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Complex Analysis and Operator Theory

Numerical Radius Inequalities Involving Commutators of G_1 Operators

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Abstract We prove numerical radius inequalities involving commutators of G_1 operators and certain analytic functions. Among other inequalities, it is shown that if A and X are bounded linear operators on a complex Hilbert space, then

$$w(f(A)X + X\bar{f}(A)) \leq \frac{2}{d_A^2} w(X - AXA^*),$$

where A is a G_1 operator with $\sigma(A) \subset \mathbb{D}$ and f is analytic on the unit disk $\mathbb{D}$ such that $\mathrm{Re}(f) > 0$ and $f(0) = 1$.

Keywords G_1 operator · Numerical radius · Commutator · Analytic function

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1 Introduction

Let $(\mathscr{H}, \langle\cdot, \cdot\rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$ with the identity I. In the case when $\dim\mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices having entries in the complex field. The numerical radius of $A \in \mathbb{B}(\mathscr{H})$ is defined by

$$w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathscr{H}, \|x\| = 1\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $A \in \mathbb{B}(\mathscr{H})$, $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$ (see [9, p. 9]). If $A^2 = 0$, then equality holds in the first inequality, and if A is normal, then equality holds in the second inequality. For further information about numerical radius inequalities, we refer the reader to [1–3,12,16,17] and references therein.

An operator $A \in \mathbb{B}(\mathscr{H})$ is called a G_1 operator if the growth condition

$$\|(z - A)^{-1}\| = \frac{1}{\operatorname{dist}(z, \sigma(A))}$$

holds for all z not in the spectrum $\sigma(A)$ of A, where $\operatorname{dist}(z, \sigma(A))$ denotes the distance between z and $\sigma(A)$. For simplicity, if z is a complex number, we write z instead of zI. It is known that hyponormal (in particular, normal) operators are G_1 operators (see, e.g., [15]). Let $A \in \mathbb{B}(\mathscr{H})$ and f be a function which is analytic on an open neighborhood Ω of $\sigma(A)$ in the complex plane. Then $f(A)$ denotes the operator defined on $\mathscr{H}$ by the Riesz–Dunford integral as

$$f(A) = \frac{1}{2\pi i}\int_C f(z)(z - A)^{-1}dz,$$

where C is a positively oriented simple closed rectifiable contour surrounding $\sigma(A)$ in Ω (see e.g., [8, p. 568]). The spectral mapping theorem asserts that $\sigma(f(A)) = f(\sigma(A))$. Throughout this note, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denotes the unit disk, $\partial\mathbb{D}$ stands for the boundary of $\mathbb{D}$ and $d_A = \operatorname{dist}(\partial\mathbb{D}, \sigma(A))$. In addition, we denote

$$\mathfrak{A} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic, } \operatorname{Re}(f) > 0 \text{ and } f(0) = 1\}.$$

The Sylvester type equations $AXB \pm X = C$ have been investigated in matrix theory (see [4]). Several perturbation bounds for the norms of sums or differences of operators have been presented in the literature by employing some integral representations of certain functions. See [5,13,14] and references therein.

In this paper, we present some upper bounds for the numerical radii of the commutators and elementary operators of the form $f(A)X \pm X\bar{f}(A)$, $f(A)X\bar{f}(B) - f(B)X\bar{f}(A)$ and $f(A)X\bar{f}(B) + 2X + f(B)X\bar{f}(A)$, where $A, B, X \in \mathbb{B}(\mathscr{H})$ and $f \in \mathfrak{A}$.

2 Main Results

To prove our first result, the following lemma concerning numerical radius inequalities and an equality is required.

Lemma 2.1 [10,11] *Let* $A, B, X, Y \in \mathbb{B}(\mathscr{H})$. *Then*

(a) $w(A^*XA) \leq \|A\|^2 w(X)$.

(b) $w\left(AX \pm XA^*\right) \leq 2\|A\|w(X)$.

(c) $w\left(A^*XB \pm B^*YA\right) \leq 2\|A\|\|B\|\, w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)$.

(d) $w\left(\begin{bmatrix} 0 & AXB^* \\ BYA^* & 0 \end{bmatrix}\right) \leq \max\{||A||^2, ||B||^2\} w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)$.

(e) $w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \frac{w(X+Y)+w(X-Y)}{2}$.

(f) $w\left(\begin{bmatrix} 0 & X \\ e^{i\theta}X & 0 \end{bmatrix}\right) = w(X)\ for\ \theta \in \mathbb{R}$.

Proof Since all parts, except part (d), have bee shown in [10,11], we prove only part (d). If we take $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $S = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$, then $CSC^* = \begin{bmatrix} 0 & AXB^* \\ BYA^* & 0 \end{bmatrix}$. Now, using part (a), we have

$$\begin{aligned} w\left(\begin{bmatrix} 0 & AXB^* \\ BYA^* & 0 \end{bmatrix}\right) &= w(CSC^*) \\ &\leq \|C\|^2 w(S) \\ &= \max\{||A||^2, ||B||^2\} w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right), \end{aligned}$$

as required. □

Now, we are in position to demonstrate the main results of this section by using some ideas from [13,14].

Theorem 2.2 *Let* $A \in \mathbb{B}(\mathscr{H})$ *be a* G_1 *operator with* $\sigma(A) \subset \mathbb{D}$ *and* $f \in \mathfrak{A}$. *Then for every* $X \in \mathbb{B}(\mathscr{H})$, *we have*

$$w(f(A)X + X\bar{f}(A)) \leq \frac{2}{d_A^2} w(X - AXA^*)$$

and

$$w(f(A)X - X\bar{f}(A)) \leq \frac{4}{d_A^2} \|A\| w(X).$$

Proof Using the Herglotz representation theorem (see e.g., [7, p.21]), we have

$$f(z)=\int_0^{2\pi}\frac{e^{i\alpha}+z}{e^{i\alpha}-z}d\mu(\alpha)+i\,\mathrm{Im}\,f(0)=\int_0^{2\pi}\frac{e^{i\alpha}+z}{e^{i\alpha}-z}d\mu(\alpha),$$

where μ is a positive Borel measure on the interval $[0,2\pi]$ with finite total mass $\int_0^{2\pi}d\mu(\alpha)=f(0)=1$. Hence,

$$\bar{f}(z)=\overline{\int_0^{2\pi}\frac{e^{i\alpha}+z}{e^{i\alpha}-z}d\mu(\alpha)}=\int_0^{2\pi}\frac{e^{-i\alpha}+\bar{z}}{e^{-i\alpha}-\bar{z}}d\mu(\alpha),$$

where $\bar{f}$ is the conjugate function of f. So,

$$\begin{aligned}f(A)X+X\bar{f}(A)&=\int_0^{2\pi}\Big[\left(e^{i\alpha}+A\right)\left(e^{i\alpha}-A\right)^{-1}X\\&\quad+X\left(e^{-i\alpha}+A^*\right)\left(e^{-i\alpha}-A^*\right)^{-1}\Big]d\mu(\alpha)\\&=\int_0^{2\pi}\left(e^{i\alpha}-A\right)^{-1}\Big[\left(e^{i\alpha}+A\right)X\left(e^{-i\alpha}-A^*\right)\\&\quad+\left(e^{i\alpha}-A\right)X\left(e^{-i\alpha}+A^*\right)\Big]\left(e^{-i\alpha}-A^*\right)^{-1}d\mu(\alpha)\\&=2\int_0^{2\pi}\left(e^{i\alpha}-A\right)^{-1}(X-AXA^*)\left(e^{-i\alpha}-A^*\right)^{-1}d\mu(\alpha).\end{aligned}$$

Hence,

$$\begin{aligned}&w(f(A)X+X\bar{f}(A))\\&=w\left(\int_0^{2\pi}\Big[\left(e^{i\alpha}+A\right)\left(e^{i\alpha}-A\right)^{-1}X\right.\\&\quad\left.+X\left(e^{-i\alpha}+A^*\right)\left(e^{-i\alpha}-A^*\right)^{-1}\Big]d\mu(\alpha)\right)\\&=2\,w\left(\int_0^{2\pi}\left(e^{i\alpha}-A\right)^{-1}(X-AXA^*)\left(e^{-i\alpha}-A^*\right)^{-1}d\mu(\alpha)\right)\\&\le 2\int_0^{2\pi}w\left(\left(e^{i\alpha}-A\right)^{-1}(X-AXA^*)\left(e^{-i\alpha}-A^*\right)^{-1}\right)d\mu(\alpha)\\&\qquad\qquad(\text{since } w(\cdot) \text{ is a norm})\\&\le 2\int_0^{2\pi}\left\|\left(e^{i\alpha}-A\right)^{-1}\right\|^2 w\left(X-AXA^*\right)d\mu(\alpha)\\&\qquad\qquad(\text{by Lemma 2.1}(a)).\end{aligned}$$

Since A is a G_1 operator, it follows that

$$\left\|\left(e^{i\alpha}-A\right)^{-1}\right\| = \frac{1}{\operatorname{dist}(e^{i\alpha},\sigma(A))} \le \frac{1}{\operatorname{dist}(\partial\mathbb{D},\sigma(A))} = \frac{1}{d_A},$$

and so

$$\begin{aligned} w\left(f(A)X + X\bar{f}(A)\right) &\le \left(\frac{2}{d_A^2}\int_0^{2\pi} d\mu(\alpha)\right) w(X - AXA^*) \\ &= \left(\frac{2}{d_A^2} f(0)\right) w(X - AXA^*) \\ &= \frac{2}{d_A^2} w(X - AXA^*). \end{aligned}$$

This proves the first inequality.

Similarly, it follows from the equations

$$\begin{aligned} f(A)X - X\bar{f}(A) &= \int_0^{2\pi} \left[\left(e^{i\alpha}+A\right)\left(e^{i\alpha}-A\right)^{-1} X \right. \\ &\quad \left. - X\left(e^{-i\alpha}+A^*\right)\left(e^{-i\alpha}-A^*\right)^{-1}\right] d\mu(\alpha) \\ &= \int_0^{2\pi} \left(e^{i\alpha}-A\right)^{-1}\left[\left(e^{i\alpha}+A\right)X\left(e^{-i\alpha}-A^*\right)\right. \\ &\quad \left. - \left(e^{i\alpha}-A\right)X\left(e^{-i\alpha}+A^*\right)\right]\left(e^{-i\alpha}-A^*\right)^{-1} d\mu(\alpha) \\ &= 2\int_0^{2\pi} \left(e^{i\alpha}-A\right)^{-1}\left(e^{-i\alpha}AX - e^{i\alpha}XA^*\right) \\ &\quad \times \left(e^{-i\alpha}-A^*\right)^{-1} d\mu(\alpha) \\ &= 2\int_0^{2\pi} \left(e^{i\alpha}-A\right)^{-1}\left(\left(e^{-i\alpha}A\right)X \right. \\ &\quad \left. - X\left(e^{-i\alpha}A\right)^*\right)\left(e^{-i\alpha}-A^*\right)^{-1} d\mu(\alpha) \end{aligned}$$

that

$$\begin{aligned} & w(f(A)X - X\bar{f}(A)) \\ &\quad = 2w\left(\int_0^{2\pi}\left(e^{i\alpha}-A\right)^{-1}\left(\left(e^{-i\alpha}A\right)X - X\left(e^{-i\alpha}A\right)^*\right)\left(e^{-i\alpha}-A^*\right)^{-1} d\mu(\alpha)\right) \\ &\quad \le 2\int_0^{2\pi} w\left(\left(e^{i\alpha}-A\right)^{-1}\left(\left(e^{-i\alpha}A\right)X - X\left(e^{-i\alpha}A\right)^*\right)\left(e^{-i\alpha}-A^*\right)^{-1}\right) d\mu(\alpha) \\ &\qquad\qquad\qquad (\text{since } w(\cdot) \text{ is a norm}) \end{aligned}$$

$$\le 2\int_0^{2\pi} \left\| \left(e^{i\alpha} - A\right)^{-1} \right\|^2 w\left(\left(e^{-i\alpha}A\right) X - X\left(e^{-i\alpha}A\right)^*\right) d\mu(\alpha)$$

(by Lemma 2.1 (a))

$$\le 4\int_0^{2\pi} \left\| \left(e^{i\alpha} - A\right)^{-1} \right\|^2 \|e^{-i\alpha}A\| w(X) d\mu(\alpha)$$

(by Lemma 2.1 (b))

$$\le \frac{4}{d_A^2}\|A\|w(X)\int_0^{2\pi} d\mu(\alpha)$$

$$\le \frac{4}{d_A^2}\|A\|w(X).$$

This proves the second inequality and completes the proof of the theorem. □

If we take $X = I$ in Theorem 2.2, we get the following result. Observe that $\bar{f}(A) = (f(A))^*$.

Corollary 2.3 *Let $A \in \mathbb{B}(\mathscr{H})$ be a G_1 operator with $\sigma(A) \subset \mathbb{D}$ and $f \in \mathfrak{A}$. Then*

$$\|Re(f(A))\| \le \frac{1}{d_A^2}\|I - AA^*\|$$

and

$$\|Im(f(A))\| \le \frac{2}{d_A^2}\|A\|.$$

Theorem 2.4 *Let $A, B \in \mathbb{B}(\mathscr{H})$ be G_1 operators with $\sigma(A)\cup\sigma(B) \subset \mathbb{D}$ and $f \in \mathfrak{A}$. Then for every $X \in \mathbb{B}(\mathscr{H})$, we have*

$$\begin{aligned} &w(f(A)X\bar{f}(B) - f(B)X\bar{f}(A)) \\ &\quad\le \frac{2}{d_Ad_B}\left[2w\left(X\right) + w\left(AXB^* + BXA^*\right) + w\left(AXB^* - BXA^*\right)\right] \end{aligned}$$

and

$$\begin{aligned} &w(f(A)X\bar{f}(B) + 2X + f(B)X\bar{f}(A)) \\ &\quad\le \frac{2}{d_Ad_B}\left[2w\left(X\right) + w\left(AXB^* + BXA^*\right) + w\left(AXB^* - BXA^*\right)\right]. \end{aligned}$$

Proof We have

$$\begin{aligned} &f(A)X\bar{f}(B) - f(B)X\bar{f}(A) \\ &\quad= \int_0^{2\pi}\int_0^{2\pi}\left[\left(e^{i\alpha} - A\right)^{-1}(e^{i\alpha} + A)X(e^{-i\beta} + B^*)\left(e^{-i\beta} - B^*\right)^{-1}\right. \\ &\qquad\left. - \left(e^{i\beta} - B\right)^{-1}(e^{i\beta} + B)X(e^{-i\alpha} + A^*)\left(e^{-i\alpha} - A^*\right)^{-1}\right] d\mu(\alpha)d\mu(\beta). \end{aligned}$$

Using the equations

$$\begin{aligned}
&\left(e^{i\alpha}-A\right)^{-1}(e^{i\alpha}+A)X(e^{-i\beta}+B^*)\left(e^{-i\beta}-B^*\right)^{-1} \\
&\quad-\left(e^{i\beta}-B\right)^{-1}(e^{i\beta}+B)X(e^{-i\alpha}+A^*)\left(e^{-i\alpha}-A^*\right)^{-1} \\
&\quad=\left(e^{i\alpha}-A\right)^{-1}(e^{i\alpha}+A)X(e^{-i\beta}+B^*)\left(e^{-i\beta}-B^*\right)^{-1}+X \\
&\quad-X-\left(e^{i\beta}-B\right)^{-1}(e^{i\beta}+B)X(e^{-i\beta}+A^*)\left(e^{-i\alpha}-A^*\right)^{-1} \\
&\quad=\left(e^{i\alpha}-A\right)^{-1}\Big[(e^{i\alpha}+A)X(e^{-i\beta}+B^*) \\
&\quad+(e^{i\alpha}-A)X(e^{-i\beta}-B^*)\Big]\left(e^{-i\beta}-B^*\right)^{-1} \\
&\quad-\left(e^{i\beta}-B\right)^{-1}\Big[(e^{i\beta}-B)X(e^{-i\alpha}-A^*) \\
&\quad+(e^{i\beta}+B)X(e^{-i\alpha}+A^*)\Big]\left(e^{-i\alpha}-A^*\right)^{-1} \\
&\quad=2(e^{i\alpha}-A)^{-1}(e^{i\alpha}e^{-i\beta}X+AXB^*)(e^{-i\beta}-B^*)^{-1} \\
&\quad-2(e^{i\beta}-B)^{-1}(e^{-i\alpha}e^{i\beta}X+BXA^*)(e^{-i\alpha}-A^*)^{-1},
\end{aligned}$$

we have

$$\begin{aligned}
&w(f(A)X\bar{f}(B)-f(B)X\bar{f}(A)) \\
&\quad=2w\left(\int_0^{2\pi}\int_0^{2\pi}(e^{i\alpha}-A)^{-1}(e^{i\alpha}e^{-i\beta}X+AXB^*)(e^{-i\beta}-B^*)^{-1}\right. \\
&\qquad\left.-(e^{i\beta}-B)^{-1}(e^{-i\alpha}e^{i\beta}X+BXA^*)(e^{-i\alpha}-A^*)^{-1}d\mu(\alpha)d\mu(\beta)\right) \\
&\quad\le 2\int_0^{2\pi}\int_0^{2\pi}w\Big((e^{i\alpha}-A)^{-1}(e^{i\alpha}e^{-i\beta}X+AXB^*)(e^{-i\beta}-B^*)^{-1} \\
&\qquad-(e^{i\beta}-B)^{-1}(e^{-i\alpha}e^{i\beta}X+BXA^*)(e^{-i\alpha}-A^*)^{-1}\Big)d\mu(\alpha)d\mu(\beta) \\
&\qquad\qquad\text{(since } w(\cdot) \text{ is a norm)} \\
&\quad\le 4\int_0^{2\pi}\int_0^{2\pi}\|(e^{i\alpha}-A)^{-1}\|\|(e^{i\beta}-B)^{-1}\| \\
&\qquad\times w\left(\begin{bmatrix}0 & e^{i\alpha}e^{-i\beta}X+AXB^* \\ e^{-i\alpha}e^{i\beta}X+BXA^* & 0\end{bmatrix}\right)d\mu(\alpha)d\mu(\beta) \\
&\qquad\qquad\text{(by Lemma 2.1 (c))} \\
&\quad\le\frac{4}{d_Ad_B}\int_0^{2\pi}\int_0^{2\pi}\left[w\left(\begin{bmatrix}0 & e^{i\alpha}e^{-i\beta}X \\ e^{-i\alpha}e^{i\beta}X & 0\end{bmatrix}\right)\right. \\
&\qquad\left.+w\left(\begin{bmatrix}0 & AXB^* \\ BXA^* & 0\end{bmatrix}\right)\right]d\mu(\alpha)d\mu(\beta)
\end{aligned}$$

$$= \frac{4}{d_A d_B} \int_0^{2\pi} \int_0^{2\pi} \left[w \left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \right) + w \left(\begin{bmatrix} 0 & AXB^* \\ BXA^* & 0 \end{bmatrix} \right) \right] d\mu(\alpha) d\mu(\beta)$$
$$\leq \frac{2}{d_A d_B} \left[2w\left(X\right) + w\left(AXB^* + BXA^*\right) + w\left(AXB^* - BXA^*\right) \right]$$
(by Lemma 2.1 (e) and (f)).

This proves the first inequality.

Similarly, we have

$$\begin{aligned} &f(A)X\bar{f}(B) + 2X + f(B)X\bar{f}(A) \\ &= \int_0^{2\pi} \int_0^{2\pi} \left[\left(e^{i\alpha} - A\right)^{-1} (e^{i\alpha} + A)X(e^{-i\beta} + B^*) \left(e^{-i\beta} - B^*\right)^{-1} + 2X \right. \\ &\quad \left. + \left(e^{i\beta} - B\right)^{-1} (e^{i\beta} + B)X(e^{-i\alpha} + A^*) \left(e^{-i\alpha} - A^*\right)^{-1} \right] d\mu(\alpha) d\mu(\beta). \end{aligned}$$

Using the equations

$$\begin{aligned} &\left(e^{i\alpha} - A\right)^{-1} (e^{i\alpha} + A)X(e^{-i\beta} + B^*) \left(e^{-i\beta} - B^*\right)^{-1} + 2X \\ &\quad + \left(e^{i\beta} - B\right)^{-1} (e^{i\beta} + B)X(e^{-i\alpha} + A^*) \left(e^{-i\alpha} - A^*\right)^{-1} \\ &= \left(e^{i\alpha} - A\right)^{-1} (e^{i\alpha} + A)X(e^{-i\beta} + B^*) \left(e^{-i\beta} - B^*\right)^{-1} + X \\ &\quad + X + \left(e^{i\beta} - B\right)^{-1} (e^{i\beta} + B)X(e^{-i\beta} + A^*) \left(e^{-i\alpha} - A^*\right)^{-1} \\ &= \left(e^{i\alpha} - A\right)^{-1} \left[(e^{i\alpha} + A)X(e^{-i\beta} + B^*) \right. \\ &\quad \left. + (e^{i\alpha} - A)X(e^{-i\beta} - B^*) \right] \left(e^{-i\beta} - B^*\right)^{-1} \\ &\quad + \left(e^{i\beta} - B\right)^{-1} \left[(e^{i\beta} - B)X(e^{-i\alpha} - A^*) \right. \\ &\quad \left. + (e^{i\beta} + B)X(e^{-i\alpha} + A^*) \right] \left(e^{-i\alpha} - A^*\right)^{-1} \\ &= 2(e^{i\alpha} - A)^{-1}(e^{i\alpha}e^{-i\beta}X + AXB^*)(e^{-i\beta} - B^*)^{-1} \\ &\quad + 2(e^{i\beta} - B)^{-1}(e^{-i\alpha}e^{i\beta}X + BXA^*)(e^{-i\alpha} - A^*)^{-1}, \end{aligned}$$

we have

$$\begin{aligned} &w(f(A)X\bar{f}(B) + 2X + f(B)X\bar{f}(A)) \\ &= 2w \left(\int_0^{2\pi} \int_0^{2\pi} (e^{i\alpha} - A)^{-1}(e^{i\alpha}e^{-i\beta}X + AXB^*)(e^{-i\beta} - B^*)^{-1} \right. \\ &\quad \left. + (e^{i\beta} - B)^{-1}(e^{-i\alpha}e^{i\beta}X + BXA^*)(e^{-i\alpha} - A^*)^{-1} d\mu(\alpha) d\mu(\beta) \right) \end{aligned}$$

$$\leq 2\int_0^{2\pi}\int_0^{2\pi} w\Big((e^{i\alpha}-A)^{-1}(e^{i\alpha}e^{-i\beta}X+AXB^*)(e^{-i\beta}-B^*)^{-1}$$
$$+(e^{i\beta}-B)^{-1}(e^{-i\alpha}e^{i\beta}X+BXA^*)(e^{-i\alpha}-A^*)^{-1}\Big)\,d\mu(\alpha)d\mu(\beta)$$
$$\text{(since } w(\cdot) \text{ is a norm)}$$
$$\leq 4\int_0^{2\pi}\int_0^{2\pi} \|(e^{i\alpha}-A)^{-1}\|\|(e^{i\beta}-B)^{-1}\|$$
$$\times\, w\left(\begin{bmatrix} 0 & e^{i\alpha}e^{-i\beta}X+AXB^* \\ e^{-i\alpha}e^{i\beta}X+BXA^* & 0\end{bmatrix}\right)d\mu(\alpha)d\mu(\beta)$$
$$\text{(by Lemma 2.1 (c))}$$
$$\leq \frac{4}{d_A d_B}\int_0^{2\pi}\int_0^{2\pi}\left[w\left(\begin{bmatrix} 0 & e^{i\alpha}e^{-i\beta}X \\ e^{-i\alpha}e^{i\beta}X & 0\end{bmatrix}\right)\right.$$
$$\left.+\,w\left(\begin{bmatrix} 0 & AXB^* \\ BXA^* & 0\end{bmatrix}\right)\right]d\mu(\alpha)d\mu(\beta)$$
$$=\frac{4}{d_A d_B}\int_0^{2\pi}\int_0^{2\pi}\left[w\left(\begin{bmatrix} 0 & X \\ X & 0\end{bmatrix}\right)+w\left(\begin{bmatrix} 0 & AXB^* \\ BXA^* & 0\end{bmatrix}\right)\right]d\mu(\alpha)d\mu(\beta)$$
$$\leq \frac{2}{d_A d_B}\left[2w\left(X\right)+w\left(AXB^*+BXA^*\right)+w\left(AXB^*-BXA^*\right)\right]$$
$$\text{(by Lemma 2.1 (e) and (f)).}$$

This proves the second inequality and completes the proof of the theorem. ◻

Remark 2.5 Under the assumptions of Theorem 2.4 and the hypothesis that X is self-adjoint, we have

$$\|f(A)X\bar{f}(B)-f(B)X\bar{f}(A)\|$$
$$\leq \frac{4}{d_A d_B}\max\{\|\,|X|\,\|+\|\,|AXB^*|\,\|, \|\,|X|\,\|+\|\,|BXA^*|\,\|\}$$

and

$$\|f(A)X\bar{f}(B)+2X+f(B)X\bar{f}(A)\|$$
$$\leq \frac{4}{d_A d_B}\max\{\|\,|X|\,\|+\|\,|AXB^*|\,\|, \|\,|X|\,\|+\|\,|BXA^*|\,\|\}.$$

To see this, first note that if X is self-adjoint, then the operator matrix

$$T=\begin{bmatrix} 0 & e^{i\alpha}e^{-i\beta}X+AXB^* \\ e^{-i\alpha}e^{i\beta}X+BXA^* & 0\end{bmatrix}$$

is self-adjoint, hence $w(T)=\|T\|$. Moreover, $T=M+N$, where

$$M=\begin{bmatrix} 0 & e^{i\alpha}e^{-i\beta}X \\ e^{-i\alpha}e^{i\beta}X & 0\end{bmatrix},\quad N=\begin{bmatrix} 0 & AXB^* \\ BXA^* & 0\end{bmatrix}$$

are self-adjoint operators. Using the fact that $\|C+D\| \leq \| \, |C|+|D| \, \|$ for any normal operators C and D (see [6]), we have

$$\begin{aligned} w(T) &= \|M+N\| \leq \| \, |M|+|N| \, \| \\ &= \max\{\| \, |X| \, \| + \| \, |AXB^*| \, \|, \| \, |X| \, \| + \| \, |BXA^*| \, \|\}. \end{aligned}$$

Hence, we get the required inequalities by the same arguments as in the proof of Theorem 2.4.

If we take $X = I$ in Theorem 2.4, we get the following result.

Corollary 2.6 *Let $A, B \in \mathbb{B}(\mathscr{H})$ be G_1 operators with $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$ and $f \in \mathfrak{A}$. Then*

$$\|Im(f(A)\bar{f}(B))\| \leq \frac{2}{d_A d_B} \left(1 + \|AB^*\|\right)$$

and

$$\|Re(f(A)\bar{f}(B)) + I\| \leq \frac{2}{d_A d_B} \left(1 + \|AB^*\|\right).$$

Remark 2.7 If instead of applying Lemma 2.1 (c) we use Lemma 2.1 (d) and (f) in the proof Theorem 2.4, we obtain the related inequalities

$$w(f(A)X\bar{f}(B) - f(B)X\bar{f}(A)) \leq \frac{4}{d_A d_B} \left[1 + \max\{\|A\|^2, \|B\|^2\}\right] w(X)$$

and

$$w(f(A)X\bar{f}(B) + 2X + f(B)X\bar{f}(A)) \leq \frac{4}{d_A d_B} \left[1 + \max\{\|A\|^2, \|B\|^2\}\right] w(X).$$

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