

REVERSE YOUNG-TYPE INEQUALITIES FOR MATRICES AND OPERATORS

MOJTABA BAKHERAD, MARIO KRNIĆ AND
 MOHAMMAD SAL MOSLEHIAN

ABSTRACT. We present some reverse Young-type inequalities for the Hilbert-Schmidt norm as well as any unitarily invariant norm. Furthermore, we give some inequalities dealing with operator means. More precisely, we show that if $A, B \in \mathfrak{B}(\mathcal{H})$ are positive operators and $r \geq 0$, $A\nabla_{-r}B + 2r(A\nabla B - A\sharp_r B) \leq A\sharp_{-r}B$. We also prove that equality holds if and only if $A = B$. In addition, we establish several reverse Young-type inequalities involving trace, determinant and singular values. In particular, we show that if A and B are positive definite matrices and $r \geq 0$, then $\text{tr}((1+r)A - rB) \leq \text{tr}|A^{1+r}B^{-r}| - r(\sqrt{\text{tr}A} - \sqrt{\text{tr}B})^2$.

1. Introduction and preliminaries. Let \mathcal{H} be a Hilbert space, and let $\mathfrak{B}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators on \mathcal{H} with the operator norm $\|\cdot\|$ and the identity $I_{\mathcal{H}}$. If $\dim \mathcal{H} = n$, then we identify $\mathfrak{B}(\mathcal{H})$ with the space \mathcal{M}_n of all $n \times n$ complex matrices and denote the identity matrix by I_n . For an operator $A \in \mathfrak{B}(\mathcal{H})$, we write $A \geq 0$ if A is positive (positive semi-definite for matrices), and $A > 0$ if A is positive and invertible (positive definite for matrices). For $A, B \in \mathfrak{B}(\mathcal{H})$, we say $A \geq B$ if $A - B \geq 0$. Let $\mathfrak{B}^+(\mathcal{H})$ (respectively, \mathcal{P}_n) denote the set of all positive invertible operators (respectively, positive definite matrices). A norm $\|\cdot\|$ on \mathcal{M}_n is called unitarily invariant if $\|UAV\| = \|A\|$ for all $A \in \mathcal{M}_n$ and all unitary matrices $U, V \in \mathcal{M}_n$. The Hilbert-Schmidt norm is defined by

$$\|A\|_2 = \left(\sum_{j=1}^n s_j^2(A) \right)^{1/2},$$

where $s(A) = (s_1(A), \dots, s_n(A))$

2010 AMS *Mathematics subject classification.* Primary 15A60, 47A30, 47A60.
Keywords and phrases. Young inequality, positive operator, operator mean, unitarily invariant norm, determinant, trace.
 Received by the editors on June 27, 2014, and in revised form on September 29, 2014.

denotes the singular values of A , that is, the eigenvalues of the positive semi-definite matrix $|A| = (A^*A)^{1/2}$, arranged in decreasing order with their multiplicities counted. This norm is unitarily invariant. It is known that, if $A = [a_{ij}] \in \mathcal{M}_n$, then

$$\|A\|_2 = \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}.$$

The weighted operator arithmetic ∇_ν , geometric \sharp_ν and harmonic $!_\nu$ means, for $\nu \in [0, 1]$ and $A, B \in \mathfrak{B}^+(\mathcal{H})$ are defined as follows:

$$\begin{aligned} A \nabla_\nu B &= (1 - \nu)A + \nu B, \\ A \sharp_\nu B &= A^{1/2} (A^{-1/2} B A^{-1/2})^\nu A^{1/2}, \\ A !_\nu B &= ((1 - \nu)A^{-1} + \nu B^{-1})^{-1}. \end{aligned}$$

If $\nu = 1/2$, we denote the arithmetic, geometric and harmonic means, respectively, by ∇ , \sharp and $!$, for brevity.

The classical Young inequality states that

$$a^\nu b^{1-\nu} \leq \nu a + (1 - \nu)b,$$

when $a, b \geq 0$ and $\nu \in [0, 1]$. If $\nu = 1/2$, we obtain the arithmetic-geometric mean inequality $\sqrt{ab} \leq (a + b)/2$. The operator Young inequality reads as follows:

$$(1.1) \quad A !_\nu B \leq A \sharp_\nu B \leq A \nabla_\nu B, \quad \nu \in [0, 1],$$

where $A, B \in \mathfrak{B}^+(\mathcal{H})$ and $\nu \in [0, 1]$, cf., [6]. For other generalizations of the Young inequality see [15, 16]. A Young inequality matrix due to Ando [1] asserts that

$$s_j(A^\nu B^{1-\nu}) \leq s_j(\nu A + (1 - \nu)B),$$

in which $A, B \in \mathcal{M}_n$ are positive semidefinite, $j = 1, 2, \dots, n$, and $\nu \in [0, 1]$. The above singular value inequality entails the following unitarily invariant norm inequality,

$$\| \|A^\nu B^{1-\nu}\| \| \leq \| \| \nu A + (1 - \nu)B \| \|,$$

where $A, B \in \mathcal{M}_n$ are positive semidefinite and $0 \leq \nu \leq 1$. Kosaki [13] proved that the inequality,

$$(1.2) \quad \|A^\nu X B^{1-\nu}\|_2 \leq \|\nu AX + (1 - \nu)XB\|_2,$$

holds for matrices $A, B, X \in \mathcal{M}_n$ such that A and B are positive semidefinite, and for $0 \leq \nu \leq 1$. It should be mentioned here that, for $\nu \neq 1/2$, inequality (1.2) may not hold for other unitarily invariant norms. Hirzallah and Kittaneh [7] gave a refinement of (1.2) by showing that

$$(1.3) \quad \|A^\nu X B^{1-\nu}\|_2^2 + r_0^2 \|AX - XB\|_2^2 \leq \|\nu AX + (1 - \nu)XB\|_2^2,$$

in which $A, B, X \in \mathcal{M}_n$ such that A and B are positive semi-definite, $0 \leq \nu \leq 1$ and $r_0 = \min\{\nu, 1 - \nu\}$. A determinant version of the Young inequality is also known (see [9, page 467]):

$$\det(A^\nu B^{1-\nu}) \leq \det(\nu A + (1 - \nu)B),$$

where $A, B, X \in \mathcal{M}_n$ such that A and B are positive semi-definite and $0 \leq \nu \leq 1$. This determinant inequality was recently improved in [12]. Further, Kittaneh [10] proved that

$$(1.4) \quad \|||A^{1-\nu} X B^\nu\||| \leq \|||AX\|||^{1-\nu} \|||XB\|||^\nu,$$

in which $\|||\cdot\|||$ is any unitarily invariant norm, $A, B, X \in \mathcal{M}_n$ such that A and B are positive semidefinite and $0 \leq \nu \leq 1$. Conde [2] showed that

$$2\|||A^{1-\nu} X B^\nu\||| + (\|||AX\|||^{1-\nu} - \|||XB\|||^\nu)^2 \leq \|||AX\|||^{2(1-\nu)} + \|||XB\|||^{2\nu},$$

where $\|||\cdot\|||$ is a unitarily invariant norm, $A, B, X \in \mathcal{M}_n$ such that A and B are positive semidefinite and $0 \leq \nu \leq 1$. Tominaga [21, 22] employed Specht’s ratio for the Young inequality. In addition, some reverses of the Young inequality are established in [4].

For $a, b \in \mathbb{R}$, the number $x = \nu a + (1 - \nu)b$ belongs to the interval $[a, b]$ for all $\nu \in [0, 1]$, and is outside the interval for all $\nu > 1$ or $\nu < 0$. Exploiting this obvious fact, Fujii [3] showed that if f is an operator concave function on an interval J , then the inequality,

$$f(C^*XC - D^*YD) \leq |C|f(V^*XV)|C| - D^*f(Y)D,$$

holds for all self-adjoint operators X and Y and operators C and D in $\mathfrak{B}(\mathcal{H})$ with spectra in J , such that $C^*C - D^*D = I_{\mathcal{H}}$, $\sigma(C^*XC - D^*YD) \subseteq J$, and $C = V|C|$ is the polar decomposition of C .

In this direction, by using some numerical inequalities, we obtain reverses of (1.1)–(1.4) under some mild conditions. We also aim to give some reverses of the Young inequality dealing with positive operator means. Finally, we present some singular value inequalities of Young-type involving trace and determinant.

2. Reverses of the Young inequality for the Hilbert-Schmidt norm. In this section, we deal with reverses of the Young inequality for the Hilbert-Schmidt norm. To this end, we need the following lemma.

Lemma 2.1. *Let $a, b > 0$. If $r \geq 0$ or $r \leq -1$, then*

$$(2.1) \quad (1+r)a - rb \leq a^{1+r}b^{-r}.$$

Proof. Let $a, b > 0$. If $a = b$, then (2.1) is trivial. Let $a \neq b$. Bernoulli's inequality [17] states that $1 + \nu x \leq (1+x)^\nu$, where $0 \neq x > -1$ and $\nu \notin (0, 1)$. If we replace ν by $1+r$ and x by $t-1$, respectively, then $(1+r)t - r \leq t^{1+r}$, where $1 \neq t > 0$ and $r \notin (-1, 0)$. Letting $t = a/b$, we obtain the desired inequality. \square

Remark 2.2. By virtue of Lemma 2.1, it follows that the inequality

$$(2.2) \quad ((1+r)a - rb)^2 \leq (a^{1+r}b^{-r})^2$$

holds if $a \geq b > 0$ and $r \geq 0$, or $b \geq a > 0$ and $r \leq -1$.

Our first result reads as follows.

Theorem 2.3. *Let $A, B, X \in \mathcal{M}_n$ and let m and m' be positive scalars. If $A \geq mI_n \geq B > 0$ and $r \geq 0$, or $B \geq m'I_n \geq A > 0$ and $r \leq -1$, then the following inequality holds:*

$$\|(1+r)AX - rXB\|_2 \leq \|A^{1+r}XB^{-r}\|_2.$$

Proof. It follows from the spectral decomposition [23] that there are unitary matrices $U, V \in \mathcal{M}_n$ such that $A = U\Lambda U^*$ and $B = VTV^*$,

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$, and $\lambda_j, \gamma_j, j = 1, 2, \dots, n$, are positive. If $Z = U^*XV = [z_{ij}]$, then

$$(2.3) \quad (1+r)AX-rXB=U((1+r)\Lambda Z-rZ\Gamma)V^*=U[((1+r)\lambda_i-r\gamma_j)z_{ij}]V^*$$

and

$$(2.4) \quad A^{1+r}XB^{-r}=U\Lambda^{1+r}U^*XV\Gamma^{-r}V^*=U\Lambda^{1+r}Z\Gamma^{-r}V^*=U[(\lambda_i^{1+r}\gamma_j^{-r})z_{ij}]V^*$$

Suppose first that $A \geq mI_n \geq B > 0$ and $r \geq 0$. Then, it follows that

$$(2.5) \quad \lambda_i \geq \gamma_j, \quad 1 \leq i, j \leq n,$$

so, utilizing (2.3) and (2.4), we have

$$\|(1+r)AX-rXB\|_2^2 = \sum_{i,j=1}^n ((1+r)\lambda_i-r\gamma_j)^2 |z_{ij}|^2 \leq \sum_{i,j=1}^n (\lambda_i^{1+r}\gamma_j^{-r})^2 |z_{ij}|^2$$

(by inequalities (2.2) and (2.5))

$$= \|A^{1+r}XB^{-r}\|_2^2.$$

The same conclusion can be drawn for the cases of $B \geq m'I_n \geq A > 0$ and $r \leq -1$. □

Generally speaking, Theorem 2.3 does not hold for arbitrary positive definite matrices A and B . The reason for this lies in the fact that inequality (2.2) is not true for arbitrary positive numbers a and b . To see this, let $a = 1, b = 4$ and $r = 2$.

Our next intention is to derive a result related to Theorem 2.3, which holds for all positive definite matrices. Observe that the inequality:

$$\begin{aligned} ((1+r)a-rb)^2-r^2(a-b)^2 &= (1+2r)a^2-2rab \leq (a^2)^{1+2r}(ab)^{-2r} \\ &= (a^{1+r}b^{-r})^2 \end{aligned}$$

yields an appropriate relation instead of (2.2), for arbitrary positive numbers a and b and $r \geq 0$ or $r \leq -1/2$, as follows:

$$((1+r)a-rb)^2 \leq (a^{1+r}b^{-r})^2 + r^2(a-b)^2 \quad a, b > 0.$$

Note also that, if $a = b$, then the equality holds.

Now, utilizing this inequality and the same argument as in the proof of Theorem 2.3, i.e., the spectral theorem for positive definite matrices, we obtain the corresponding result.

Theorem 2.4. *Suppose that $A, B \in \mathcal{P}_n$ and $X \in \mathcal{M}_n$. Then the inequality:*

$$(2.6) \quad \|(1+r)AX - rXB\|_2^2 \leq \|A^{1+r}XB^{-r}\|_2^2 + r^2 \|AX - XB\|_2^2$$

holds for $r \geq 0$ or $r \leq -1/2$.

3. Reverse Young-type inequalities involving unitarily invariant norms. It has been shown in [8] that the inequality,

$$(3.1) \quad \|A^{1+r}XB^{1+r}\| \geq \|X\|^{-r} \|AXB\|^{1+r},$$

holds for $A, B \in \mathcal{P}_n$, $0 \neq X \in \mathcal{M}_n$ and $r \geq 0$. Replacing B by B^{-1} and X by XB in (3.1), respectively, yields the relation

$$(3.2) \quad \|A^{1+r}XB^{-r}\| \geq \|AX\|^{1+r} \|XB\|^{-r},$$

where $r \geq 0$, $A, B \in \mathcal{P}_n$ and $X \in \mathcal{M}_n$ with $X \neq 0$.

Next we show that inequality (3.2) holds for every unitarily invariant norm. This can be done by virtue of inequality (1.4). In fact, the following result is, in some way, complementary to inequality (1.4).

Lemma 3.1. *Suppose that $A, B \in \mathcal{P}_n$, $X \in \mathcal{M}_n$ such that $X \neq 0$. If $r \geq 0$ or $r \leq -1$, then the inequality,*

$$\| \|AX\| \| \|^{1+r} \| \|XB\| \|^{-r} \leq \| \|A^{1+r}XB^{-r}\| \|,$$

holds for any unitarily invariant norm $\| \| \cdot \| \|$.

Proof. First, let $r \geq 0$. Set $\alpha = r + 1$. Utilizing inequality (1.4), it follows that

$$\begin{aligned} \| \|AX\| \| &= \| \| (A^\alpha)^{1/\alpha} (XB^{1-\alpha}) (B^\alpha)^{(\alpha-1)/\alpha} \| \| \\ &\leq \| \| A^\alpha XB^{1-\alpha} \| \|^{1/\alpha} \| \| XB^{1-\alpha} B^\alpha \| \|^{(\alpha-1)/\alpha} \\ &= \| \| A^\alpha XB^{1-\alpha} \| \|^{1/\alpha} \| \| XB \| \|^{(\alpha-1)/\alpha}, \end{aligned}$$

that is,

$$\| \|AX\| \| \| \|XB\| \| \|^{(1-\alpha)/\alpha} \leq \| \| A^\alpha XB^{1-\alpha} \| \|^{1/\alpha}.$$

Hence,

$$\| \|AX\| \|XB\|^{1-\alpha} \leq \|A^\alpha XB^{1-\alpha}\|,$$

whence

$$\| \|AX\|^{1+r} \|XB\|^{-r} \leq \|A^{1+r} XB^{-r}\|.$$

On the other hand, if $r \leq -1$, set $\alpha = -r$. By a similar argument, we obtain the desired result. \square

An application of Lemmas 2.1 and 3.1 yields the Young-type inequality,

$$(3.3) \quad (1+r)\| \|AX\| - r\| \|XB\| \leq \| \|A^{1+r} XB^{-r}\|,$$

which holds for matrices $A, B \in \mathcal{P}_n$, $X \in \mathcal{M}_n$ such that $X \neq 0$ and $r \geq 0$ or $r \leq -1$. It is interesting that inequality (3.3) can be improved. But first we have to improve the scalar inequality (2.1).

Lemma 3.2. *Let $a, b > 0$ and $r \geq 0$ or $r \leq -1/2$. Then,*

$$(3.4) \quad (1+r)a - rb + r(\sqrt{a} - \sqrt{b})^2 \leq a^{1+r}b^{-r}.$$

Proof. Due to Lemma 2.1, it follows that

$$\begin{aligned} (1+r)a - rb + r(\sqrt{a} - \sqrt{b})^2 &= -2r\sqrt{ab} + (1+2r)a \leq (\sqrt{ab})^{-2r} a^{1+2r} \\ &= a^{1+r}b^{-r}. \end{aligned} \quad \square$$

Obviously, if $r \geq 0$, inequality (3.4) represents an improvement of inequality (2.1). Finally, we give an improvement of matrix inequality (3.3).

Theorem 3.3. *Let $A, B \in \mathcal{P}_n$, $X \in \mathcal{M}_n$ be such that $X \neq 0$ and let $r \geq 0$. Then the inequality*

$$(1+r)\| \|AX\| - r\| \|XB\| + r(\sqrt{\| \|AX\|} - \sqrt{\| \|XB\|})^2 \leq \| \|A^{1+r} XB^{-r}\|$$

holds for any unitarily invariant norm $\| \cdot \|$.

Proof.

$$\begin{aligned}
 (1+r) \||AX\|| - r \||XB\|| + r(\sqrt{\||AX\||} - \sqrt{\||XB\||})^2 \\
 \leq \||AX\||^{1+r} \||XB\||^{-r} \quad (\text{by Lemma 3.2}) \\
 \leq \||A^{1+r}XB^{-r}\|| \quad (\text{by Lemma 3.1}). \quad \square
 \end{aligned}$$

Remark 3.4. It should be noted here that Theorem 3.3 is also true in the case of $r \leq -1/2$. However, in this case, the corresponding inequality is less precise than relation (3.3) and does not represent its refinement.

4. Reverse Young-type inequalities related to operator means. The matrix Young inequality can be considered in a more general setting. Namely, this inequality also holds for self-adjoint operators on a Hilbert space. The main objective of this section is to derive inequalities which are complementary to the mean inequalities in (1.1), presented in the introduction.

The main tool in obtaining inequalities for self-adjoint operators on Hilbert spaces is the following monotonicity property for operator functions. If X is a self-adjoint operator with the spectrum $\text{sp}(X)$, then

$$(4.1) \quad f(t) \geq g(t), \quad t \in \text{sp}(X) \implies f(X) \geq g(X).$$

For more details about this property the reader is referred to [19].

Since $A, B \in \mathfrak{B}^+(\mathcal{H})$, the expressions $A\nabla_\nu B$ and $A\sharp_\nu B$ are also well defined when $\nu \in \mathbb{R} \setminus [0, 1]$. In this case, we obtain the reverse of the second inequality in (1.1).

Theorem 4.1. *If $A, B \in \mathfrak{B}^+(\mathcal{H})$ and $r \geq 0$ or $r \leq -1$, then*

$$(4.2) \quad A\nabla_{-r} B \leq A\sharp_{-r} B.$$

Proof. By virtue of Lemma 2.1, it follows that $f(x) = x^{-r} + rx - (1+r) \geq 0$, $x > 0$. Moreover, since $B \in \mathfrak{B}^+(\mathcal{H})$, it follows that $A^{-1/2}BA^{-1/2} \in \mathfrak{B}^+(\mathcal{H})$, that is, $\text{sp}(A^{-1/2}BA^{-1/2}) \subseteq (0, \infty)$.

Thus, applying monotonicity property (4.1) to the above function f , we have that

$$\left(A^{-1/2}BA^{-1/2} \right)^{-r} + rA^{-1/2}BA^{-1/2} - (1+r)I_{\mathcal{H}} \geq 0.$$

Finally, multiplying both sides of this relation by $A^{1/2}$, we have

$$A^{1/2} \left(A^{-1/2}BA^{-1/2} \right)^{-r} A^{1/2} + rB - (1+r)A \geq 0,$$

and the proof is complete. □

If $A, B \in \mathfrak{B}^+(\mathcal{H})$ such that $A \leq B$, the expression $A!_{-r}B$ is well defined for $r \geq 0$. Namely, due to operator monotonicity of the function $h(x) = -1/x$ on $(0, \infty)$ (for more details, see [19]), $A \leq B$ implies that $B^{-1} \leq A^{-1}$, so that $(r+1)A^{-1} - rB^{-1} \in \mathfrak{B}^+(\mathcal{H})$. Therefore, the operator $A!_{-r}B = ((r+1)A^{-1} - rB^{-1})^{-1}$ is well defined for $r \geq 0$.

Now, we give the reverse of the first inequality in (1.1).

Corollary 4.2. *Let $A, B \in \mathfrak{B}^+(\mathcal{H})$ be such that $A \leq B$. If $r \geq 0$, then $A\sharp_{-r}B \leq A!_{-r}B$.*

Proof. From Theorem 4.1 with operators A and B replaced by A^{-1} and B^{-1} , respectively, it follows that

$$(4.3) \quad A^{-1}\nabla_{-r}B^{-1} \leq A^{-1}\sharp_{-r}B^{-1}.$$

Now, applying the monotonicity operator of the function $h(x) = -1/x, x \in (0, \infty)$, to relation (4.3), we have that

$$(A^{-1}\sharp_{-r}B^{-1})^{-1} \leq (A^{-1}\nabla_{-r}B^{-1})^{-1}.$$

Finally, the result follows by virtue of $(A^{-1}\sharp_{-r}B^{-1})^{-1} = A\sharp_{-r}B$. □

Kittaneh et al. [11] obtained the following relation (see also [14]):

$$(4.4) \quad \begin{aligned} 2 \max\{\nu, 1 - \nu\}(A\nabla B - A\sharp B) &\geq A\nabla_{\nu}B - A\sharp_{\nu}B \\ &\geq 2 \min\{\nu, 1 - \nu\}(A\nabla B - A\sharp B). \end{aligned}$$

Clearly, the left inequality in (4.4) represents the converse, while the right inequality represents a refinement of the arithmetic-geometric mean operator inequality in (1.1).

Our next goal is to derive a refinement of inequality (4.2) which is, in some way, complementary to the above relations in (4.4). Clearly, this will be carried out by virtue of Lemma 3.2.

Theorem 4.3. *If $A, B \in \mathfrak{B}^+(\mathcal{H})$ and $r \geq 0$, then the following inequality holds:*

$$(4.5) \quad A\nabla_{-r}B + 2r(A\nabla B - A\sharp B) \leq A\sharp_{-r}B.$$

Proof. By virtue of Lemma 3.2, it follows that

$$(4.6) \quad (1 + r) - rx + r(x - 2\sqrt{x} + 1) \leq x^{-r}$$

holds for all $x > 0$. Now, applying functional calculus, i.e., property (4.1), to this scalar inequality, we have

$$(1+r)I_{\mathcal{H}} - rA^{-1/2}BA^{-1/2} + r(A^{-1/2}BA^{-1/2} - 2(A^{-1/2}BA^{-1/2})^{1/2} + I_{\mathcal{H}}) \leq (A^{-1/2}BA^{-1/2})^{-r}.$$

Finally, multiplying both sides of this operator inequality by $A^{1/2}$, we obtain (4.5). □

Corollary 4.4. *Let $A, B \in \mathfrak{B}^+(\mathcal{H})$ and $r > 0$. Then, $A\nabla_{-r}B = A\sharp_{-r}B$ if and only if $A = B$.*

Proof. It follows from Theorem 4.3 and the fact that $A\nabla B = A\sharp B$ if and only if $A = B$. □

Remark 4.5. Keeping in mind that scalar inequality (4.6) also holds for $r \leq -1/2$ (see Lemma 3.2), it follows that inequality (4.5) also holds for $r \leq -1/2$. However, if $r < -1$, relation (4.5) is less precise than the original inequality (4.2) and does not represent its refinement. On the other hand, it is interesting to consider the case when $-1 \leq r \leq -1/2$. Namely, denoting $\nu = -r$, where $1/2 \leq \nu \leq 1$ and (4.5) reduces to

$$A\nabla_{\nu}B - 2\nu(A\nabla B - A\sharp B) \leq A\sharp_{\nu}B,$$

this relation coincides with the converse of the arithmetic-geometric mean inequality, that is, with the left inequality in (4.4).

Remark 4.6. In [11], the authors considered the operator version of the classical Heinz mean, i.e., the operator,

$$(4.7) \quad H_\nu(A, B) = \frac{A \sharp_\nu B + A \sharp_{1-\nu} B}{2},$$

where $A, B \in \mathfrak{B}^+(\mathcal{H})$ and $\nu \in [0, 1]$. As in the real case, this mean interpolates between the arithmetic and geometric means, that is,

$$(4.8) \quad A \sharp B \leq H_\nu(A, B) \leq A \nabla B.$$

On the other hand, since $A, B \in \mathfrak{B}^+(\mathcal{H})$, expression (4.7) is also well defined for $\nu \in \mathbb{R} \setminus [0, 1]$. Moreover, due to Theorem 4.1, we obtain the inequality,

$$\begin{aligned} H_{-r}(A, B) &= \frac{A \sharp_{-r} B + A \sharp_{1+r} B}{2} \geq \frac{A \nabla_{-r} B + A \nabla_{1+r} B}{2} \\ &= A \nabla B, \quad r \geq 0 \quad \text{or} \quad r \leq -1, \end{aligned}$$

complementary to (4.8).

In order to conclude this section, we mention yet another inequality closely connected to the Young inequality, namely, in [5], the equivalence between the Young and the Hölder-McCarthy inequalities was shown. This asserts that

$$(4.9) \quad \langle Ax, x \rangle^{-r} \leq \langle A^{-r} x, x \rangle, \quad x \in \mathcal{H}, \quad \|x\| = 1,$$

holds for all $A \in \mathfrak{B}^+(\mathcal{H})$ and $r > 0$ or $r < -1$. If $-1 < r < 0$, then the sign of the inequality in (4.9) is reversed.

Now, we give a refinement of the Hölder-McCarthy inequality, once again by exploiting Lemma 3.2.

Theorem 4.7. *Let $A \in \mathfrak{B}^+(\mathcal{H})$ and $r > 0$. Then the inequality*

$$(4.10) \quad 0 \leq 2r \left(1 - \langle A^{1/2} x, x \rangle \langle Ax, x \rangle^{-1/2} \right) \leq \langle A^{-r} x, x \rangle \langle Ax, x \rangle^r - 1$$

holds for any unit vector $x \in \mathcal{H}$.

Proof. By virtue of equation (4.6), it follows that the inequality $2r(1 - \sqrt{x}) \leq x^{-r} - 1$ holds for all $x > 0$. Now, applying functional calculus to this inequality and the positive operator $\lambda^{1/r} A$, $\lambda > 0$, we

obtain

$$2r \left(I_{\mathcal{H}} - \lambda^{1/(2r)} A^{1/2} \right) \leq \lambda^{-1} A^{-r} - I_{\mathcal{H}}.$$

Further, fix a unit vector $x \in \mathcal{H}$. Then we have

$$2r \left(1 - \lambda^{1/(2r)} \langle A^{1/2} x, x \rangle \right) \leq \lambda^{-1} \langle A^{-r} x, x \rangle - 1.$$

Finally, putting $\lambda = \langle Ax, x \rangle^{-r}$ in the last inequality, we obtain the second inequality in (4.10). Clearly, the first inequality in (4.10) holds due to (4.9) since $\langle A^{1/2} x, x \rangle \leq \langle Ax, x \rangle^{1/2}$. □

Remark 4.8. Since relation (4.6) holds for $r \leq -1/2$, it follows that the second inequality in (4.10) also holds for $r \leq -1/2$. Clearly, the case of $r < -1$ is not interesting since, in this case, we obtain a less precise relation than the original Hölder-McCarthy inequality (4.9). On the other hand, the case of $-1 < r < -1/2$ yields the converse of (4.9).

5. Reverse Young-type inequalities for the trace and the determinant. In this section, we derive some Young-type inequalities for the trace and the determinant of a matrix. The starting point for this direction was used in Lemma 3.2.

In [12], Kittaneh and Manasrah obtained the inequality

$$(5.1) \quad \text{tr} \left| A^\nu B^{1-\nu} \right| + r_0 (\sqrt{\text{tr} A} - \sqrt{\text{tr} B})^2 \leq \text{tr} (\nu A + (1 - \nu) B),$$

which holds for positive semi-definite matrices $A, B \in \mathcal{M}_n$, $0 \leq \nu \leq 1$, and $r_0 = \min\{\nu, 1 - \nu\}$.

By virtue of Lemma 3.2, we can accomplish the inequality complementary to (5.1). To do this, we also need the following inequality regarding singular values of complex matrices:

$$(5.2) \quad \sum_{j=1}^n s_j(A) s_{n-j+1}(B) \leq \sum_{j=1}^n s_j(AB) \leq \sum_{j=1}^n s_j(A) s_j(B).$$

Now, we have the following result.

Theorem 5.1. *If $A, B \in \mathcal{P}_n$ and $r \geq 0$, then the following inequality holds:*

$$(5.3) \quad \text{tr} ((1 + r)A - rB) \leq \text{tr} \left| A^{1+r} B^{-r} \right| - r (\sqrt{\text{tr} A} - \sqrt{\text{tr} B})^2.$$

Proof. Applying Theorem 3.3 with $X = I_n$ and with the trace norm $\|\cdot\|_1$, that is, $\|A\|_1 = \sum_{i=1}^n s_j(A) = \text{tr}|A|$, it follows that

$$(1+r)\|A\|_1 - r\|B\|_1 + r(\sqrt{\|A\|_1} - \sqrt{\|B\|_1})^2 \leq \|A^{1+r}B^{-r}\|_1.$$

Now, since $A, B \in \mathcal{P}_n$, it follows that $\|A\|_1 = \text{tr}A$ and $\|B\|_1 = \text{tr}B$, that is, $(1+r)\|A\|_1 - r\|B\|_1 = \text{tr}((1+r)A - rB)$, so we have inequality (5.3). □

Our next intention is to obtain an analogous reverse relation for the determinant of a matrix. In [12], the authors obtained the inequality,

$$\det(A^\nu B^{1-\nu}) + r_0^n \det(2A\nabla B - 2A\sharp B) \leq \det(\nu A + (1-\nu)B),$$

where $0 \leq \nu \leq 1$, $r_0 = \min\{\nu, 1-\nu\}$, and A, B are positive definite matrices. The corresponding complementary result can also be established by virtue of Lemma 3.2.

Theorem 5.2. *Let $r \geq 0$, and let $A, B \in \mathcal{P}_n$ be such that $A \geq r/(r+1)B$. Then the following inequality holds:*

$$(5.4) \quad \det((1+r)A - rB) \leq \det(A^{r+1}B^{-r}) - r^n \det(2A\nabla B - 2A\sharp B).$$

Proof. The starting point is Lemma 3.2 with $a = s_j(B^{-1/2}AB^{-1/2})$ and $b = 1$, i.e., the inequality

$$s_j^{r+1}(B^{-1/2}AB^{-1/2}) \geq (1+r)s_j(B^{-1/2}AB^{-1/2}) - r + r(s_j^{1/2}(B^{-1/2}AB^{-1/2}) - 1)^2.$$

Furthermore, since $A \geq r/(r+1)B$, it follows that $B^{-1/2}AB^{-1/2} \geq r/(r+1)I_n$, which means that $s_j(B^{-1/2}AB^{-1/2}) \geq r/(r+1)$. Consequently, we have that

$$(1+r)s_j(B^{-1/2}AB^{-1/2}) - r \geq 0.$$

Hence, by virtue of the above two relations and the well-known properties of the determinant, we have

$$\begin{aligned}
 \det \left(B^{-1/2} A B^{-1/2} \right)^{r+1} &= \prod_{j=1}^n s_j^{r+1} \left(B^{-1/2} A B^{-1/2} \right) \\
 &\geq \prod_{j=1}^n \left[(1+r) s_j \left(B^{-1/2} A B^{-1/2} \right) - r \right. \\
 &\quad \left. + r \left(s_j^{1/2} \left(B^{-1/2} A B^{-1/2} \right) - 1 \right)^2 \right] \\
 &\geq \prod_{j=1}^n \left[(1+r) s_j \left(B^{-1/2} A B^{-1/2} \right) - r \right] \\
 &\quad + r^n \prod_{j=1}^n \left[\left(s_j^{1/2} \left(B^{-1/2} A B^{-1/2} \right) - 1 \right)^2 \right] \\
 &= \det \left((1+r) B^{-1/2} A B^{-1/2} - r I_n \right) \\
 &\quad + r^n \det \left(\left(B^{-1/2} A B^{-1/2} \right)^{1/2} - I_n \right)^2.
 \end{aligned}$$

Finally, multiplying both sides of the inequality obtained by $\det(B^{1/2})$ and utilizing the well known Binet-Cauchy theorem, we obtain (5.4), as claimed. □

6. Reverses of the Young inequality dealing with singular values. Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ be such that $0 \leq x_1 \leq \dots \leq x_n$ and $0 \leq y_1 \leq \dots \leq y_n$. Then x is said to be log majorized by y , denoted by $x \prec_{\log} y$, if

$$\prod_{j=1}^k x_j \leq \prod_{j=1}^k y_j, \quad 1 \leq k < n \quad \text{and} \quad \prod_{j=1}^n x_j = \prod_{j=1}^n y_j.$$

For $X \in \mathcal{M}_n$ and $k = 1, \dots, n$, the k th compound of X is defined as the $\binom{n}{k} \times \binom{n}{k}$ complex matrix $C_k(X)$, whose entries are defined by $C_k(X)_{r,s} = \det X[(r_1, r_2, \dots, r_k) \mid (s_1, s_2, \dots, s_k)]$, where (r_1, r_2, \dots, r_k) , $(s_1, s_2, \dots, s_k) \in P_{k,n} = \{(x_1, \dots, x_k) \mid 1 \leq x_1 < \dots < x_k \leq n\}$ are arranged in lexicographical order and (r_1, r_2, \dots, r_k) and (s_1, s_2, \dots, s_k) are the r th and s th elements in $P_{k,n}$, respectively. $X[r, s]$ is the $k \times k$ matrix that contains the elements in the intersection of rows

$(r_1, r_2, \dots, r_k) \in P_{k,n}$ and columns $(s_1, s_2, \dots, s_k) \in P_{k,n}$ (for more details, see [18]). For example, if $n = 3$ and $k = 2$, then $(1, 2), (1, 3)$ and $(2, 3)$ are the first, second and third elements of $P_{k,n}$, respectively. So,

$$C_2(X) = \begin{pmatrix} \det X[1, 2 | 1, 2] & \det X[1, 2 | 1, 3] & \det X[1, 2 | 2, 3] \\ \det X[1, 3 | 1, 2] & \det X[1, 3 | 1, 3] & \det X[1, 3 | 2, 3] \\ \det X[2, 3 | 1, 2] & \det X[2, 3 | 1, 3] & \det X[2, 3 | 2, 3] \end{pmatrix}.$$

In the general case, for $A, B \in \mathcal{M}_n$, we have

$$(6.1) \quad C_k(AB) = C_k(A)C_k(B)$$

and

$$s_1(C_k(A)) = \prod_{j=1}^k s_j(A), \quad 1 \leq k \leq n.$$

Finally, we use the corresponding ideas from [20] to present our last result.

Theorem 6.1. *Suppose that $A, B \in \mathcal{P}_n$ and $X \in \mathcal{M}_n$. If $r \geq 0$, then*

- (i) $s(A^{1+r}XB^{1+r}) \succ_{\log} s^{1+r}(AXB)s^{-r}(X)$,
- (ii) $s(A^{1+r}XB^{-r}) \succ_{\log} s^{1+r}(AX)s^{-r}(XB)$.

Proof. (i) Let $C_k(X) \in \mathbb{C}^{\binom{n}{k} \times \binom{n}{k}}$ denote the k th component of X , $1 \leq k \leq n$. Then, we have

$$\begin{aligned} \prod_{i=1}^k s_i(A^{1+r}XB^{1+r}) &= s_1(C_k(A^{1+r}XB^{1+r})) && \text{by (6.1)} \\ &= s_1(C_k(A)^{1+r}C_k(X)C_k(B)^{1+r}) && \text{by (6.1)} \\ &\geq s_1^{-r}(C_k(X))s_1^{1+r}(C_k(AXB)) && \\ &&& \text{(by inequality (3.1))} \\ &= \prod_{i=1}^k s_i^{-r}(X) \prod_{i=1}^k s_i^{1+r}(AXB). \end{aligned}$$

Moreover, if $k = n$, we have

$$\begin{aligned} \prod_{i=1}^n s_i(A^{1+r}XB^{1+r}) &= |\det(A^{1+r}XB^{1+r})| \\ &= (\det A)^{1+r}|\det X|(\det B)^{1+r} \end{aligned}$$

and

$$\begin{aligned} \prod_{i=1}^n s_i(X)^{-r} \prod_{i=1}^n s_i(AXB)^{1+r} &= |\det X^{-r}| |\det(AXB)^{1+r}| \\ &= (\det A)^{1+r}|\det X|(\det B)^{1+r}. \end{aligned}$$

(ii) The second conclusion can be accomplished by a similar argument as in (i) and by utilizing inequality (3.2). \square

Acknowledgments. The authors would like to express their sincere thanks to the referees for their warm comments. M.S. Moslehian (the corresponding author) and M. Bakherad would like to thank the Tusi Mathematical Research Group (TMRG).

REFERENCES

1. T. Ando, *Matrix Young inequality*, J. Oper. Theor. Adv. Appl. **75** (1995), 33–38.
2. C. Conde, *Young type inequalities for positive operators*, Ann. Funct. Anal. **4** (2013), 144–152.
3. J.I. Fujii, *An external version of the Jensen operator inequality*, Sci. Math. Japonica **73** (2011), 125–128.
4. S. Furuichi and N. Minculete, *Alternative reverse inequalities for Young's inequality*, J. Math. Inequal. **5** (2011), 595–600.
5. T. Furuta, *The Hölder-McCarthy and the Young inequalities are equivalent for Hilbert space operators*, Amer. Math. Month. **108** (2001), 68–69.
6. T. Furuta and M. Yanagida, *Generalized means and convexity of inversion for positive operators*, Amer. Math. Month. **105** (1998), 258–259.
7. O. Hirzallah and F. Kittaneh, *Matrix Young inequalities for the Hilbert-Schmidt norm*, Linear Algebra Appl. **308** (2000), 77–84.
8. E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415–438.
9. R.A. Horn and C.R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1991.
10. F. Kittaneh, *Norm inequalities for fractional powers of positive operators*, Lett. Math. Phys. **27** (1993), 279–285.

11. F. Kittaneh, M. Krnić, N. Lovričević and J. Pečarić, *Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators*, Publ. Math. Debrecen **80** (2012), 465–478.
12. F. Kittaneh and Y. Manasrah, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl. **361** (2010), 262–269.
13. H. Kosaki, *Arithmetic-geometric mean and related inequalities for operators*, J. Funct. Anal. **156** (1998), 429–451.
14. M. Krnić, N. Lovričević and J. Pečarić, *Jensen's operator and applications to mean inequalities for operators in Hilbert space*, Bull. Malaysian Math. Sci. Soc. **35** (2012), 1–14.
15. S.M. Manjegani, *Hölder and Young inequalities for the trace of operators*, Positivity **11** (2007), 239–250.
16. ———, *Spectral dominance and Young's inequality in type III factors*, J. Inequal. Pure Appl. Math. **7** (2006), 8 pages.
17. D.S. Mitrinovic, J. Pecaric and A.M Fink, *Classical and new inequalities in analysis*, in *Mathematics and its applications* **61**, Kluwer Academic Publishers Group, Dordrecht, 1993.
18. R. Merris, *Multilinear algebra*, Gordon and Breach Science Publishers, Amsterdam, 1997.
19. J. Pečarić, T. Furuta, J. Mičić Hot and Y. Seo, *Mond-Pečarić method in operator inequalities*, Element, Zagreb, 2005.
20. T.Y. Tam, *Heinz-Kato's inequalities for semisimple Lie groups*, J. Lie Theory **18** (2008), 919–931.
21. M. Tominaga, *Specht's ratio in the Young inequality*, Sci. Math. Japonica **55** (2002), 583–588.
22. ———, *Specht's ratio and logarithmic mean in the Young inequality*, Math. Inequal. Appl. **7** (2004), 113–125.
23. F. Zhang, *Matrix theory*, Springer-Verlag, New York, 2011.

DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P.O. BOX 1159, MASHHAD, 91775 IRAN

Email address: bakherad@member.ams.org

UNIVERSITY OF ZAGREB, FACULTY OF ELECTRICAL ENGINEERING AND COMPUTING, UNSKA 3, 10000 ZAGREB, CROATIA

Email address: mario.krnic@fer.hr

DEPARTMENT OF PURE MATHEMATICS, CENTER OF EXCELLENCE IN ANALYSIS ON ALGEBRAIC STRUCTURES (CEAAS), FERDOWSI UNIVERSITY OF MASHHAD, P.O. BOX 1159, MASHHAD, 91775 IRAN

Email address: moslehian@um.ac.ir, moslehian@member.ams.org