \qquad \text{(by inequality (2.1))}.
\end{aligned}$$

Hence

$$\mathbf{ber}\,(X) = \sup_{\lambda\in\Omega}\left|\langle X\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle\right| \leqslant \mathbf{ber}\,(Y).$$

□

Corollary 2.4. *Let $A_i, B_i, X_i \in \mathbb{B}(\mathcal{H}(\Omega))$ $(1 \leqslant i \leqslant n)$, $r \geqslant 1$ and f, g be nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t\,(t \in [0,\infty))$. Then*

$$\mathbf{ber}^r\left(\sum_{i=1}^n A_i^* X_i B_i\right) \leqslant \mathbf{ber}\left(\frac{1}{p}\left[\sum_{i=1}^n B_i^* f^2(|X_i|)B_i\right]^{\frac{rp}{2}} + \frac{1}{q}\left[\sum_{i=1}^n A_i^* g^2(|X_i^*|)A_i\right]^{\frac{rq}{2}}\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $pr \geqslant qr \geqslant 2$.
In particular, if $\sum_{i=1}^n A_i^* A_i \leqslant I$ and $\sum_{i=1}^n B_i^* B_i \leqslant I$, then

$$\mathbf{ber}^r\left(\sum_{i=1}^n A_i^* X_i B_i\right) \leqslant \mathbf{ber}\left(\frac{1}{p}\sum_{i=1}^n B_i^* f^{rp}(|X_i|)B_i + \frac{1}{q}\sum_{i=1}^n A_i^* g^{rq}(|X_i^*|)A_i\right).$$

Proof. If we replace A, B and X by operator matrices

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ B_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_n & 0 & \cdots & 0 \end{bmatrix} \text{ and } \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_n \end{bmatrix},$$

respectively, in Theorem 2.3, then get

$$\mathbf{ber}^r\left(\begin{bmatrix} \sum_{i=1}^n A_i^* X_i B_i & 0 \\ 0 & 0 \end{bmatrix}\right)$$
$$\leqslant \mathbf{ber}\left(\begin{bmatrix} \frac{1}{p}\left[\sum_{i=1}^n B_i^* f^2(|X_i|)B_i\right]^{\frac{rp}{2}} + \frac{1}{q}\left[\sum_{i=1}^n A_i^* g^2(|X_i^*|)A_i\right]^{\frac{rq}{2}} & 0 \\ 0 & 0 \end{bmatrix}\right).$$

Now, using Lemma 2.3 we have

$$\mathbf{ber}^r\left(\sum_{i=1}^n A_i^* X_i B_i\right) \leqslant \mathbf{ber}\left(\frac{1}{p}\left[\sum_{i=1}^n B_i^* f^2(|X_i|)B_i\right]^{\frac{rp}{2}} + \frac{1}{q}\left[\sum_{i=1}^n A_i^* g^2(|X_i^*|)A_i\right]^{\frac{rq}{2}}\right)$$

the first inequality. The second inequality follows from inequality (2.2) and this completes the proof. □

In the next theorem we present an inequality involving the generalized Euclidean Berezin number for off-diagonal operator matrices.

Theorem 2.4. *Let $T_i = \begin{bmatrix} 0 & X_i \\ Y_i & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$ $(1 \leqslant i \leqslant n)$. Then*

$$\mathbf{ber}_p^p(T_1, T_2, \ldots, T_n)$$
$$\leqslant 2^{p-2}\sum_{i=1}^n \mathbf{ber}^{\frac{1}{2}}\left(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|)\right)\mathbf{ber}^{\frac{1}{2}}\left(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)\right)$$

for $p \geqslant 1$.

Proof. For every $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$, let $\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel of $\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2)$ (i.e., $\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2 = 1$). Then

$$
\begin{aligned}
&\sum_{i=1}^{n} |\langle T_i \hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}, \hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}\rangle|^p \\
&= \sum_{i=1}^{n} |\langle X_i k_{\lambda_2}, k_{\lambda_1}\rangle + \langle Y_i k_{\lambda_1}, k_{\lambda_2}\rangle|^p \\
&\leqslant \sum_{i=1}^{n} \left(|\langle X_i k_{\lambda_2}, k_{\lambda_1}\rangle| + |\langle Y_i k_{\lambda_1}, k_{\lambda_2}\rangle|\right)^p \qquad \text{(by the triangular inequality)} \\
&\leqslant \frac{2^p}{2} \sum_{i=1}^{n} |\langle X_i k_{\lambda_2}, k_{\lambda_1}\rangle|^p + |\langle Y_i k_{\lambda_1}, k_{\lambda_2}\rangle|^p \qquad \text{(by the convexity } f(t) = t^p\text{)} \\
&\leqslant \frac{2^p}{2} \sum_{i=1}^{n} \langle f^2(|X_i|) k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{p}{2}} \langle g^2(|X_i^*|) k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{p}{2}} + \langle f^2(|Y_i|) k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{p}{2}} \langle g^2(|Y_i^*|) k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{p}{2}} \\
&\qquad \text{(by Lemma 2.2)} \\
&\leqslant \frac{2^p}{2} \sum_{i=1}^{n} \langle f^{2p}(|X_i|) k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} \langle g^{2p}(|X_i^*|) k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}} + \langle f^{2p}(|Y_i|) k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}} \langle g^{2p}(|Y_i^*|) k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} \\
&\qquad \text{(by Lemma 2.1)} \\
&\leqslant \frac{2^p}{2} \sum_{i=1}^{n} \left(\langle f^{2p}(|X_i|) k_{\lambda_2}, k_{\lambda_2}\rangle + \langle g^{2p}(|Y_i^*|) k_{\lambda_2}, k_{\lambda_2}\rangle\right)^{\frac{1}{2}} \\
&\times \left(\langle f^{2p}(|Y_i|) k_{\lambda_1}, k_{\lambda_1}\rangle + \langle g^{2p}(|X_i^*|) k_{\lambda_1}, k_{\lambda_1}\rangle\right)^{\frac{1}{2}} \\
&\qquad \text{(by the Cauchy-Schwarz inequality)} \\
&\leqslant \frac{2^p}{2} \sum_{i=1}^{n} \mathbf{ber}^{\frac{1}{2}} \left(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|)\right) \mathbf{ber}^{\frac{1}{2}} \left(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)\right) \|k_{\lambda_1}\| \|k_{\lambda_2}\| \\
&= \frac{2^p}{2} \sum_{i=1}^{n} \mathbf{ber}^{\frac{1}{2}} \left(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|)\right) \mathbf{ber}^{\frac{1}{2}} \left(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)\right) \left(\frac{\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2}{2}\right) \\
&= \frac{2^p}{4} \sum_{i=1}^{n} \mathbf{ber}^{\frac{1}{2}} \left(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|)\right) \mathbf{ber}^{\frac{1}{2}} \left(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbf{ber}_p^p(T_1, T_2, \ldots, T_n) &= \sup_{(\lambda_1,\lambda_2) \in \Omega_1 \times \Omega_2} \sum_{i=1}^{n} |\langle T_i \hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}, \hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}\rangle|^p \\
&\leqslant 2^{p-2} \sum_{i=1}^{n} \mathbf{ber}^{\frac{1}{2}} \left(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|)\right) \mathbf{ber}^{\frac{1}{2}} \left(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)\right)
\end{aligned}
$$

as required. □

Theorem 2.5. *Let* $T_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$ $(1 \leqslant i \leqslant n)$ *and* $p \geqslant 1$. *Then*

$$\begin{aligned}&\mathbf{ber}_p^p(T_1, \ldots, T_n)\\ &\leqslant 2^{-p} \sum_{i=1}^{n} \left(\mathbf{ber}\,(A_i) + \mathbf{ber}\,(D_i) + \sqrt{(\mathbf{ber}\,(A_i) - \mathbf{ber}\,(D_i))^2 + (\|B_i\| + \|C_i\|)^2} \right)^p.\end{aligned}$$

Proof. For every $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$, let $\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel of $\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2)$. It follows from

$$\begin{aligned}&\left|\left\langle T_i \hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}, \hat{\mathbf{k}}_{(\lambda_1,\lambda_2)} \right\rangle\right|\\ &= \left|\left\langle \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}, \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix} \right\rangle\right|\\ &= \left|\left\langle \begin{bmatrix} A_i k_{\lambda_1} + B_i k_{\lambda_2} \\ C_i k_{\lambda_1} + D_i k_{\lambda_2} \end{bmatrix}, \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix} \right\rangle\right|\\ &= |\langle A_i k_{\lambda_1}, k_{\lambda_1}\rangle + \langle B_i k_{\lambda_2}, k_{\lambda_1}\rangle + \langle C_i k_{\lambda_1}, k_{\lambda_2}\rangle + \langle D_i k_{\lambda_2}, k_{\lambda_2}\rangle|\\ &\leqslant |\langle A_i k_{\lambda_1}, k_{\lambda_1}\rangle| + |\langle B_i k_{\lambda_2}, k_{\lambda_1}\rangle| + |\langle C_i k_{\lambda_1}, k_{\lambda_2}\rangle| + |\langle D_i k_{\lambda_2}, k_{\lambda_2}\rangle|\end{aligned}$$

that

$$\begin{aligned}&\mathbf{ber}_p^p(T_1, \ldots, T_n)\\ &= \sup_{(\lambda_1,\lambda_2)\in\Omega_1\times\Omega_2} \sum_{i=1}^{n} \left|\left\langle T_i \hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}, \hat{\mathbf{k}}_{(\lambda_1,\lambda_2)} \right\rangle\right|^p\\ &\leqslant \sup_{(\lambda_1,\lambda_2)\in\Omega_1\times\Omega_2} \sum_{i=1}^{n} \left(|\langle A_i k_{\lambda_1}, k_{\lambda_1}\rangle| + |\langle B_i k_{\lambda_2}, k_{\lambda_1}\rangle| + |\langle C_i k_{\lambda_1}, k_{\lambda_2}\rangle| + |\langle D_i k_{\lambda_2}, k_{\lambda_2}\rangle|\right)^p\\ &\leqslant \sum_{i=1}^{n} \left(\sup_{(\lambda_1,\lambda_2)\in\Omega_1\times\Omega_2} \left(|\langle A_i k_{\lambda_1}, k_{\lambda_1}\rangle| + |\langle B_i k_{\lambda_2}, k_{\lambda_1}\rangle| + |\langle C_i k_{\lambda_1}, k_{\lambda_2}\rangle| + |\langle D_i k_{\lambda_2}, k_{\lambda_2}\rangle|\right) \right)^p\\ &\leqslant \sum_{i=1}^{n} \left(\sup_{(\lambda_1,\lambda_2)\in\Omega_1\times\Omega_2} \left(\mathbf{ber}\,(A_i) \|k_{\lambda_1}\|^2 + \mathbf{ber}\,(D_i) \|k_{\lambda_2}\|^2 + (\|B_i\| + \|C_i\|) \|k_{\lambda_1}\| \|k_{\lambda_2}\|\right) \right)^p\\ &\leqslant \sum_{i=1}^{n} \left(\sup_{\theta\in[0,2\pi]} \left(\mathbf{ber}\,(A_i)\cos^2\theta + \mathbf{ber}\,(D_i)\sin^2\theta + (\|B_i\| + \|C_i\|)\cos\theta\sin\theta\right) \right)^p\\ &= 2^{-p} \sum_{i=1}^{n} \left(\mathbf{ber}\,(A_i) + \mathbf{ber}\,(D_i) + \sqrt{(\mathbf{ber}\,(A_i) - \mathbf{ber}\,(D_i))^2 + (\|B_i\| + \|C_i\|)^2} \right)^p.\end{aligned}$$

This completes the proof. □

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