DEPENDENT RANDOM VARIABLES AND RELATED TOPICS

Chapter 1

Different Concepts of Dependence

1.1 Quadrant dependence

This section contains basic concepts, properties, theorems together with several concerning quadrant dependent (QD), in particular NQD random variables The following Definitions due to Lehmman (1966).

Definition 1.1.1. The random variables X and Y are said positive quadrant dependence (PQD)if for every $x, y \in R$

$$P[X \leq x, Y \leq y] \geq P[X \leq x]P[Y \leq y] \quad (1)$$

negative quadrant dependent (NQD) if for every $x, y \in R$

$$P[X \leq x, Y \leq y] \leq P[X \leq x]P[Y \leq y] \quad (2)$$

Remark 1.1.2. The inequalities of 1 and 2 equivalent to the following inequalities respectively,

$$P[X > x, Y > y] \geq P[X > x]P[Y > y]$$

and

$$P[X > x, Y > y] \leq P[X > x]P[Y > y]$$

Example 1.1.3. i) (X, X) is PQD, ii) $(X, -X)$ is NQD.
iii) If X_1, X_2, X_3 are iid with distribution of exp(1), then $X = X_1 + X_3$ and $Y = X_2 + X_3$ are PQD.

Question: What it is relationships between QD and correlated.?

The following Lemma which due to Hoeffding (1940) give us answer this question.

Lemma 1.1.4. *Let (X, Y) be a joint distribution F and marginal distributions F_1 and F_2 such that $E|XY| < \infty,\ E|X| < \infty, and\ \ E|Y| < \infty$. Then*

$$Cov(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(x, y) - F_1(x)F_2(y)]dxdy.$$

Proof. Let $(X_1, Y_1), (X_2, Y_2) \sim^{iid} F$, then

$$\begin{aligned} 2(E(X_1Y_1) - EX_1EY_1) &= E[(X_1 - X_2)(Y_1 - Y_2)] \\ &= E \int \int (I_{(-\infty, X_1]}(u) - I_{(-\infty, X_2]}(u))(I_{(-\infty, Y_1]}(v) - I_{(-\infty, Y_2]}(v))dudv \\ &= 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(x, y) - F_1(x)F_2(y)]dxdy. \end{aligned}$$

Where for all $i, j = 1, 2$

$$E(X_iY_j) = E \int_0^{X_i} \int_0^{Y_j} dudv = E \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} I_{(X_i > u)} I_{(Y_j > v)} dudv.$$

Guardas (2002) extended above Lemma as the following and Matula (2006) proved some applications of this Lemma.

Lemma 1.1.5. *Let $\alpha(.)$ and $\beta(.)$ be two real valued function such that*

$$E|\alpha(X)\beta(Y)| < \infty, E|\alpha(X)| < \infty, E|\beta(Y)| < \infty.$$

Then

$$Cov(\alpha(X), \beta(Y)) = \int \int [F(x, y) - F_1(x)F_2(y)]d\alpha(x)d\beta(y)$$

Definition 1.1.6. The two real-valued functions f and g of n arguments are concordance for the ith coordinate if, considered as functions of the ith coordinate (with all other coordinates held fixed), they are monotone in the same direction, i.e. either both non-decreasing or both non-increasing. Similarly f and g will be called discordant for the ith coordinate if they are monotone in opposite directions.

Theorem 1.1.7. *Let f and g be two real valued functions and concordance ,then*

i) If X and Y are PQD, then $f(X)$ and $g(Y)$ are PQD.

ii) If X and Y are NQD, then $f(X)$ and $g(Y)$ are NQD.

Proof. ii) Let f and g be nondecreasing functions and X , Y be NQD random variables then for all $x, y \in R$ we have

$$\begin{aligned} P[f(X) \leq x, g(Y) \leq y] &= P[X \in f^{-1}(-\infty, x], Y \in g^{-1}(-\infty, y]] \\ &\leq P[X \in f^{-1}(-\infty, x]]P[Y \in g^{-1}(-\infty, y]] \\ &= P[f(X) \leq x]P[g(Y) \leq y]. \end{aligned}$$

Hence $f(X)$, $g(Y)$ are NQD random variables. Similar argument works when f and g are non-increasing functions.

Theorem 1.1.8. *Let f and g be two real valued functions and concordance (all increasing or all decreasing),then*

i) X and Y are PQD, if and only if $Cov(f(X), g(Y)) \geq 0$.

ii) X and Y are NQD, if and only if $Cov(f(X), g(Y)) \leq 0$.

Proof. ii) (Necessary) By Hoeffding's Lemma and Theorem 1.2 we have

$$Cov(f(X), g(Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(P[f(X) \leq t, g(Y) \leq s] - P[f(X) \leq t]P[g(Y) \leq s])dtds \leq 0.$$

(Sufficiency): Let (1.3) hold. Define $f_x(t) = I_{[t>x]}$ and

$g_y(s) = I_{[s>y]}$,where

$$I_{[u>v]} = \begin{cases} 1 & \text{if } u > v \\ 0 & \text{if } u \leq v \end{cases}$$

for every $x, y \in R$ we have

$$Cov(f_x(X), g_y(Y)) = P[X > x, Y > y] - P[X > x]P[Y > y] \leq 0,$$

hence

$$P[X > x, Y > y] \leq P[X > x]P[Y > y],$$

and this completes the proof. □

Corollary 1.1.9. *i) If X and Y are PQD, then $Cov(X, Y) \geq 0$.*

ii) If X and Y are NQD, then $Cov(X, Y) \leq 0$.

iii) If X and Y are PQD, and $Cov(X, Y) = 0$ then X and Y are independence.

iv) If X and Y are NQD, and $Cov(X, Y) = 0$ then X and Y are independence.

Theorem 1.1.10. *Let $(X_i, Y_i) \sim^{id} F_i(x, y), i = 1, 2, ..., n,$ $f : R^n \to R,$ $g : R^n \to R$ and $X = f(X_1, X_2, ..., X_n),$ $Y = g(Y_1, Y_2, ..., Y_n),$ then*

i) X and Y are PQD, if:

1- X_i and $Y_i, i = 1, 2, ..., n$ are PQD, and $f,$ g concordance .

2- X_i and $Y_i, i = 1, 2, ..., n$ are NQD, and $f,$ g dis-concordance.

ii) X and Y are NQD, if:

1- X_i and $Y_i, i = 1, 2, ..., n$ are NQD, and $f,$ g concordance .

2- X_i and $Y_i, i = 1, 2, ..., n$ are PQD, and f, g dis-concordance.

iii) Let U and V be independence r.v.'s and moreover independence of $(X_i, Y_i), i = 1, 2, ...n$, then $X = f(U, X_1, X_2, ..., X_n)$, $\ Y = g(V, Y_1, Y_2, ..., Y_n)$ satisfy in $i)$ and $ii)$ without any behavior f and g in u and v respectively.

Example 1.1.11. The following are some pairs of random variables (X, Y) with PQD (NQD); the property in each case follows from Theorem 1.9.

i) X and $f(X)$ are PQD for any r.v. X and any non-decreasing f (NQD for any non-increasing f).

ii) $X = U + aZ$, and $Y = V + bZ$ are PQD for any independent r.v. U, V, Z and $ab > 0$ (NQD if $ab < 0$..

iii) $X = f(U, Z)$ and $Y = g(V, Z)$ are PQD where U, V, Z are independent and f and g are non-decreasing in Z but otherwise arbitrary, (NQD if f and g are non-increasing).

Example 1.1.12. Let $F(x, y) = F_1(x)F_2(y)[1 + \alpha(1 - F_1(x))(1 - F_2(y))]$, $\ -1 \leq \alpha \leq 1$.

i) Show that X and Y are PQD iff $0 \leq \alpha \leq 1$.

ii) Show that X and Y are NQD iff $-1 \leq \alpha \leq 0$.

Corollary 1.1.13. *Let X and Y be NQD(PQD) random variables and absolute continuous then Kendal's τ and Spearman's ρ_s are negative(positive).*

Proof. Let (X_1, Y_1) and (X_2, Y_2) be independent and distributed identically as

(X,Y). Define

$$U = sgn(X_2 - X_1) = \begin{cases} 1 & \text{if } \; X_2 > X_1 \\ -1 & \text{if } \; X_2 < X_1 \end{cases}$$

and

$$V = sgn(Y_2 - Y_1) = \begin{cases} 1 & \text{if } \; Y_2 > Y_1 \\ -1 & \text{if } \; Y_2 < Y_1 \end{cases}$$

by Theorem 1.3 and 1.9 we have

$$\tau = Cov(U,V) \leq (\geq) 0.$$

Now let (X_1,Y_1) , (X_2,Y_2) and (X_3,Y_3) be independent and distributed identically as (X,Y), we have

$$\frac{\rho_s}{3} = Cov[sgn(X_2 - X_1), sgn(Y_3 - Y_1)],$$

the result follows from Lemma 1.3 and Theorem 1.9 (i) by putting
$n = 3$ and

$$f(x_1,x_2,x_3) = sgn(x_2 - x_1) \quad , \quad g(y_1,y_2,y_3) = sgn(y_3 - y_1).$$

Theorem 1.1.14. *Let* f *be the joint and* f_X , f_Y *be the marginal densities of* X *and* Y *, and let for every* $x,y \in R$*,*

$$f(x,y) \leq f_X(x) f_Y(y), \tag{1.4}$$

then

i) X *and* Y *are NQD.*

ii) If X , Y *are nonnegative random variables then,*

a) $E[X|Y=y] \leq E[X]$ *W.P.1,*

b) $E[Y|X=x] \leq E[Y]$ *W.P.1.*

iii) $F_{X+Y} \leq F_X \star F_Y$.

proof i) Suppose X and Y are absolute continuous then, for each $x, y \in R$,

$$\begin{aligned} F(x,y) &= \int_{-\infty}^{x}\int_{-\infty}^{y} f(t,s)dtds \\ &\leq \int_{-\infty}^{x}\int_{-\infty}^{y} f(t)f(s)dtds = F_X(x)F_Y(y), \end{aligned}$$

hence X and Y are NQD.

ii) To prove (a),we have

$$\begin{aligned} E[X|Y=y] &= \int_{0}^{\infty} xf(x|y)dx \\ &= \int_{0}^{\infty} x\frac{f(x,y)}{f(y)}dx \\ &\leq \int_{0}^{\infty} x\frac{f(x)f(y)}{f(y)}dx = E[X]. \end{aligned}$$

Similarly we obtain (b).

iii)

$$\begin{aligned} F_{X+Y}(t) &= \int\int_{[x+y\leq t]} f(x,y)dxdy \\ &\leq \int\int_{[x+y\leq t]} f(x)f(y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{t-y} f(x)f(y)dxdy \\ &= \int_{-\infty}^{\infty} F_X(t-y)f_Y(y)dy = F_X \star F_Y(t). \qquad \square \end{aligned}$$

In the following example we will show that inverse of part i of Theorem 1.13 dose not hold, and the NQD properties are not valid for absolute value and square of random variables.

Example 1.1.15. Let (X, Y) have the following probability density function.

X_2 X_1	-1	0	1
-1	0	$\frac{1}{9}$	$\frac{2}{9}$
0	$\frac{1}{9}$	$\frac{1}{9}$	0
1	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

i) Random variables X and Y are NQD random variables since for each $x, y \in R$ we have

$$F(x,y) \leq F(x)F(y).$$

ii) The density functions f , f_X and f_Y do not satisfy condition (1.4) since

$$P[X=1, Y=-1] = 2/9 >$$

$$P[X=1]P[Y=-1] = (3/9)(3/9).$$

iii) X and $V = Y^2$ are not NQD random variables because for $-1 \leq x < 0$ and $0 \leq v < 1$ we have

$$F(x,v) = 1/9 > F_X(x)F_V(v) = (3/9)(2/9).$$

iv) $U = X^2$ and Y are not NQD random variables because for $0 \leq u < 1$ and $0 \leq y < 1$ we have

$$F(u,y) = 2/9 > F_U(u)F_Y(y) = (3/9)(5/9).$$

v) $U = X^2$ and $V = Y^2$ are not NQD random variables as well as $|X|$ and $|Y|$ since for $0 \leq u < 1$, $0 \leq v < 1$ we have

$$F(u,v) = 1/9 > F_U(u)F_V(v) = (2/9)(3/9). \qquad \square$$

Remark 1.1.16. The above example also show that if X , Y are NQD random variables, then part *ii* of Theorem 1.13 may not hold, because in the above example we have

$$E[X|Y=-1] = 2/9 > E[X] = 0$$

and

$$E[Y|X=-1] = 2/9 > E[Y] = 1/9.$$

Hence the condition (1.4) is a necessary condition for part *ii* of Theorem 1.13.

Lemma 1.1.17. *Let* Y *be a random variable with* $P[Y > 0] = 1$, $E[\frac{1}{Y}] < \infty$ *, then*

$$E[\frac{1}{Y}] \geq \frac{1}{E[Y]}.$$

Proof *In Caushy Schurz inequality , for* $U = \sqrt{Y}$, $V = \frac{1}{\sqrt{Y}}$ *we have*

$$1 = E^2[UV] \leq E[U^2]E[V^2] = E[Y]E[\frac{1}{Y}],$$

hence

$$E[\frac{1}{Y}] \geq \frac{1}{E[Y]}. \qquad \Box$$

Theorem 1.6 *i) If* X *and* Y *are NQD random variables with* $P[Y > 0] = 1$, $E|X| < \infty$, $E[\frac{1}{Y}] < \infty$ *, then,*

a) X *and* $\frac{1}{Y}$ *are PQD random variables,*

b) $E[\frac{X}{Y}] \geq \frac{E(X)}{E(Y)}$ *if* $E(X) \geq 0$.

ii) If X *and* Y *are PQD random variables then ,*

a) X *and* $\frac{1}{Y}$ *are NQD random variables,*

b) $E[\frac{X}{Y}] \leq \frac{E(X)}{E(Y)}$ *if* $E(X) \leq 0$.

Proof i) Part ii of Theorem 1.1 implies that X and $\frac{1}{Y}$ are PQD ,and by Lemma 1.3 we have

$$Cov(X, \frac{1}{Y}) \geq 0 \Longrightarrow E[\frac{X}{Y}] \geq E[X]E[\frac{1}{Y}],$$

hence Lemma 1.3 implies that

$$E[\frac{X}{Y}] \geq \frac{E[X]}{E[Y]} \quad if \quad E(X) \geq 0.$$

ii) Part i of Theorem 1.9 implies that X and $\frac{1}{Y}$ are NQD ,and by Lemma 1.1 we have

$$Cov(X, \frac{1}{Y}) \leq 0 \Longrightarrow E[\frac{X}{Y}] \leq E[X]E[\frac{1}{Y}],$$

hence Lemma 1.3 implies that

$$E[\frac{X}{Y}] \leq \frac{E[X]}{E[Y]} \quad if \quad E(X) \leq 0. \qquad \Box$$

1.2 Weakley Negatively Dependent

In following we present a new definition of dependence which is assumed in this section.

Definition 1.2.1. The random variables X_1 and X_2 are said Weakley Negatively Dependent (WND) if there exist a $C > 1$ such that, $f(x_1, x_2) \leq C.f_1(x_1).f_2(x_2)$ where $f(x_1, x_2)$, $f_1(x_1)$ and $f_2(x_2)$ are joint density and marginal densities of X_1 and X_2, respectively.

The class of WND random variables is well defined and a large class of these random variables can be found. Some examples of this class will present in following.

Example 1.2.2. The following examples are evidence of WND random variables:

i) Suppose that X_1 and X_2 have half-normal distribution, then

$$f_{X_1,X_2}(x_1, x_2) = \frac{2}{\pi\sqrt{1-\rho^2}} exp\left[-\frac{1}{2(1-\rho^2)}\{x_1^2 + x_2^2 - 2\rho x_1 x_2\}\right]; x_1, x_2 > 0,$$

$$f_{X_i}(x_i) = \sqrt{\frac{1}{\pi}} exp\{-\frac{1}{2}x_i^2\}; i = 1, 2.$$

If $-1 < \rho \leq 0$, then X_1 and X_2 are NQD r.v.'s (Ebrahimi and Ghosh. (1981)). Moreover,

$$\frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_1(x_1)} f_{X_2(x_2)}} = \frac{1}{\sqrt{1-\rho^2}} \exp\left[\frac{-\rho^2}{2(1-\rho^2)}(x_1^2 + x_2^2) + \frac{\rho}{1-\rho^2} x_1 x_2\right] \leq \frac{1}{\sqrt{1-\rho^2}}.$$

Then $f(x_1, x_2) \leq C.f_1(x_1).f_2(x_2)$, where $C = 1/\sqrt{1-\rho^2} \geq 1$. So, X_1 and X_2 are WND.

ii) Let X and Y be two random variables with joint FGM (Farlie-Gumbel-Morgenstern) distribution, we have

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\left[1+\alpha(1-2F_X(x))(1-2F_Y(y))\right].$$

On the other hand, it's obvious that

$$|1+\alpha(1-2F_X(x))(1-2F_Y(y)| \leq 1+|\alpha|,$$

and

$$f_{X,Y}(x,y) \leq [1+|\alpha|]f_X(x)f_Y(y).$$

Therefore, the random variables X and Y are WND with $C = 1+|\alpha| \geq 1$. Moreover, we know if $-1 < \alpha \leq 0$, then X and Y are NQD ([3]).(For more details see: Ranjbar et al. 2008).

Lemma 1.2.3. *Let X_1 and X_2 be two WND random variables with distribution functions $F_i, i = 1,2$, then*
i) For every $x_1, x_2 \in R$ we have, $F_{X_1,X_2}(x_1,x_2) \leq C.F_{X_1}(x_1)F_{X_2}(x_2)$.
ii) For all positive value of x, $P(X_1+X_2 > x) \leq C.\int_0^\infty \overline{F}_1(x-u)dF_2(u)$.
iii)If $h_1(.), h_2(.)$ are monotone measurable functions then $h_1(X_1), h_2(X_2)$ are WND.

1.3 Dependent events

Definition 1.3.1. The events A and B are NQD if their indicator functions are NQD.

Theorem 1.18 The events A and B are NQD if and only if

$$P(A \cap B) \leq P(A)P(B). \tag{1.5}$$

Proof (Necessary) Let $X = I_A$ and $Y = I_B$, then by Definition 1.2 X and Y are NQD ,thus the inequalities (1.1) and (1.2) hold ,and for every $x, y \in R$ such that $0 \leq x, y < 1$ we have

$$\begin{aligned} P(A \cap B) &= P(X = 1, Y = 1) \\ &= P(X > x, Y > y) \\ &\leq P(X > x)P(Y > y) = P(A)P(B). \end{aligned}$$

Moreover we can show that the inequality (1.5) is equivalent with the following inequality

$$P(A^c \cap B^c) \leq P(A^c)P(B^c). \tag{1.6}$$

(Sufficiency): Let the inequality (1.5) is hold.

i) If $0 \leq x, y < 1$ then

$$\begin{aligned} P(X \leq x, Y \leq y) &= P(X = 0, Y = 0) \\ &= P(A^c \cap B^c) \leq P(A^c)P(B^c) \\ &= P(X = 0)P(Y = 0) = P(X \leq x)P(Y \leq y), \end{aligned}$$

ii) If $x < 0$ or $y < 0$ then

$$P(X \leq x, Y \leq y) = 0 \leq P(X \leq x)P(Y \leq y),$$

iii) If $x \geq 1$ and $0 \leq y < 1$ then

$$P(X \leq x, Y \leq y) = P(Y \leq y),$$

and since $P(X \leq x) = 1$, thus the inequality (1.1) is hold,similarly for $0 \leq x < 1$ and $y \geq 1$.

iv) If $x \geq 1$, $y \geq 1$ then

$$P(X \leq x, Y \leq y) = 1 \leq P(X \leq x)P(Y \leq y) = 1.$$

Hence for all real numbers x , y the inequality (1.1) holds for random variables $X = I_A$ and $Y = I_B$, thus by definition X , Y are NQD and A , B are ND. □

1.4 Association

Let X and Y be two random variables with joint distribution function $F(x, y)$.
i)-The random variables X and Y are said positive correlated if

$$Cov(X, Y) \geq 0, \qquad (1)$$

ii)-(Lehmann, 1966) The random variables X and Y are said positive associated if for all nondecreasing f, g

$$Cov(f(X), g(Y)) \geq 0, \qquad (2)$$

iii)-(Esary and Proschan, 1967) The random variables X and Y are said positive associated (PA) if for all nondecreasing f, g

$$Cov(f(X, Y), g(X, Y)) \geq 0, \qquad (3)$$

Remark 1.4.1. It is easy to show that (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (2) $\Leftrightarrow$ PQD.
iv)-(Joage and Proschan, 1983) The random variables X and Y are said negative associated (NA) if for all nondecreasing f, g

$$Cov(f(X), g(Y)) \leq 0, \qquad (4).$$

Properties of NA random variables

P_1-The non decreasing functions of NA random variables are NA.
P_2-If for all binary function γ_1 and γ_2 $Cov(\gamma_1(X), \gamma_1(Y)) \leq 0$ then X and Y are NA.
Proof For all non decreasing functions f and g set

$$\gamma_f(x) = I_{(x,\infty)}(f(X)), \gamma_g(y) = I_{(y,\infty)}(g(Y))$$

Now, Hoefdding's Lemma implies that

$$Cov(f(X), g(Y)) = \int\int Cov(\gamma_{f(X)}(x), \gamma_{g(Y)}(y))dxdy,$$

this complete the proof.
P_3-If X and Y are binary NA random variables then $1 - X$ and $1 - Y$ are NA.

P_4-If X and Y are binary NA random variables then X and Y are NA if and only if $Cov(X, Y) \leq 0$.
P_5-The random variables X and Y are NQD if and only if are NA.
For more details see (Joage and Proschan, 1983.)

Definition 1.4.2. The random variables X and Y are said linear negative dependence (LIND) if for all $\lambda_1, \lambda_2 > 0$ the random variables $\lambda_1 X$ and $\lambda_2 Y$ are NQD. Similarly we LIPD.

Corollary 1.4.3. *It is easy to show that in bivariate case* $LIND(X, Y) \Leftrightarrow NA(X, Y) \Leftrightarrow NQD(X, Y)$.

Corollary 1.4.4. *If* X *and* Y *are NA(PA) and* $Cov(X, Y) = 0$*, then* X *and* Y *are independent. Joage and Proschan 1983.*

1.5 Regression Dependence and Stochastic monotonicity

Let X and Y be two random variables with joint distribution function $F(x, y)$ and mrginals F_1 and F_2 respectively
$i)$- The random variable Y is said stochastically increasing in $X, (SI(Y|X))$ if for all y: $P[Y > y|X = x]$ is non decreasing in x, equivalently $P[Y \leq y|X = x]$ is non increasing in x.
$ii)$-The random variable Y is said stochastically decreasing in $X, (SD(Y|X))$ if for all y: $P[Y > y|X = x]$ is non increasing in x, equivalently $P[Y \leq y|X = x]$ is non decreasing in x
Lehmann (1966) introduced these concepts as the following:
Positive regression dependence $(PRD(Y|X)$ if $P[Y \leq y|X = x]$ is non increasing in x. Similarly we can define $SI(X|Y)$ *or* $PRD(X|Y)$.
Negative regression dependence $NRD(Y|X)$, if $P[Y \leq y|X = x]$ is non decreas-

ing in x. Similarly we can define $SD(X|Y)$ *or* $NRD(X|Y)$.

Example 1.5.1. Let $Y = \alpha + \beta X + U$ and X and U are independent, then
i) PRD(X,Y) $\Leftrightarrow \beta \geq 0$
ii) NRD(X,Y) $\Leftrightarrow \beta \leq 0$.
For all $x, t \in R$ we have

$$\begin{aligned} P[Y \leq t|X = x] &= P[\alpha + \beta X + U \leq t|X = x] \\ &= P[U \leq t - \alpha - \beta x] = F_U(t - \alpha - \beta x), \end{aligned}$$

that is increasing(decreasing) function in x iff $\beta \leq (\geq)0$

Theorem 1.5.2. *If for all* $x, y \in R$, $P[Y \leq y|X = x]$ *is a nondecreasing function in* x *, then* X *and* Y *are NQD random variables.*

Proof For each $x_1, x_2 \in R$, $(x_1 < x_2)$ and with assumptions $P[X \leq x_1] \neq 0$, $P[X \leq x_2] \neq 0$ we have

$$\begin{aligned} P[Y > y|X \leq x_1] &= \frac{\int_{-\infty}^{x_1} P[Y > y|X = u] d(F_X(u))}{P[X \leq x_1]} \\ &= \frac{\int_{-\infty}^{=\infty} P[Y > y|X = u] I_{[-\infty, x_1]}(u) d(F_X(u))}{P[X \leq x_1]} \\ &\geq \frac{\int_{-\infty}^{+\infty} P[Y > y|X = u] I_{[-\infty, x_2]}(u) d(F_X(u))}{P[X \leq x_2]} \\ &= P[Y > y|X \leq x_2], \end{aligned}$$

The inequality holds since $h(u) = P[Y > y|X = u]$ is decreasing in u, so, for all $x_1 < x_2$,

$$g_1(u) = h(u) I_{[-\infty, x_1]}(u) \geq h(u) I_{[-\infty, x_2]}(u) = g_2(u),$$

Therefore, $\int_{-\infty}^{+\infty} g_1(u) dF(u) \geq \int_{-\infty}^{+\infty} g_2(u) dF(u)$. Thus for every $x_1, x_2 \in R, (x_1 < x_2)$

$$P[Y \leq y|X \leq x_1] \leq P[Y \leq y|X \leq x_2].$$

If $x_2 \longrightarrow \infty$ we have

$$P[Y \leq y|X \leq x_1] \leq P[Y \leq y],$$

and for all $x_1, y \in R$

$$P[X \leq x_1, Y \leq y] \leq P[X \leq x_1]P[Y \leq y]$$

hence X and Y are NQD. $\square$

Example 1.5.3. i) Let random vector $(X_1, \cdots, X_n)$ to have multivariate distribution Bure (B_{12}) and $g(u) = \frac{(u+v)^{c_2}}{1+u^{c_1}} \nearrow in\ u$ for some $v \in R$. Then X_i and $X_j - X_i$, $(i \neq j)$ are NQD random variables. We have

$$f_{i,j}(x, y) = \frac{k(k+1)c_1c_2x^{c_1-1}y^{c_2-1}}{[1 + x^{c_1} + y^{c_2}]^{k+2}}, \qquad x, y, c_1, c_2 > 0$$

and

$$f_i(x) = \frac{kc_1x^{c_1-1}}{[1 + x^{c_1}]^{c_1+1}}, \qquad x, c_1 > 0$$

let $U = X_i$ and $V = X_j - X_i$, $(i \neq j)$ we obtain

$$P[V > v|U = u] = [1 + g(u)]^{-(k+1)}, \quad u, v > 0$$

Now by the above assumption $P[V \leq v|U = u]$ is increasing in u. Hence by Theorem 1.4 X_i and $X_j - X_i$, $(i \neq j)$ are NQD.

ii) let $X_1, \cdots, X_n$ be a random sample of F distribution ,and $X_{u_1} < X_{u_2} < \cdots < X_{u_m}, (m < n)$ are upper records of above sample and $h(x) = \frac{1-F(x+a)}{1-F(x)} \nearrow in\ x$ for every $a > 0$, then X_{u_m} and $X_{u_n} - X_{u_m}$ are NQD.

Define

$$V = X_{u_m}\ ,\ W = X_{u_n} - X_{u_m}\ \ and\ \ R(x) = -\ln(1 - F(x))$$

we have

$$f_V(v) = \frac{[R(v)]^{m-1}f(v)}{(m-1)!}\ , \qquad v \in R$$

and

$$f(v, w) = \frac{[R(v)]^{m-1}[R(v+w) - R(v)]^{n-m-1}f(v)f(v+w)}{(m-1)!(n-m-1)!(1-F(v))},$$

thus

$$P[W \leq w|V = v] = \frac{1}{(n-m)!}\int_0^w \frac{1 - F(v+u)}{1 - F(v)} d[R(u+v) - R(v)]^{n-m}.$$

Now by assumption ($h(x) \nearrow in\ x$) we obtain

$$P[W \leq w|V = v] \nearrow in\ v.$$

Hence V and W are NQD.

1.6 Right-tail increasing and Left-tail decreasing

Let X and Y be two random variables with joint distribution function $F(x, y)$ and mrginals F_1 and F_2 respectively

i)- The random variable Y is said left tail decreasing in X ($LTD(Y|X)$)
if $P[Y \leq y|X \leq x] = \frac{F(x,y)}{F_1(x)}$ is non increasing in x, and similarly we can define $LTD(X|Y)$.

ii)- The random variable Y is said right tail increasing in X ($RTI(Y|X)$)
if $P[Y > y|X > x] = \frac{\bar{F}(x,y)}{\bar{F}_1(x)}$ is non decreasing in x, and similarly we can define $RTI(X|Y)$.

(for more detail see Esary and Proschan, 1972)

iii)-The random variable Y is said left tail increasing in X ($LTI(Y|X)$)
if $P[Y \leq y|X \leq x] = \frac{F(x,y)}{F_1(x)}$ is non decreasing in x, and similarly we can define $LTI(X|Y)$.

iv)-The random variable Y is said right tail decreasing in X ($RTD(Y|X)$)
if $P[Y > y|X > x] = \frac{\bar{F}(x,y)}{\bar{F}_1(x)}$ is non decreasing in x, and similarly we can define $LTD(X|Y)$.

Theorem 1.6.1. *It is easy to prove that:*
i)-$SI(Y|X) \Rightarrow LTD(Y|X)$ *and* $RTI(Y|X) \Rightarrow PQD(X, Y)$
ii)-$SD(Y|X) \Rightarrow LTI(Y|X)$ *and* $RTD(Y|X) \Rightarrow NQD(X, Y)$.

Proof. By Theorem 1.5.1 $SD(Y|X) \Rightarrow LTI(Y|X), RTD(Y|X)$, but for the second implication we have,
$LTI(Y|X) \Leftrightarrow \frac{F(x_1,y)}{F_1(x_1)} \leq \frac{F(x_2,Y)}{F_1(x_2)}$, for all $X_1 < x_2$. Now, if $x_2 \to \infty$, we obtain

$F(x_1, y) \leq F_1(x_1).F_2(y)$. This completes proof of (ii), and similarly we can prove (i). □

Example 1.6.2. Let for all $x, y \geq 0$ and $0 \leq \theta \leq a + 1, a > 0$.

$$\bar{F}(x, y) = [1 + x + y + \theta xy]^{-a}$$

. Then $\frac{\bar{F}(x,y)}{\bar{F}_1(x)} = [1 + y.\frac{1=\theta x}{1+x}]^{-\alpha}$, is increasing if $0 \leq \theta \leq 1$ (in this case this family of distributions is PQD)and decreasing if $1 \leq \theta \leq \alpha$. (in this case this family of distribution is NQD).

1.7 The Likelihood ratio dependence and corner set monotonicity

Definition 1.7.1. i).(Karlin 1968) A non negative function f defined on R^2 is totally positive of order 2 (denoted by $TP2$) if for all $x_1 < x_2, y_1 < y_2$,

$$f(x_1, y_1).f(x_2, y_2) \geq f(x_1, y_2).f(x_2, y_1).$$

ii).The non negative function f is said reverse regular of order 2 (or reverse rule of order 2 $RR2$) if for all $x_1 < x_2, y_1 < y_2$,

$$f(x_1, y_1).f(x_2, y_2) \leq f(x_1, y_2).f(x_2, y_1).$$

The following definition give a concept of dependence which introduced by Lehmann(1966).

Definition 1.7.2. Let X and Y be continuous random variables with joint density function $f(x, y)$, then
i) X and Y are said positive likelihood ratio dependence $PLRD(X, Y)$ if $f(x, y)$ is $TP2$. or equivalently if $\frac{f(y|x')}{f(y|x)}$ is increasing in y for all $x < x'$.
ii) X and Y are said negative likelihood ratio dependence $NLRD(X, Y)$ if $f(x, y)$ is $TP2$. or equivalently if $\frac{f(y|x')}{f(y|x)}$ is decreasing in y for all $x < x'$.

Theorem 1.7.3. *Let X and Y be continuous random variables with joint density function $f(x,y)$, then*
i) $PLRD(X,Y) \Rightarrow PRD(Y|X)$ and $PRD(X|Y)$. But inverse implication is not true. (Lehmann (1966).
ii) $NLRD(X,Y) \Rightarrow NRD(Y|X)$ and $NRD(X|Y)$. But inverse implication is not true

Proof *For all y and fixed y_0, define*

$$\psi_{y_0}(y) = I_{(-\infty,y_0)}(y) \quad and \quad E_x\phi(Y) = \int \phi(y)f(y|x)dy.$$

Let Y_1, Y_2 be iid copy of Y for all $x < x'$ we have

$$\begin{aligned} 0 \le I &= \frac{1}{2}E_x\{(\psi(Y_1) - \psi(Y_2))(\frac{f(Y_1|x')}{f(Y_1|x)} - \frac{f(Y_2|x')}{f(Y_2|x)})\} \\ &= \frac{1}{2}\{E_x(\psi(Y_1)\frac{f(Y_1|x')}{f(Y_1|x)}) - E_x(\psi(Y_1))E_x(\frac{f(Y_2|x')}{f(Y_2|x)})\} \\ &- \frac{1}{2}\{E_x(\psi(Y_2))E_x(\frac{f(Y_1|x')}{f(Y_1|x)}) + E_x(\psi(Y_2))E_x(\frac{f(Y_2|x')}{f(Y_2|x)})\} \\ &= E_x(\psi(Y)\frac{f(Y|x')}{f(Y|x)}) - E_x\psi(Y)E_x(\frac{f(Y|x')}{f(Y|x)}) \end{aligned}$$

this implies that

$$\int_{-\infty}^{y_0} f(y|x)dy \le \int_{-\infty}^{y_0} f(y|x')dy$$

if and only if

$$P[Y \le y_0|X = x] \le P[Y \le y_0|X = x'] \quad \forall \quad x < x' \Leftrightarrow NRD(Y|X)$$

Similarly we can show that $NLRD(X,Y) \Rightarrow NRD(X|Y)$.
The following concepts on dependence due to Harris (1970).

Definition 1.7.4. The random variables X and Y are said
$i)$ Left corner set decreasing $LCSD(X,Y)$ if for all $x < x', y < y'$

$$P[X \le x, Y \le y|X \le x', Y \le y'] \searrow \quad in \quad x', y'.$$

$ii)$ Right corner set increasing $RCSI(X,Y)$ if for all $x < x', y < y'$

$$P[X > x, Y > y|X > x', Y > y'] \nearrow \quad in \quad x', y'.$$

$iii)$ Left corner set increasing $LCSI(X,Y)$ if for all $x < x', y < y'$

$$P[X \le x, Y \le y | X \le x', Y \le y'] \nearrow \quad in \quad x', y'.$$

$iv)$ Right corner set decreasing $RCSD(X,Y)$ if for all $x < x', y < y'$

$$P[X > x, Y > y | X > x', Y > y'] \searrow \quad in \quad x', y'.$$

Theorem 1.7.5. *Let X and Y be continuous random variables, then i)* $LCSD(X,Y) \Rightarrow LTD(Y|X), LTD(X|Y)$.
$ii)$ $RCSI(X,Y) \Rightarrow RTI(Y|X), RTI(X|Y)$.
$iii)$ $LCSI(X,Y) \Rightarrow LTI(Y|X), LTI(X|Y)$.
$iv)$ $RCSD(X,Y) \Rightarrow RTD(Y|X), RTD(X|Y)$.

Proof. For part (i), set $x = \infty$ and $y' = \infty$ to obtain $LTD(Y|X)$, and set $y = \infty$ and $x' = \infty$ to obtain $LTD(X|Y)$. Parts (ii), (III) and (iV) are similar. □

Theorem 1.7.6. *Let (X,Y) be an absolutely continuous random vector with distribution function $F(x,y)$ and survival function $\bar{F}(x,y)$. Then,*
$i)$ $LCSD(X,Y) \Leftrightarrow F(x,y) \;\; is \;\; TP2$.
$ii)$ $RCSI(X,Y) \Leftrightarrow \bar{F}(x,y) \;\; is \;\; TP2$.
$iii)$ $LCSI(X,Y) \Rightarrow F(x,y) \;\; is \;\; RR2$.
$iv)$ $RCSD(X,Y) \Rightarrow \bar{F}(x,y) \;\; is \;\; RR2$.

Proof. The part (iv) is proved, the other parts are similar.
$RCSD(X,Y) \Rightarrow \bar{F}(x,y)$ is RR_2 : In this case, taking $y = -\infty$,
$P(X > x | X > x', Y > y')$ is decreasing in x' and in y', for all $x \in I\!R$. So, if $x > x'$, then $P(X > x | X > x', Y > y') = \frac{P(X>x,Y>y')}{P(X>x',Y>y')}$ is decreasing in y', consequently for all $y' < y$, we obtain

$$\frac{P(X > x, Y > y)}{P(X > x', Y > y)} \le \frac{P(X > x, Y > y')}{P(X > x', Y > y')}, \tag{1.1}$$

this implies that $\bar{F}(x,y)$ is RR_2.

$\bar{F}(x,y)$ is $RR_2 \Rightarrow RCSD(X,Y)$: In this case, for all $x > x'$ and $y > y'$, (1.1) valid and for all $x > x'$, $P(X > x|X > x', Y > y') = \frac{P(X>x,Y>y')}{P(X>x',Y>y')}$ is decreasing in y' . Similarly for all $y > y'$ we have,

$$P(Y > y|X > x', Y > y') \geq P(Y > y|X > x, Y > y')$$

i.e. $P(Y > y|X > x', Y > y')$ is decreasing in x' . Now, if $x > x'$, $y < y'$

$$\begin{aligned} P(X > x, Y > y|X > x', Y > y') &= \frac{P(X > x, Y > y')}{P(X > x', Y > y')} \\ &\leq \frac{P(X > x, Y > y)}{P(X > x', Y > y)} \\ &= P(X > x, Y > y|X > x', Y > y), \end{aligned}$$

then, $P(X > x, Y > y|X > x', Y > y')$ is decreasing in y'. Similarly for $x \leq x'$, $y > y'$, $P(X > x, Y > y|X > x', Y > y')$ is decreasing in x'. Also for $x < x'$, $y < y'$, $P(X > x, Y > y|X > x', Y > y') = 1$. Therefore (X,Y) is $RCSD$. □

Example 1.7.7. i) Let for all $x, y \geq 0$ and $\lambda_1, \lambda_2, \lambda_{12} \geq 0$

$$\bar{F}(x,y) = \exp[-\lambda_1 x - \lambda_2 y - \lambda_{12} \max\{x,y\}].$$

Show that $\bar{F}(x,y)$ is $TP2$ and so is we have $RCSI(X,Y)$.

ii)-Let for all $x, y \geq 0$ and $0 \leq \theta \leq a+1, a > 0$.

$$\bar{F}(x,y) = [1 + x + y + \theta xy]^{-a}$$

. *i*) Show that $\bar{F}(x,y)$ is $TP2$ if $0 \leq \theta \leq 1$.

ii) $\bar{F}(x,y)$ is $RR2$ if $1 \leq \theta \leq a+1$.

Theorem 1.7.8. *Let X and Y be continuous random variables with joint distribution F, and density f.*

i) $PLRD(X,Y) \Rightarrow SI(Y|X), SI(X|Y), LCSD(X,Y), RCSI(X,Y)$

ii) $NLRD(X,Y) \Rightarrow SD(Y|X), SD(X|Y), LCSI(X,Y), RCSD(X,Y)$

Proof *For part i) refer to Joe(1997) and Nelsen (2006).For part ii) we have two* *method*

Method 1 :(based on Joe,1997)

$$
\begin{aligned}
NLRD(X,Y) &\Leftrightarrow f(x,y).f(x',y') - f(x,y').f(x',y) \leq 0 \quad \forall \quad x<x', y<y' \\
&\rightarrow \int_{-\infty}^{x}\int_{-\infty}^{y}\int_{x}^{x'}\int_{y}^{y'}[f(t,s).f(t',s') - f(t,s').f(t',s)]dt'ds'dtds \leq 0 \\
&\rightarrow F(x,y)[F(x',y') - F(x',y) - F(x,y') + F(x,y)] \\
&\leq [F(x,y') - F(x,y)].[F(x',y) - F(x,y)] \\
&\rightarrow F(x,y).F(x',y') \leq F(x,y').F(x',y) \Leftrightarrow LCSI(X,Y).
\end{aligned}
$$

Similarly we can show that $NLRD(X,Y) \Rightarrow RCSD(X,Y)$.

Method 1 :(based on Nelsen,2006)

$$
\begin{aligned}
NLRD(X,Y) &\Leftrightarrow f(y_1|x_1).f(y_2|x_2) \leq f(y_1|x_2).f(y_2|x_1), \quad \forall \quad x_1<x_2, y_1<y_2 \\
&\rightarrow \int_{y_2}^{+\infty}\int_{-\infty}^{y_1} f(t|x_1).f(s|x_2)dtds \leq \int_{y_2}^{+\infty}\int_{-\infty}^{y_1} f(t|x_1).f(s|x_2)dtds \\
&\rightarrow P[Y \leq y|X=x_1].P[Y>y|X=x_2] \leq P[Y \leq y|X=x_2].P[Y>y|X=x_1],
\end{aligned}
$$

adding $P[Y>y|X=x_1].P[Y>y|X=x_2]$ *to both side final inequality. We get*

$$P[Y>y|X=x_1] \leq P[Y>y|X=x_2] \Leftrightarrow SD(Y|X).$$

1.8 Negatively hazard and Local dependence

Let X and Y be absolutely continuous random variables having joint density $f(x,y)$ and survival function $\bar{F}(x,y)$. Basu [3] introduced bivariate hazard function by $r(x,y) = f(x,y)/\bar{F}(x,y)$. In the independent case the bivariate hazard function is equal to product of conditional hazard functions, $\frac{\partial}{\partial x}[-\log \bar{F}(x,y)]$ and $\frac{\partial}{\partial y}[-\log \bar{F}(x,y)]$. If equality failed we deal with dependent (positive or negative) random variables. Oluyede [17] and [18] has obtained some properties and inequalities for positively hazard and local dependence. More details about notions of dependence are in Lehmann [14], Karlin [13], Esary and Proschan [5], Joe [10] and Shaked and Shanthikumar [20]. In this paper we use notions of negatively hazard and local dependence, say HND, LND, and investigate relationship between these concepts with some other concepts of dependence. Finally, we obtain measures of association as Θ-measure (known as Clayton-Oakes measure), φ-measure and γ-measure for some bivariate distributions family, then evaluate the relationship between these measures and $HND(LND)$.

Let (X, Y) be an absolutely continuous random vector with distribution function $F(x, y)$ and survival function$\bar{F}(x, y)$. Next, we need the following definitions.

Definition 1.8.1. ([17]) Absolutely continuous random variables X and Y having a joint density function $f(x, y)$ are hazard negative (positive)dependence, HND(HPD), if and only if

$$\frac{f(x,y)}{\bar{F}(x,y)} \leq (\geq) \int_x^\infty \frac{f(u,y)du}{\bar{F}(x,y)} \int_y^\infty \frac{f(x,v)dv}{\bar{F}(x,y)} \tag{1.2}$$

where $\frac{f(x,y)}{\bar{F}(x,y)}$ is the bivariate hazard rate function, and

$$\int_x^\infty \frac{f(u,y)du}{\bar{F}(x,y)} = \frac{\partial}{\partial y}[-\log \bar{F}(x,y)], \quad and \quad \int_y^\infty \frac{f(x,v)dv}{\bar{F}(x,y)} = \frac{\partial}{\partial x}[-\log \bar{F}(x,y)]$$

are conditional hazard functions. Note that, equality holds in (1) if and only if X and Y are independent.

Definition 1.8.2. ([18]) Absolutely continuous random variables X and Y having a joint density function $f(x, y)$ are locally negative (positive) dependence, LND(LPD), if and only if

$$F(x,y)f(x,y) \leq (\geq) \int_{-\infty}^x f(u,y)du \int_{-\infty}^y f(x,v)dv, \tag{1.3}$$

Note that, equality holds in (2) if and only if X and Y are independent

Definition 1.8.3. A non-negative function h on A^2 , where $A \subseteq I\!R$, is reverse rule of order 2 (RR_2) if for all $x_1 < x_2$ and $y_1 < y_2$, with $x_i, y_j \in A$ $i = 1, 2$ $j = 1, 2$

$$h(x_1, y_1)h(x_2, y_2) \leq \ h(x_1, y_2)h(x_2, y_1). \tag{1.4}$$

Definition 1.8.4. Let X and Y be continuous random variables.Then;

- X and Y are right corner set decreasing,(which we denote $RCSD(X,Y)$), if

$$P(X > x, Y > y | X > x', Y > y') \tag{1.5}$$

is decreasing (non-increasing) in x' and in y' , for all x and y .

- X and Y are left corner set increasing, $LCSI(X,Y)$, if

$$P(X \leq x, Y \leq y | X \leq x', Y \leq y') \tag{1.6}$$

is increasing (non-decreasing) in x' and in y' , for all x and y .

Definition 1.8.5. Let $F_\theta(x)$ be a family of distribution functions. This family is called monotone decreasing likelihood ratio, (MDLR)(monotone increasing likelihood ratio, (MILR)) if for all $\eta > \theta$, $\frac{F_\eta(x)}{F_\theta(x)}$ is decreasing (increasing) in x.

1.8.1 Some results

In this section, we obtain some useful results about HND and LND which show relation of these concepts with other notions of dependence.

Proposition 1.8.6. *Let (X,Y) be an absolutely continuous random vector with distribution $F(x,y)$ and survival function $\bar{F}(x,y)$ .Then*

i) *$\bar{F}(x,y)$ is RR_2 if and only if for all $x_1 < x_2$ and $y_1 < y_2$,*

$$\begin{aligned} P(X > x_2, Y > y_2) \; & P(x_1 < X \leq x_2, y_1 < Y \leq y_2) \\ & \leq P(x_1 < X \leq x_2, Y > y_2) P(X > x_2, y_1 < Y \leq y_2) \end{aligned} \tag{1.7}$$

ii) *$F(x,y)$ is RR_2 if and only if for all $x_1 < x_2$ and $y_1 < y_2$,*

$$\begin{aligned} P(X \leq x_1, Y \leq y_1) \; & P(x_1 < X \leq x_2, y_1 < Y \leq y_2) \\ & \leq P(X \leq x_1, y_1 < Y \leq y_2) P(x_1 < X \leq x_2, Y \leq y_1) \end{aligned} \tag{1.8}$$

Proof. We prove part (i). The part of (ii) is similar. Note that $\bar{F}(x, y)$ is RR_2 , *i.e.* for all $x_1 < x_2$ and $y_1 < y_2$

$$\begin{vmatrix} P(X > x_1, Y > y_1) & P(X > x_1, Y > y_2) \\ P(X > x_2, Y > y_1) & P(X > x_2, Y > y_2) \end{vmatrix} \leq 0. \tag{1.9}$$

It is easy to show that (8) is equivalent to

$$\begin{vmatrix} P(x_1 < X \leq x_2, y_1 < Y \leq y_2) & P(x_1 < X \leq x_2, Y > y_2) \\ P(X > x_2, y_1 < Y \leq y_2) & P(X > x_2, Y > y_2) \end{vmatrix} \leq 0. \tag{1.10}$$

and (9) is equivalent to (6). This completes the proof. □

The following proposition gives a relationship between RR_2 and HND(LND).

Proposition 1.8.7. *Let (X, Y) be an absolutely continuous random vector with distribution function $F(x, y)$ and survival function $\bar{F}(x, y)$. Then,*

i) *$\bar{F}(x, y)$ is $RR_2 \Rightarrow$ HND(X, Y).*

ii) *$F(x, y)$ is RR_2 , $\Rightarrow$ LND(X, Y).*

Proof.

i) Let $x_1 = x$, $x_2 = x + \Delta x$, $y_1 = y$, $y_2 = y + \Delta y$ where $\Delta x, \Delta y > 0$. By using (6) and dividing the result by $\Delta x \Delta y$ and letting $\Delta x \to 0, \Delta y \to 0$, the result follows.

ii) The proof is similar, to (i).

□

Corollary 1.8.8. *Under the assumptions of Theorem 2.3 and Proposition 2.2*

i) $RCSD(X, Y) \Rightarrow HND(X, Y)$.

ii) $LCSI(X, Y) \Rightarrow LND(X, Y)$.

Theorem 1.8.9. *Let $F_\theta(x)$ and $G_\theta(y)$ be two families of distribution functions. For any mixing distribution K , consider the distribution*

$$H(x,y) = \int_\Omega F_\theta(x)G_\theta(y)dK(\theta),$$

where Ω is a Borel set in $I\!R^n$ and K is a probability measure on Ω .

(i) If one of the family is $MILR$ and the other is $MDLR$, then $H(x,y)$ is LND.

(ii) If $F_\theta(x)$ and $G_\theta(y)$ are both $MDLR$ or $MILR$, then $H(x,y)$ is LPD.

Proof. We prove part (i) . The proof of part (ii) is similar . Let $F_\theta(x)$ be $MDLR$ and $G_\theta(y)$ be $MILR$, so that for $x < x'$, $y < y'$ and $\eta > \theta$ $(\eta, \theta \in \Omega)$, we have

$$[F_\eta(x)F_\theta(x') - F_\eta(x')F_\theta(x)][G_\eta(y)G_\theta(y') - G_\eta(y')G_\theta(y)] \leq 0.$$

After some simple calculation we obtain $H(x,y)H(x',y') \leq H(x,y')H(x',y)$. Therefore the distribution function H is RR_2, and hence H is LND. □

Assadian et al. (2009).

1.9 Dependence DTP(m,n) and DRR(m,n)

5. The bivariate failure rate (Basu (1971)). The failure rate of a random vector (X,Y) having joint density $f(x,y)$ and distribution function $F(x,y)$ is given by

$$r(x,y) = \frac{f(x,y)}{\bar{F}(x,y)}.$$

Johnson and Kotz (1975) defined the hazard gradient as a vector $(r(x|Y>y), r(y|X>x))$ where $r(x|Y>y)$ is the hazard rate of the conditional distribution of X given $Y>y$. Similarly $r(y|X>x)$ is the hazard rate of the conditional distribution of Y given $X>x$.

6. Arnold and Zahedi (1988) defined The vector $(m(x|Y>y), m(y|X>x))$ where $m(y|X>x) = E[Y-y|X>x, Y>y]$ is the mean residual life function of Y with the additional information that $X>x$. In general $m(y|X \in A) =$

$E[Y-y|X\in A, Y>y]$. $m(x|Y\in A)$ is defined similarly.

7. Dependent by total positivity of order two.
Shaked (1977b) proposes some nested definitions of dependence. Let
$\psi_{m,n}(x,y)=\int_y^\infty\int_{y_{n-1}}^\infty...\int_{y_1}^\infty\int_x^\infty\int_{x_{m-1}}^\infty...\int_{x_1}^\infty f(x_\circ,y_\circ)dx_\circ dx_1...dx_{m-1}dy_\circ dy_1...dy_{n-1}$, for $m,n>0$, and for $m=0,n=0$ define $\psi_{0,0}(x,y)=f(x,y)$. For $m,n\geq 0$ the random vector (X,Y) , or its distribution function F , or its Survival function $\bar{F}$ is said to be *dependent by total positivity of order two with degree* (m,n) (denoted by DTP (m,n)) if $\psi_{m,n}(x,y)$ is TP_2 in x and y $(x,y\in R)$.

Remark 1.9.1. Let (X,Y) be a random vector with joint distribution function F and joint density function f and suppose that (X,Y) is absolutely continuous, based on definition and Proposition 3.3 of Shaked (1977b), the random vector (X,Y) is:
(i) DTP(0,0) or equivalently LRD, when the joint density $f(x,y)$ is TP_2 .
(ii) DTP(0,1), when $-\frac{\partial}{\partial x}\bar{F}(x,y)$ is TP_2 , similarly DTP(1,0) when $-\frac{\partial}{\partial y}\bar{F}(x,y)$ is TP_2.
(iii) DTP(1,1), when $\bar{F}(x,y)$ is TP_2 .
(iv) DTP(0,2) (DTP(2,0)), When the mean residual life function,

$$m(y|X=x)=E[Y-y|X=x,Y>y]\,(m(x|Y=y)=E[X-x|X>x,Y=y])$$

increasing in $y(x)$ for all $x(y)$
(v) DTP(1,2) (DTP(2,1)), when $m(y|X>x)$ $(m(x|Y>y))$ is increasing in x (y) for all y (x) .

Bivariate decreasing failure rate.

Brindley and Tompson (1972) proved that if X , Y are non-negative random variables with joint distribution function $F(x,y)$ then $F(x,y)$ is decreasing failure rate (DFR) if $\frac{\bar{F}(x+\Delta,y+\Delta)}{\bar{F}(x,y)}$ is increasing in x and y for each $\Delta>0$ and all x , $y\geq 0$, such that $\bar{F}(x,y)>0$.

9. Let X,Y be non-negative random variables, denote the conditional hazard function of X given $Y \in A$ by

$$R(x|Y \in A) = \int_0^x r(t|Y \in A)dt = -\log P(X > x|Y \in A).$$

1.10 Copula function and dependence

It is well known and easily verified that $F_1(X)$ and $F_2(Y)$, where F_1 and F_2 are the marginals distributions of X and Y respectively, are two uniform variables if F_1 and F_2 are continuous. Hence if the marginals F_1 and F_2 of the bivariate distribution F are continuous, there exists a unique copula, which is a cumulative distribution function, with its marginals being uniform. Formally a function $C : [0,1]^2 \rightarrow [0,1]$ such that

$$F(x, y) = C(F_1(x), F_2(y))$$

is a copula. on other hand, if $C(u_1, u_2)$ and continuous F_1 and F_2 are given, then there exists and F such that:

$$C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2)).$$

$F_i(t), i = 1, 2$ is continuous and non decreasing, but could be constant on some intervals. In that case, one defines a quasi-inverse by

$$F_i^{-1}(t) = \inf\{x : F_i(x) \geq t\}.$$

Using copulas allows us to separate the study of dependence from the study of the marginals, since one is then reduced to study of the relation between two uniform variables. The purpose of this section is to present results on copulas scattered in diverse literature with the emphasis on dependence concepts and properties.

Definition 1.10.1. A bivariate copulas is a function $C : [0,1]^2 \rightarrow [0,1]$ subject to

i) $C(x, 0) = C(0, y) = 0$, for all $x, y \in [0,1]$.

ii) $C(x, 1) = C(1, y) = y$, for all $x, y \in [0,1]$.

iii) C is joint-increasing i.e. for every 2-box $J = [x_1, x_2] \times [y_1, y_2] \in [0,1]^2$, the associated C-volume $V_C(J)$ satisfies

$$V_C(J) = C(x_2, y_2) + C(x_1, y_1) - C(x_1, y_2) - C(x_2, y_1) \geq 0.$$

Theorem 1.10.2. *(Sklor, 1959) Let F be a joint distribution function with marginal F_1 and F_2. Then, there exists a copula C subject to*

$$\forall x, y \in \bar{R}; \quad F(x,y) = C(F_1(x), F_2(y)). \tag{1.11}$$

If F_1 and F_2 are continuous, then C is unique. Otherwise, C is uniquely determined on Rain $F_1\times$ Rain F_2. Conversely, if C is copula and F_1 and F_2 are distribution functions, then the function F as defined in 1.11 is a joint distribution function with marginal F_1 and F_2.

Corollary 1.10.3. *Under the assumptions of Theorem 1, we have*

$$C(u,v) = F(F_1^{-1}(u), F_2^{-1}(v)),$$

where, $F^{-1}(u) = \sup[x : \ F(x) \leq u] = \inf[x : \ F(x) \geq u]$.

Example 1.10.4. 1) FGM: $F(x,y) = F_1F_2[1+\theta\bar{F}_1\bar{F}_2]$ then, $C(u,v) = uv[1+\theta(1-u)(1-v)]$

2) Gumbel, $\bar{F}(x,y) = \exp -(x+y+\theta xy),\ x \geq 0,\ y \geq 0,\ 0 \leq \theta \leq 1$. Then, we can show that

$$C_\theta(u,v) = u+v-1+(1-u)(1-v)\exp\{-\theta \ln(1-u)(1-v)\}$$

Corollary 1.10.5. *Under the assumptions of Theorem 1, we obtain that*

i) $f(x,y) = f_1f_2C(F_1,F_2)$,

ii) $c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$.

1.11 Some elementary properties

1) Continuity: The copulas C satisfies in Lipschitz's condition as the following

$$|C(u_2,v_2) - C(u_1,v_1)| \leq |u_2-u_1| + |v_2-v_1|.$$

2) Differentiability:

i) $0 \leq \frac{\partial C(u,v)}{\partial u} \leq 1$, $0 \leq \frac{\partial C(u,v)}{\partial v} \leq 1$,

ii) $c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$.

3) The survival function of a Copula:

$$\begin{aligned}\bar{C}(u,v) = P(U > u, V > v) &= 1 - P(U \leq u) - P(V \leq v) + P(U \leq u, V \leq v) \\ &= 1 - u - v - C(u,v).\end{aligned}$$

We have

$$\bar{C}(u,1) = \bar{C}(1,v) = 0 \;\; and \;\; \bar{C}(u,0) = \bar{C}(0,v) = 1.$$

3)

$\bar{F}(x,y) = \hat{C}(\bar{F}_1(x), \bar{F}_2(y))$ where

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v).$$

$$\begin{aligned}\bar{F}(x,y) &= 1 - F_1(x) - \bar{F}_2(y) + F(x,y) \\ &= \bar{F}_1(x) + \bar{F}_2(y) - 1 + C(1 - \bar{F}_1(x), 1 - \bar{F}_2(y)).\end{aligned}$$

So if we define, $\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$, we have

$$\bar{F}(x,y) = \hat{C}(\bar{F}_1(x), \bar{F}_2(y)).$$

Then,

$$\hat{C}(u,v) = \bar{F}(\bar{F_1}^{-1}(u), \bar{F_2}^{-1}(v)).$$

The $\hat{C}(u,v)$ is a copula and we refer to $\hat{C}$ as the survival Copula of X and Y.

Remark 1.11.1. If $\hat{C}(u,v) = 1 - u - v + C(u,v) = \hat{C}(1-u, 1-v)$ then

$$C(u,v) = u + v - 1 + \hat{C}(1-u, 1-v).$$

Example 1.11.2.

$$\bar{F}(x,y) = [1 + x + y + \theta xy]^{-a}, \;\; 0 \leq \theta \leq a + 10, \; a > 0, \; x, y \geq 0.$$

Assume that $u = \bar{F}_1(x) = (1+x)^{-a}$, $v = \bar{F}_2(y) = (1+y)^{-a}$. Then, we have

$$1 + x = u^{-1/a} \;\; and \;\; 1 + y = v^{-1/a}.$$

So,

$$x = u^{-1/a} - 1 \ \ and \ \ y = v^{-1/a} - 1.$$

Therefore,

$$\begin{aligned}
\hat{C}(u,v) &= \bar{F}(u^{-1/a} - 1, v^{-1/a} - 1) \\
&= [1 + u^{-1/a} - 1 + v^{-1/a} - 1 + \theta(1 - u^{-1/a})(1 - v^{-1/a})]^{-a} \\
&= [u^{-1/a} + v^{-1/a} - 1 + \theta(1 - u^{-1/a})(1 - v^{-1/a})]^{-a}, \quad 0 < u < 1, \quad 0 < v < 1.
\end{aligned}$$

Then,

$$C(u,v) = u+v-1+\{(1-u)^{-1/a}+(1-v)^{-1/a}-1+[1-(1-u)^{-1/a}][1-(1-v)^{-1/a}]\}^{-1/a}.$$

Theorem 1.11.3. *Let X and Y be continuous r.v.'s with copula C_{xy}. If α and β are strictly increasing on Rain X and Rain Y, respectively. Then, $C_{\alpha(X)}, \beta(Y) = C_{XY}$. Thus, C_{XY} is invariant under strictly increasing transformation of X and Y.*

Proof. Let $\alpha(X) \sim G_1$ and $\beta(Y) \sim G_2$. Then $G_1(x) = F_1(\alpha^{-1}(x))$ and $G_2(y) = F_2(\beta^{-1}(y))$. Since $\alpha(.)$ and $\beta(.)$ are strictly increasing, we have

$$\begin{aligned}
G_{\alpha(X),\beta(Y)}(t,s) &= P[\alpha(x) \leq t, \beta(Y) \leq s] = P[X \leq \alpha^{-1}(t), Y \leq \beta^{-1}(s)] \\
&= F(\alpha^{-1}(t), \beta^{-1}(s)) = C_{X,Y}[F_1(\alpha^{-1}(t)), F_2(\beta^{-1}(s))] \\
&= G_{X,Y}(G_1(t), G_2(s)). \qquad (1.12)
\end{aligned}$$

Also,

$$G_{\alpha(X),\beta(Y)}(t,s) = C_{\alpha(X),\beta(Y)}(G_1(t), G_2(s)). \qquad (1.13)$$

Then from (1.12) and (1.13)

$$C_{\alpha(X),\beta(Y)}(u,v) = G_{X,Y}(u,v), \ \forall (u,v) \in I^2. \qquad (1.14)$$

Since X and Y are continuous, hence $RanG_1 = RanG_2 = I = [0,1]$.

Theorem 1.11.4. *Let X and Y are continuous random variables with Copula $C_{X,Y}$. Let α and β be strictly monotone on Ran(x) and Ran(Y). Then i) If α is strictly increasing and β is strictly decreasing, then*

$$C_{\alpha(X),\beta(Y)}(u,v) = U - G_{X,Y}(u, 1-v).$$

ii) If $\alpha(.)$ is strictly decreasing and β is strictly increasing, then

$$C_{\alpha(U,V),\beta(Y)}(u,v) = V - G_{X,Y}(1-u, v).$$

iii) If α and β are both strictly decreasing, then

$$C_{\alpha(X),\beta(Y)}(u,v) = U + V - 1 - G_{X,Y}(1-u, 1-v).$$

1.12 Copula function and Dependence

1- We define $\pi = u.v$. Let X and Y are continuous random variables with joint distribution function F and with Copula function $C(u,v)$. Then

i) If $C(u,v) \geq \pi(u,v) = u.v \Rightarrow PQD(X,Y)$

ii) If $C(u,v) \leq \pi(u,v) = u.v \Rightarrow NQD(X,Y)$

Example 1.12.1. In FGM family, we have

i) $C_\theta(u,v) = uv[1 + \theta(1-u)(1-v)]$

ii) C_θ is PQD if $\theta \geq 0$

iii) C_θ is NQD if $\theta \leq 0$

Exercise: Prove that

i) $\rho_C = 12\int_{I^2}$\{C(u,v)-uv\}dudv.

ii) $\tau_C = 4\int_{I^2}$C(u,v)dC(u,v)-1.

Theorem 1.12.2. *i) If X and Y are PQD random variables, then $3\tau \geq \rho \geq 0$.*

ii) If X and Y are NQD random variables, then $3\tau \leq \rho \leq 0$.

Proof. In FGM family, we have

$$C_\theta(u,v) = uv + \theta uv(1-u)(1-v), \quad 0 \leq u \leq 1,\ 0 \leq v \leq 1.$$

Then

$$\begin{aligned}\rho_C &= 12\int_0^1\int_0^1 \{C(u,v)-uv\}dudv\\ &= 12\int_0^1\int_0^1 \theta uv(1-u)(1-v)dudv\\ &= 12\theta\int_0^1\{\int_0^1(u-u^2)du\}v(1-v)dv\\ &= 12\theta[\frac{1}{2}-\frac{1}{3}][\frac{1}{2}-\frac{1}{3}]=\frac{\theta}{3}.\end{aligned}$$

Since $dC_\theta(u,v)=[1+\theta(2u-1)(2v-1)dudv]$, we obtain that

$$\begin{aligned}\int_0^1\int_0^1 C_\theta(u,v)dC_\theta(u,v) &= \int_0^1\int_0^1\{uv+\theta uv(1-u)(1-v)\}\{1+\theta(2u-1)(2v-1)\}dudv\\ &= \frac{1}{4}+\frac{\theta}{18}.\end{aligned}$$

So,

$$\begin{aligned}\tau_C &= 4\int_0^1\int_0^1 C_\theta(u,v)dC_\theta(u,v)-1\\ &= \frac{2\theta}{9}.\end{aligned}$$

Then

$$\begin{aligned}3\tau_C-\rho_C &= \frac{2\theta}{3}-\frac{\theta}{3}\\ &= \frac{\theta}{3}.\end{aligned}$$

Therefore, if $0\le\theta\le 1$ then X and Y are PQD random variables and so $3\tau_C-\rho_C\ge 0$. Also, if $-1\le\theta\le 0$ then X and Y are NQD random variables and so $3\tau_C-\rho_C\le 0$.

Theorem 1.12.3. *Let X and Y are continuous random variables with copula function C. Then*

i) LTD$(Y|X)$ $\Leftrightarrow$ $\forall v\in[0,1]$: $\frac{C(u,v)}{u}$ $\searrow$ in u.
ii) LTD$(X|Y)$ $\Leftrightarrow$ $\forall u\in[0,1]$: $\frac{C(u,v)}{v}$ $\searrow$ in v.
iii) RTI$(Y|X)$ $\Leftrightarrow$ $\frac{v-C(u,v)}{1-u}$ $\searrow$ in u, $\forall v\in[0,1]$.
iv) RTI$(X|Y)$ $\Leftrightarrow$ $\frac{u-C(u,v)}{1-v}$ $\searrow$ in v, $\forall u\in[0,1]$.

Proof.

LTD$(Y|X) \Leftrightarrow \frac{F(x,y)}{F_1(x)} \searrow$ in x, $\forall y \Leftrightarrow \frac{C(u,v)}{u} \searrow$ in u, $\forall v$.

$\{U = F_1(X),\ V = F_2(Y)\}= \frac{C(F_1(x),F_2(y))}{F_1(x)}$

$$
\begin{aligned}
RTI(Y|X) \Leftrightarrow \frac{\bar{F}(x,y)}{\bar{F}_1(x)} &= \frac{\hat{C}(\bar{F}_1(x),\bar{F}_2(y))}{\bar{F}_1(x)} \\
&= \frac{\hat{C}(1-u,1-v)}{1-u} \\
&= \frac{1-u-v+C(u,v)}{1-u} \\
&= 1-\frac{v-C(u,v)}{1-u}.
\end{aligned}
$$

So, we conclude

$\frac{\bar{F}(x,y)}{\bar{F}_1(x)} \nearrow$ in x, $\forall y \Leftrightarrow \frac{v-C(u,v)}{1-u} \searrow$ in u, $\forall v \in [0,1]$.

Corollary 1.12.4. *i) LTD*$(Y|X) \Leftrightarrow \frac{\partial C(u,v)}{\partial u} \leq \frac{C(u,v)}{u}$, *a.s.* $\forall u$.

ii) LTD$(X|Y) \Leftrightarrow \frac{\partial C(u,v)}{\partial v} \leq \frac{C(u,v)}{v}$, *a.s.* $\forall v$.

iii) RTI$(Y|X) \Leftrightarrow \frac{\partial C(u,v)}{\partial u} \leq \frac{v-C(u,v)}{1-u}$, *a.s.* $\forall u$.

iv) RTI$(X|Y) \Leftrightarrow \frac{\partial C(u,v)}{\partial v} \geq \frac{u-C(u,v)}{1-v}$, *a.s.* $\forall v$.

Proof. i) If $\frac{C(u,v)}{u} \searrow$ in u, then

$$
\frac{\partial}{\partial u}[\frac{C(u,v)}{u}] = \frac{u\frac{\partial}{\partial u}C(u,v) - C(u,v)}{u^2} \leq 0 \Leftrightarrow \frac{\partial C(u,v)}{\partial u} \leq \frac{C(u,v)}{u}.
$$

The other parts are prove as the same method.

Theorem 1.12.5. *Under the assumptions of Theorem 1.12.3, we have*

i) SI$(Y|X) \Leftrightarrow \forall v \in [0,1]$, *for almost all u:* $\frac{\partial C(u,v)}{\partial u} \searrow$ *in u.*

ii) SI$(X|Y) \Leftrightarrow \forall u \in [0,1]$, *for almost all v:* $\frac{\partial C(u,v)}{\partial v} \searrow$ *in v.*

Proof. i)

SI$(Y|X) \Leftrightarrow P[y \leq y|X = x] \searrow$ in x, $\forall y$.

And

$$\begin{aligned} P[y \leq y|X = x] &= \int_{-\infty}^{y} f(t|x)dt \\ &= \frac{1}{f_1(x)} \int_{-\infty}^{y} f(t,x)dt \\ &= \frac{1}{f_1(x)} \frac{\partial F(x,y)}{\partial x} \\ &= \frac{1}{f_1(x)} \frac{\partial F(x,y)}{\partial F_1(x)} \frac{\partial F_1(x)}{\partial x} \\ &= \frac{\partial C(u,v)}{\partial u} \searrow in \quad u, \quad \forall v \in [0,1]. \end{aligned}$$

Corollary 1.12.6. *Under the assumptions of Theorem 1.12.5, we have i) SI*$(Y|X)$ $\Leftrightarrow \forall v \in [0,1]$, $C(u,v)$ *is concave function with respect to* u.

ii) SI$(X|Y) \Leftrightarrow \forall u \in [0,1]$, $C(u,v)$ *is concave function with respect to* v.

Theorem 1.12.7. *We have the similar results for* ND *case:*

i) LTI$(Y|X) \Leftrightarrow \frac{C(u,v)}{u} \nearrow$ *in* u.

ii) RTD$(Y|X) \Leftrightarrow \frac{v-C(u,v)}{1-u} \nearrow$ *in* u.

iii) LTI$(Y|X) \Leftrightarrow \frac{\partial C(u,v)}{\partial u} \geq \frac{C(u,v)}{u}$.

iv) RTD$(Y|X) \Leftrightarrow \frac{\partial C(u,v)}{\partial u} \leq \frac{v-C(u,v)}{1-u}$.

v) LSD$(Y|X) \Leftrightarrow C(u,v)$ *is a convex function with respect to* u.

vi) RTI$(Y|X) \Leftrightarrow C(u,v)$ *is a convex function with respect to* u.

vii) SD$(Y|X) \Leftrightarrow \frac{\partial C(u,v)}{\partial u} \nearrow$ *in* u, $\forall v \in [0,1]$.

iix) SD$(Y|X) \Leftrightarrow C(u,v)$ *is a convex function with respect to* u.

Example 1.12.8. Let $C(u,v) = uv[1+\theta(1-u)^a(1-v)^a], \quad 0 \leq u \leq 1, \ 0 \leq v \leq 1$. Then

$$C_\theta(u,v) = uv[1+2\theta(1-u)(1-v)(1+u+v-2uv)]. \quad (Hutchinson-Lai, 1990)$$

So, we obtain that

i)

$$\begin{aligned}
\frac{C_\theta(u,v)}{u} &= v + 2\theta v(1-v)(1-u)[1 + u(1-v) + v(1-u)] \\
&= v + 2\theta v(1-v)[(1-u) + (1-v)u(1-u) + v(1-u)^2] \\
&= v + 2\theta v(1-v)(1-u) + 2\theta v(1-v)^2 u(1-u) + 2\theta v^2(1-v)(1-u)^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial}{\partial u}[\frac{C_\theta(u,v)}{u}] &= -2\theta v(1-v) + 2\theta v(1-v)^2 - 4\theta v^2(1-u)(1-v) \\
&= -2\theta v(1-v)[1 - (1-v)(1-2u) + 2v(1-v)] \\
&= -2\theta v(1-v)[v + 2u - 2uv + 2v - 2uv] \\
&= -2\theta v(1-v)[v + 2u(1-v) + 2v(1-u)].
\end{aligned}$$

Then if $-\frac{1}{4} \leq \theta \leq 0$, we have $\frac{\partial}{\partial u}[\frac{C_\theta(u,v)}{u}] \geq 0$. On the other hand if $0 \leq \theta \leq \frac{1}{4}$, we get $\frac{\partial}{\partial u}[\frac{C_\theta(u,v)}{u}] \leq 0$.

So, LTD$(Y|X) \Leftrightarrow 0 \leq \theta \leq \frac{1}{4}$ and LTI$(Y|X) \Leftrightarrow -\frac{1}{4} \leq \theta \leq 0$.

ii) RTI$(Y|X) \Leftrightarrow \frac{v - C(u,v)}{1-u} \searrow$ in u. And

$$\begin{aligned}
\frac{v - C(u,v)}{1-u} &= v - 2\theta uv(1-v)[1 + u + v - 2uv] \\
&= v - 2\theta uv(1-v) - 2\theta u^2 v(1-v)^2 - 2\theta uv^2(1-v)(1-u).
\end{aligned}$$

So, we obtain that

$$\begin{aligned}
\frac{\partial}{\partial u}[\frac{v - C(u,v)}{1-u}] &= -2\theta v(1-v) - 4\theta uv(1-v) - 2\theta v^2(1-v)(1-2u) \\
&= -2\theta v(1-v)[1 + 2u + v(1-2u)] \\
&= -2\theta v(1-v)[1 + v + 2u(1-v)].
\end{aligned}$$

So, RTI$(Y|X) \Leftrightarrow 0 \leq \theta \leq \frac{1}{4}$ and RTD$(Y|X) \Leftrightarrow -\frac{1}{4} \leq \theta \leq 0$.

iii) LCSD$(Y|X) \Leftrightarrow C$ is $TD \Leftrightarrow \frac{\partial^2 \ln C}{\partial u \partial v} \geq 0$.

Theorem 1.12.9. *Let X and Y be continuous random variables with copula function C. Then*

i) LSCD$(X,Y) \Leftrightarrow C$ is TP_2.

ii) RCSI$(X,Y) \Leftrightarrow \hat{C}$ *is* TP_2.

iii) LCSI$(X,Y) \Leftrightarrow C$ *is* RR_2.

iv) RCSD$(X,Y) \Leftrightarrow \hat{C}$ *is* RR_2.

Lemma 1.12.10. *i)* f *is* $TP_2 \Leftrightarrow \frac{\partial^2 \ln f}{\partial x \partial y} \geq 0$.

ii) f *is* $RR_2 \Leftrightarrow \frac{\partial^2 \ln f}{\partial x \partial y} \leq 0$.

Proof. Let (X,Y) has a joint distribution function F with Copula function C. Then

$$
\begin{aligned}
i) LCSD(X,Y) &\Leftrightarrow F \; is \; TP_2 \\
&\Leftrightarrow F(x,y)F(\acute{x},\acute{y}) \geq F(x,\acute{y})F(\acute{x},y) \; for \; all \; x < \acute{x} and y < \acute{y} \\
&\Leftrightarrow C(F_1(x),F_2(y)).C(F_1(\acute{x}),F_2(\acute{y})) \geq C(F_1(x),F_2(\acute{y})).C(F_1(\acute{x}),F_2(y)) \\
&\Leftrightarrow C(u,v).C(\acute{u}),C(\acute{v}) \geq C(u,\acute{v}).C(\acute{u},v) \\
&\Leftrightarrow C(u,v) \; is \; TP_2.
\end{aligned}
$$

$$
\begin{aligned}
ii) RCSD(X,Y) &\Leftrightarrow \bar{F} \; is \; RR_2 \\
&\Leftrightarrow \; for \; all \; x < \acute{x} and y < \acute{y} \\
&\Leftrightarrow \bar{F}(x,y).\bar{F}(\acute{x},\acute{y}) \leq \bar{F}(\acute{x},y).\bar{F}(x,\acute{y}) \\
&\Leftrightarrow \hat{C}(\bar{F}_1(x),\bar{F}_2(y)).\hat{C}(\bar{F}_1(\acute{u}),\bar{F}_2(\acute{v}) \leq \hat{C}(\bar{F}_1(\acute{x},\bar{F}_2(y)).\hat{C}(\bar{F}_1(x),\bar{F}_2(\acute{y})) \\
&\Leftrightarrow \hat{C}(u,v).\hat{C}(\acute{u},\acute{v}) \leq \hat{C}(\acute{u},v).\hat{C}(u,\acute{v}) \\
&\Leftrightarrow \hat{C}(u,v) \; is \; RR_2.
\end{aligned}
$$

Corollary 1.12.11. *Under the assumptions of Theorem 1.12.12, we have*

i) LSCD$(X,Y) \Leftrightarrow \frac{\partial^2 C(u,v)}{\partial u \partial v} \geq 0$ *and RCSI*$(X,Y) \Leftrightarrow \frac{\partial^2 \hat{C}(u,v)}{\partial u \partial v} \geq 0$.

ii) LCSI$(X,Y) \Leftrightarrow \frac{\partial^2 C(u,v)}{\partial u \partial v} \leq 0$ *and RCSI*$(X,Y) \Leftrightarrow \frac{\partial^2 \hat{C}(u,v)}{\partial u \partial v} \leq 0$.

Theorem 1.12.12. *Let* X *and* Y *be continuous random variables with joint density function* F *and Copula function* C*. Then*

i) PLRD$(X,Y) \Leftrightarrow \frac{C(v,\acute{u})}{C(v,u)} \nearrow$ *in* v, $\forall u < \acute{u}$.

ii) NLRD$(X,Y) \Leftrightarrow \frac{C(v,\acute{u})}{C(v,u)} \searrow$ *in* v, $\forall u < \acute{u}$.

Proof.

$$\begin{aligned} i)PLRD(X,Y) &\Leftrightarrow f(x,y)f(\acute{x},\acute{y}) \geq f(x,\acute{y})f(\acute{x},y) \ for \ all \ x<\acute{x} and y<\acute{y} \\ &\Leftrightarrow C(u,v).C(\acute{u},\acute{v}) \geq C(\acute{u},v).C(u,\acute{v}) \\ &\Leftrightarrow \frac{C(\acute{u}\acute{v})}{C(u,\acute{v})} \geq \frac{C(\acute{u},v)}{C(u,v)} \\ &\Leftrightarrow \frac{C(\acute{u},v)}{C(u,v)} \nearrow \ in \ v, \ \forall u<\acute{u}, \end{aligned}$$

where $u = F_1(x)$, $\acute{u} = F_2(\acute{x})$, $v = F_1(y)$ and $\acute{v} = F_2(\acute{y})$ and

$$f(x,y) = f_1(x)f_2(y).\frac{\partial^2 C(F_1(x),F_2(y))}{\partial F_1(x)\partial F_2(y)} = f_1f_2C(u,v),$$

where $C(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$.

$$\begin{aligned} ii)NLRD(X,Y) &\Leftrightarrow f(x,y)f(\acute{x},\acute{y}) \leq f(x,\acute{y})f(\acute{x},y) \ for \ all \ x<\acute{x} and y<\acute{y} \\ &\Leftrightarrow C(u,v).C(\acute{u},\acute{v}) \leq C(\acute{u},v).C(u,\acute{v}) \\ &\Leftrightarrow \frac{C(\acute{u}\acute{v})}{C(u,\acute{v})} \leq \frac{C(\acute{u},v)}{C(u,v)} \\ &\Leftrightarrow \frac{C(\acute{u},v)}{C(u,v)} \searrow \ in \ v, \ \forall u<\acute{u}. \end{aligned}$$

1.13 Archimedean Copulas

Definition 1.13.1. Let ϕ be a continuous *** decreasing function from $I \to [0,+\infty]$ such that $\phi(1) = 0$. The Pseudo-inverse of ϕ is the function $\phi^{[-1]}$ with $Dom\phi^{[-1]} = [0,+\infty]$, $R\phi^{[-1]} = I$ given by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) & , \quad 0 \leq t \leq \phi(0) \\ 0 & , \quad t \geq \phi(0) \end{cases} \tag{1.15}$$

Note that $\phi^{[-1]}$ is continuous and non-increasing on $[0,+\infty]$ and strictly decreasing on $[0,\phi(0)]$. Furthermore, $\phi^{[-1]}(\phi(u)) = u$ on I. So

$$\phi(\phi^{-1}(t)) \quad = \quad \begin{cases} t & , \quad 0 \leq t \leq \phi(0) \\ \phi(0) & , \quad \phi(0) \leq t \leq \infty \end{cases} = \min\{t,\phi(0)\}$$

Finally if $\phi(0)=+\infty$, then $\phi^{[-1]}=\phi^{-1}$.

Lemma 1.13.2. *Let ϕ be a continuous, strictly decreasing function $\phi: \ I=[0,1]\rightarrow[0,+\infty]$ subject to $\phi(1)=0$ and $\phi^{[-1]}$ be the Pseudo-Inverse of ϕ defined above. Let $C: \ I^2\rightarrow I$ given by*

$$C(u,v)=\phi^{[-1]}(\phi(u)+\phi(v)).$$

Then if and only if C is a Copula function, ϕ is convex.

In this case ϕ is to be said generator of the Copula C and C is to be said Archimedean Copula.

$C(u,v)=\phi^{-1}[\phi(u)+\phi(v)]\Rightarrow\phi[C(u,v)]=\phi(u)+\phi(v)$. Then by differentiating relative to u (v) of both side of the last equation, we have

$$\begin{cases}\phi'(C)\frac{\partial C}{\partial u}=\phi'(u)\\ \phi'(C)\frac{\partial C}{\partial v}=\phi'(v)\end{cases}\tag{1.16}$$

So,

$$\begin{cases}\frac{\partial C}{\partial u}=\frac{\phi'(u)}{\phi'(C)}\\ \frac{\partial C}{\partial v}=\frac{\phi'(v)}{\phi'(C)}\end{cases}\tag{1.17}$$

Therefore,

$$\frac{\partial^2 C}{\partial u\partial v}=-\frac{\phi''(C)\phi'(u)\phi'(v)}{\phi'^2(C)}.$$

Since $C(u,v)\geq 0$, then if and only if ϕ is convex and decreasing function, ϕ^{-1} is convex and decreasing function.

Remark 1.13.3. If $\Lambda(\theta)$ is a distribution function with $\Lambda(0)=0$ and $\Psi(t)=\int_0^{+\infty}$, then $\phi=\Psi^{-1}$. Also, if Ψ is the Laplace transform of a distribution, then

$$C(u,v)=\Psi[\Psi^{-1}(u)+\Psi^{-1}(v)],$$

where $F(u)=\exp\{-\Psi^{-1}(u)\}$ and $G(v)=\exp\{-\Psi^{-1}(v)\}$.

Let M be a univariate distribution function of a positive random variables ($M(0)=0$) and Ψ be the Laplace transform of M as:

$$\Psi(s)=Ee^{-s\theta}=\int_0^{+\infty}e^{-s\theta}dM(\theta). \quad s\geq 0$$

Then

i) *(Joe, 1997)* For an arbitrary univariate distribution function F, there exists a unique distribution G subject to

$$F(x) = \int_0^{+\infty} G^\theta(x) dM(\theta) = \Psi[-\log G(x)].$$

So, $G(t) = \exp[-\Psi^{-1}(F(t))]$.

There is a similar relationship for survival functions as

$$\bar{F}(x) = \int_0^{+\infty} \bar{H}^\theta(x) dM(\theta) = \Psi[-\log \bar{H}(x)],$$

where $\bar{H} = \exp[-\Psi^{-1}(\bar{F}(t))]$.

Next, consider the bivariate class $\mathcal{F}(F_1, F_2)$ for all $j = 1, 2$, let $G_j = \exp[-\Psi^{-1}(F_j)]$. Then the following term is a distribution function in $\mathcal{F}(F_1, F_2)$:

$$\begin{aligned} \int_0^{+\infty} G_1^\theta G_2^\theta dM(\theta) &= \Psi[-\log G_1 - \log G_2] \\ &= \Psi[\Psi^{-1}(F_1) + \Psi^{-1}(F_2)]. \end{aligned}$$

The Copula distribution function for F_1 and F_2 is

$$C(u, v) = \Psi[\Psi^{-1}(u) + \Psi^{-1}(v)],$$

where $\Psi^{-1} = \phi$. Therefore, $\phi = \Psi^{-1}$ is a generator of C.

Similarly, one could work with survival functions to get

$$\begin{aligned} \int_0^{+\infty} \bar{H}_1^\theta \bar{H}_2^\theta dM(\theta) &= \Psi[-\log \bar{H}_1 - \log \bar{H}_2] \\ &= \Psi[\Psi^{-1}(\bar{F}_1) + \Psi^{-1}(\bar{F}_2)], \end{aligned}$$

where $\bar{H}_j = \exp[-\Psi^{-1}(\bar{F}_j)], \quad j = 1, 2$.

Example 1.13.4. Let $M \sim \Gamma(\alpha = \frac{1}{\theta}, \beta = 1)$. For $v > 0$, we have

$$\begin{aligned} s = \Psi(t) &= (\frac{1}{\lambda + t})^{\frac{1}{\theta}} \\ &= (1 + t)^{-\frac{1}{\theta}}. \end{aligned}$$

Then, $s^{-\theta} = 1 + t$ and $\phi(t) = \Psi^{-1}(t) = t^{-\theta} - 1$. So

$$\begin{aligned} C_\phi(u,v) = \phi^{-1}[\phi(u)+\phi(v)] &= \phi^{-1}[u^{-\theta} - 1 + v^{-\theta} - 1] \\ &= [u^{-\theta} + v^{-\theta} - 1]^{-\frac{1}{\theta}} \ is\ a\ Copula\ function. \end{aligned}$$

Example 1.13.5. If $M(\theta) \sim EXP(\theta)$. Then $\Psi(t) = \frac{1}{\theta+t}$. So with $s = \frac{1}{\theta+t}$, we have $\Psi^{-1} = \frac{1}{t} - \theta$. Therefore

$$\begin{aligned} C_\phi(u,v) = \phi^{-1}[\phi(u)+\phi(v)] &= \phi^{-1}[\frac{1}{u} + \frac{1}{v} - 2\theta] \\ &= [\theta + \frac{1}{u} + \frac{1}{v} - \theta]^{-1} \\ &= [\frac{1}{u} + \frac{1}{v} - \theta]^{-1} = \frac{uv}{u+v-\theta uv}. \end{aligned}$$

Theorem 1.13.6. *A Copula C is Archimedean, if and only if there exists a mapping $f:\ (0,1) \to (0,+\infty)$ subject to*

$$\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{f(u)}{f(v)}.\ \forall\ 0 < u, v < 1$$

Then $\phi(t) = \int_t^1 f(u)du$.

Example 1.13.7. i) In Lomax distribution with joint distribution function

$$\bar{F}(x,y) = [1 + x + y + \theta xy]^{-a}.\ \ 0 \le \theta \le a+1.$$

Then

$$C(u,v) = [(1-u)^{-1/a} + (1-v)^{-1/a} - 1 + \theta[1-(1-u)^{-1/a}][1-(1-v)^{-1/a}]]^{-a} + u + v - 1.$$

ii) $C_\theta(u,v) = u+v-1+(1-u)(1-v)e^{-\theta\log(1-u)\log(1-v)}$, $\ 0 \le \theta \le 1$ and we have

$$\frac{\partial C}{\partial u} = 1-(1-v)e^{-\theta\log(1-u)\log(1-v)} + \frac{\theta\log(1-v)}{1-u}(1-u)(1-v)e^{-\theta\log(1-u)\log(1-v)},$$

and

$$\frac{\partial C}{\partial v} = 1-(1-u)e^{-\theta\log(1-u)\log(1-v)} + \frac{\theta\log(1-v)}{1-u}(1-u)(1-v)e^{-\theta\log(1-u)\log(1-v)}$$

.

1.14 Empirical copula function

In this section we will show that there are expressions for the sample versions of several measures of association analogous to those whose population versions were seen. The population versions can be expressed in terms of copulasthe sample versions will now be expressed in terms of empirical copulas and the corresponding empirical copula frequency function.

Definition 1. Let $\{(x_k, y_k)\}_{k=1}^{n}$ denote a sample of size n from a continuous bivariate distribution. The empirical copula is the function C_n given by

$$C_n\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{number\ of\ pairs\ (x,y)\ in\ the\ sample\ with\ x \leq x_{(i)}, y \leq y_{(j)}}{n}.$$

where $x_{(i)}$ and $y_{(j)}$, $1 \leq i, j \leq n$, denote order statistics from the sample. The empirical copula frequency c_n is given by

$$c_n\left(\frac{i}{n}, \frac{j}{n}\right) = \begin{cases} 1/n, & if\ (x_{(i)}, y_{(j)})\ is\ an\ element\ of\ the\ sample \\ 0, & otherwise \end{cases}$$

Note that C_n and c_n are related via

$$C_n\left(\frac{i}{n}, \frac{j}{n}\right) = \sum_{p=1}^{i}\sum_{q=1}^{j} c_n\left(\frac{p}{n}, \frac{q}{n}\right)$$

and

$$c_n\left(\frac{i}{n}, \frac{j}{n}\right) = C_n\left(\frac{i}{n}, \frac{j}{n}\right) - C_n\left(\frac{i-1}{n}, \frac{j}{n}\right) - C_n\left(\frac{i}{n}, \frac{j-1}{n}\right) + C_n\left(\frac{i-1}{n}, \frac{j-1}{n}\right).$$

Empirical copulas were introduced and first studied by Deheuvels (1979), who called them empirical dependence functions.

For continuous random variables X and Y with copula C, recall the population versions of Spearman's ρ, Kendall's τ, and Gini's γ, respectively,

$$\rho = 12 \int\int_{I^2} [C(u,v) - uv] du dv,$$

$$\tau = 2 \int_0^1 \int_0^1 \int_0^{v'} \int_0^{u'} \left[c(u,v)c(u',v') - c(u,v')c(u',v)\right] du dv du' dv',$$

and

$$\gamma = 4\left\{\int_0^1 C(u,1-u)du - \int_0^1 [u - C(u,u)]du\right\}.$$

In the next theorem, we present the corresponding version for a sample (we use Latin letters for the sample statistics):

Theorem 1 Let C_n and c_n denote, respectively, the empirical copula and the empirical copula frequency function for the sample $\{(x_k, y_k)\}_{k=1}^n$. If r, t and g denote, respectively, the sample versions of Spearman's rho, Kendall's tau, and Gini's gamma, then

$$r = \frac{12}{n^2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[C_n\left(\frac{i}{n},\frac{j}{n}\right) - \frac{i}{n}.\frac{j}{n}\right], \tag{1.18}$$

$$t = \frac{2n}{n-1}\sum_{i=2}^{n}\sum_{j=2}^{n}\sum_{p=1}^{i-1}\sum_{q=1}^{j-1}\left[c_n\left(\frac{i}{n},\frac{j}{n}\right)c_n\left(\frac{p}{n},\frac{q}{n}\right) - c_n\left(\frac{i}{n},\frac{q}{n}\right)c_n\left(\frac{p}{n},\frac{j}{n}\right)\right], \tag{1.19}$$

$$g = \frac{2n}{[n^2/2]}\left\{\sum_{i=1}^{n-1}C_n\left(\frac{i}{n},1-\frac{i}{n}\right) - \sum_{i=1}^{n}\left[\frac{i}{n} - C_n\left(\frac{i}{n},\frac{i}{n}\right)\right]\right\}. \tag{1.20}$$

Proof. We will show that the above expressions are equivalent to the expressions for r, t, and g that are usually encountered in the literature. The usual expression for r is (Kruskal 1958; Lehmann 1975)

$$r = \frac{12}{n(n^2-1)}\left[\sum_{k=1}^{n}kR_k - \frac{n(n+1)^2}{4}\right], \tag{1.21}$$

where $R_k = m$, whenever $(x_{(k)}, y_{(m)})$ is an element of the sample. Since

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{i}{n}\frac{j}{n}\right) = \frac{(n+1)^2}{4}$$

to show that (1) is equivalent to (4), we need only show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}C_n\left(\frac{i}{n},\frac{j}{n}\right) = \frac{1}{n}\sum_{k=1}^{n}kR_k \tag{1.22}$$

Observe that a particular pair $(x_{(k)}, y_{(m)})$ in the sample contributes 1/n to the double sum in (5) for each pair of subscripts (i,j) with $i \geq k$ and $j \geq m$. That is, the total contribution to the double sum in (5) by a particular pair $(x_{(k)}, y_{(m)})$ is 1/n times $(n-k+1)(n-m+1)$, the total number of pairs (i,j) such that $i \geq k$ and $j \geq m$. Hence, writing R_k for m and summing on k, we have ,

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} C_n\left(\frac{i}{n},\frac{j}{n}\right) &= \frac{1}{n}\sum_{k=1}^{n}(n-k+1)(n-R_k+1) \\
&= \frac{1}{n}\sum_{k=1}^{n}\left\{(n+1)^2 + kR_k - k(n+1) - R_k(n+1)\right\} \\
&= (n+1)^2 + \frac{1}{n}\sum_{k=1}^{n} kR_k - \frac{2(n+1)}{n}\sum_{k=1}^{n} k \\
&= \frac{1}{n}\sum_{k=1}^{n} kR_k
\end{aligned}$$

Next we show that (2) is equivalent to definition of Kendall's tau, i.e., the difference between number of concordant and discordant pairs in the sample divided by the total number $\binom{n}{2}$ of pairs of elements from the sample. Note that the summand in (2) reduces to $(1/n)^2$ whenever the sample contains both $(x_{(p)}, y_{(q)})$ and $(x_{(i)}, y_{(j)})$, a concordant pair because $x_{(p)} < x_{(i)}$ and $y_{(q)} < y_{(j)}$; reduces to $-(1/n)^2$ whenever the sample contains both $(x_{(p)}, y_{(j)})$ and $(x_{(i)}, y_{(q)})$, a discordant pair; and is 0 otherwise. Thus the quadruple sum in (2) is $(1/n)^2$ times the difference between the number of concordant and discordant pairs, which is equivalent to definition of Kendall's tau. Evaluating the inner double summation in (2) yields

$$\begin{aligned}
t &= \frac{2n}{n-1}\sum_{i=2}^{n}\sum_{j=2}^{n}\left[C_n\left(\frac{i}{n},\frac{j}{n}\right)C_n\left(\frac{i-1}{n},\frac{j-1}{n}\right) - C_n\left(\frac{i}{n},\frac{j-1}{n}\right)C_n\left(\frac{i-1}{n},\frac{j}{n}\right)\right] \\
&= \frac{2n}{n-1}\sum_{i=2}^{n}\sum_{j=2}^{n}\sum_{p=1}^{i-1}\sum_{q=1}^{j-1}\left[c_n\left(\frac{i}{n},\frac{j}{n}\right)c_n\left(\frac{p}{n},\frac{q}{n}\right) - c_n\left(\frac{i}{n},\frac{q}{n}\right)c_n\left(\frac{p}{n},\frac{j}{n}\right)\right].
\end{aligned}$$

If p_i and q_i denote the ranks of x_i and y_i,$i = 1, ..., n$, respectively, then(Gini 1910)

$$g = \frac{1}{[n^2/2]}\left\{\sum_{i=1}^{n}|p_i + q_i - n - 1| - \sum_{i=1}^{n}|p_i - q_i|\right\} \tag{1.23}$$

hence to show that (3) is equivalent to the sample version of Gini's gamma in (6), we need only show that

$$\sum_{i=1}^{n} |p_i - q_i| = 2n \sum_{i=1}^{n} \left[\frac{i}{n} - C_n\left(\frac{i}{n}, \frac{i}{n}\right) \right] \tag{1.24}$$

$$\sum_{i=1}^{n} |p_i + q_i - n - 1| = 2n \sum_{i=1}^{n-1} C_n\left(\frac{i}{n}, \frac{n-i}{n}\right). \tag{1.25}$$

We give only the proof for (7); the proof for (8) is similar. The sample $\{(x_k, y_k)\}_{k=1}^{n}$ can be written $\{(x_{p_i}, y_{q_i})\}_{i=1}^{n}$. Because $nC_n(i/n, i/n)$ is the number of points (x_{p_i}, y_{q_i}) in the sample for which $p_i \le i$ and $q_i \le i$, the sample point (x_{p_i}, y_{q_i}) is counted $(n - max(p_i, q_i) + 1)$ times in the sum $n \sum_{i=1}^{n} C_n(i/n, i/n)$. Thus

$$\begin{aligned}
2n \sum_{i=1}^{n} \left[\frac{i}{n} - C_n\left(\frac{i}{n}, \frac{i}{n}\right) \right] &= n(n+1) - 2\sum_{i=1}^{n} [(n+1) - max(p_i, q_i)] \\
&= 2\sum_{i=1}^{n} max(p_i, q_i) - n(n+1) \\
&= 2\sum_{i=1}^{n} max(p_i, q_i) - \frac{n(n+1)}{2} - \frac{n(n+1)}{2} \\
&= 2\sum_{i=1}^{n} max(p_i, q_i) - \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} q_i \\
&= \sum_{i=1}^{n} \left[2max(p_i, q_i) - (p_i + q_i)\right].
\end{aligned}$$

But $2max(u, v) - (u + v) = |u - v|$, and hence

$$2n \sum_{i=1}^{n} \left[\frac{i}{n} - C_n\left(\frac{i}{n}, \frac{i}{n}\right) \right] = \sum_{i=1}^{n} |p_i - q_i|. \quad \square$$

1.15 Particular cases and relationships

Implications 1:(among concepts of Positive Dependence)

$$SI(Y|X) \Rightarrow LTD(Y|X), RTI(Y|X) \Rightarrow PA(X, Y) \Rightarrow PQD(X, Y)$$

$$PLRD(X,Y) \Rightarrow SI(Y|X) \Rightarrow RTI(Y|X) \Leftarrow RCSI(X,Y) \Leftarrow PLRD(X,Y)$$

$$SI(Y|X) \Rightarrow LTD(Y|X) \Rightarrow PQD(X,Y) \Leftarrow RTI(Y|X)$$

$$SI(X|Y) \Rightarrow LTD(X|Y) \Rightarrow PQD(X,Y) \Leftarrow RTI(X|Y) \Leftarrow SI(X|Y)$$

$$PLRD(X,Y) \Rightarrow LCSD(X,Y) \Rightarrow LTD(X|Y) \Leftarrow SI(X|Y) \Leftarrow PLRD(X,Y)$$

Implications 2:(among concepts of Negative Dependence)

$$SD(Y|X) \Rightarrow LTI(Y|X), RTD(Y|X) \Rightarrow NA(X,Y) \Leftrightarrow NQD(X,Y)$$

$$NLRD(X,Y) \Rightarrow SD(Y|X) \Rightarrow RTD(Y|X) \Leftarrow RCSD(X,Y) \Leftarrow NLRD(X,Y)$$

$$SD(Y|X) \Rightarrow LTI(Y|X) \Rightarrow NQD(X,Y) \Leftarrow RTD(Y|X)$$

$$SD(X|Y) \Rightarrow LTI(X|Y) \Rightarrow NQD(X,Y) \Leftarrow RTD(X|Y) \Leftarrow SD(X|Y)$$