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A NEW ESTIMATION FOR EIGENVALUES OF MATRIX POWER FUNCTIONS

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Abstract. We give an estimation for the eigenvalues of matrix power functions. In particular, it has been shown that

$$\lambda((A+B)^p) \leq \lambda(2^{p-1}(A^p + B^p - \gamma I)) \quad (p \geq 2)$$

for all positive semi-definite matrices A, B, where γ is a positive constant. This provides a sharper bound for the known estimation for eigenvalues.

1. Introduction and preliminaries

Assume that $\mathbb{M}_n$ is the algebra of all $n \times n$ complex matrices and $\mathbb{M}_n^+$ is the subset of all positive semi-definite matrices. For $A \in \mathbb{M}_n^+$, assume that $\lambda(A)$ is the row vector $(\lambda_1(A), \ldots, \lambda_n(A))$ of eigenvalues of A arranged in decreasing order. For two positive semi-definite matrices A, B we say that $\lambda(A)$ is weakly majorized by $\lambda(B)$ and we write $\lambda(A) \prec_w \lambda(B)$ if

$$\sum_{i=1}^{k} \lambda_i(A) \leq \sum_{i=1}^{k} \lambda_i(B) \quad (k = 1, \ldots, n).$$

We say that $\lambda(A) \leq \lambda(B)$ if $\lambda_i(A) \leq \lambda_i(B)$ for every $i = 1, \ldots, n$. It is known that

$$A \leq B \implies \lambda(A) \leq \lambda(B) \implies \lambda(A) \prec_w \lambda(B).$$

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Ando and Zhan [2] showed that

$$\lambda(A^p + B^p) \prec_w \lambda((A+B)^p) \quad (p \geq 1) \tag{1}$$

and

$$\lambda((A+B)^p) \prec_w \lambda(A^p + B^p) \quad (0 \leq p \leq 1). \tag{2}$$

Aujla and Silva [3] gave an upper bound for (1) and a lower bound for (2) by showing that

$$\lambda((A+B)^p) \leq \lambda(2^{p-1}(A^p + B^p)) \quad (p \geq 1) \tag{3}$$

and

$$\lambda(2^{p-1}(A^p + B^p)) \leq \lambda((A+B)^p) \quad (0 \leq p \leq 1). \tag{4}$$

Similar results concerning the eigenvalue inequalities can be found in [5,6,9, 10]. The main aim of this paper is to present better bounds for the inequalities (3) and (4). We will provide some computational examples to give a clear result. For our purposes, we need a class of functions.

Definition 1.1 [1, Definition 2.1]. A function $f\colon [0,\infty) \to \mathbb{R}$ is superquadratic if for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$f(y) \geq f(x) + C_x(y-x) + f(|y-x|)$$

for all $y \geq 0$.

The reader is referred to [1] for properties of these functions. It is clear that a non-negative superquadratic function is convex. Moreover, Definition 1.1 implies that

$$f(tx + (1-t)y) \leq tx + (1-t)y - tf((1-t)|x-y|) - (1-t)f(t|x-y|) \tag{5}$$

for all $x, y \geq 0$ and $t \in [0,1]$.

Kian [7,8] proved a matrix Jensen type inequality for this class of functions.

Lemma 1.2 [7]. *If $f\colon [0,\infty) \to \mathbb{R}$ is a superquadratic function, then*

$$f(\langle A\xi, \xi\rangle) \leq \big\langle f(A)\xi, \xi\big\rangle - \big\langle f(|A - \langle A\xi,\xi\rangle|)\xi, \xi\big\rangle \tag{6}$$

for every $A \geq 0$ and every unit vector ξ.

Note that for $Z \in \mathbb{M}_n$, the matrix $|Z|$ is the absolute value of Z defined by $|Z| = (Z^*Z)^{1/2}$. We need the following well-known result.

LEMMA 1.3 [4]. *If $A \in \mathbb{M}_n$ is a Hermitian matrix, then*

$$\sum_{j=1}^{k} \lambda_j(A) = \max \sum_{j=1}^{k} \langle Au_j, u_j \rangle$$

for every $k = 1, \ldots, n$, where the maximum is taken over all sets of orthonormal vectors in $\mathbb{C}^n$.

2. Main results

Assume that $f\colon [0,\infty) \to \mathbb{R}$ is a continuous function. We need some notation. For every $s \in [0,1]$ and all positive semi-definite matrices A, B, C, we set

$$\begin{aligned} \alpha_s^f &= \min\big\{f((1-s)|\langle (A-B)w, w\rangle|) : w \in \mathbb{C}^n,\ \|w\| = 1\big\} \\ \alpha_C^f &= \min\big\{\langle f(|C - \langle Cw, w\rangle|)w, w\rangle : w \in \mathbb{C}^n,\ \|w\| = 1\big\} \\ \beta_s^f &= \max\big\{f((1-s)|\langle (A-B)w, w\rangle|) : w \in \mathbb{C}^n,\ \|w\| = 1\big\} \\ \beta_C^f &= \max\big\{\langle f(|C - \langle Cw, w\rangle|)w, w\rangle : w \in \mathbb{C}^n,\ \|w\| = 1\big\}. \end{aligned}$$

In the case where $f(x) = x^p$, we will use α_s, α_C, β_s and β_C, briefly.

First we give a general result.

THEOREM 2.1. *If $f\colon [0,\infty) \to \mathbb{R}$ is a continuous superquadratic function then*

$$\begin{aligned} &\lambda\big(f(tA + (1-t)B)\big) \\ &\prec_w \lambda\big(t\big(f(A) - (\alpha_t^f + \alpha_A^f)I\big) + (1-t)\big(f(B) - (\alpha_{1-t}^f + \alpha_B^f)I\big)\big) \end{aligned}$$

for every $t \in [0,1]$ and all positive semidefinite matrices A, B.

PROOF. Consider $t \in [0,1]$ and positive semi-definite matrices A and B. Assume that $\{v_j\}$ is a set of orthonormal eigenvectors corresponding to the eigenvalues λ_j of $tA + (1-t)B$, which are arranged in such a way that $f(\lambda_1) \geq f(\lambda_2) \geq \cdots \geq f(\lambda_n)$. Then for every $k = 1, \ldots, n$,

$$\sum_{j=1}^{k} \lambda_j\big(f(tA + (1-t)B)\big) = \sum_{j=1}^{k} f\big(\lambda_j(tA + (1-t)B)\big)$$

$$= \sum_{j=1}^{k} f\big(\langle (tA + (1-t)B)v_j, v_j\rangle\big) = \sum_{j=1}^{k} f\big(t\langle Av_j, v_j\rangle + (1-t)\langle Bv_j, v_j\rangle\big).$$

It follows from (5) that

$$\sum_{j=1}^{k}\lambda_j(f(tA+(1-t)B))\leq t\sum_{j=1}^{k}\big[f\big(\langle Av_j,v_j\rangle\big)-f\big((1-t)|\langle(A-B)v_j,v_j\rangle|\big)\big]$$

$$+(1-t)\sum_{j=1}^{n}\big[f\big(\langle Bv_j,v_j\rangle\big)-f\big(t|\langle(A-B)v_j,v_j\rangle|\big)\big].$$

It follows from the definition of α_s^f that

$$\sum_{j=1}^{k}\lambda_j\big(f(tA+(1-t)B)\big) \tag{7}$$

$$\leq t\sum_{j=1}^{k}\big[f(\langle Av_j,v_j\rangle)-\alpha_t^f\big]+(1-t)\sum_{j=1}^{k}\big[f(\langle Bv_j,v_j\rangle)-\alpha_{1-t}^f\big].$$

Moreover, Lemma 1.2 implies that

$$f(\langle Av_j,v_j\rangle)\leq\big\langle f(A)v_j,v_j\big\rangle-\big\langle f(|A-\langle Av_j,v_j\rangle|)v_j,v_j\big\rangle \tag{8}$$

and

$$f(\langle Bv_j,v_j\rangle)\leq\big\langle f(B)v_j,v_j\big\rangle-\big\langle f(|B-\langle Bv_j,v_j\rangle|)v_j,v_j\big\rangle. \tag{9}$$

It follows from (7), (8) and (9) that

$$\sum_{j=1}^{k}\lambda_j(f(tA+(1-t)B)) \tag{10}$$

$$\leq t\sum_{j=1}^{k}\big[\langle f(A)v_j,v_j\rangle-\big\langle f(|A-\langle Av_j,v_j\rangle|)v_j,v_j\big\rangle-\alpha_t^f\big]$$

$$+(1-t)\sum_{j=1}^{k}\big[\langle f(B)v_j,v_j\rangle-\big\langle f(|B-\langle Bv_j,v_j\rangle|)v_j,v_j\big\rangle-\alpha_{1-t}^f\big].$$

For every positive semidefinite matrix C, let

$$\alpha_C^f=\min\big\{\langle f(|C-\langle Cw,w\rangle|)w,w\rangle : w\in\mathbb{C}^n,\ \|w\|=1\big\},$$

whence (10) gives

$$\sum_{j=1}^{k}\lambda_j(f(tA+(1-t)B))$$

$$\leq t \sum_{j=1}^{k} \left[\langle f(A)v_j, v_j\rangle - \alpha_t^f - \alpha_A^f\right] + (1-t) \sum_{j=1}^{k} \left[\langle f(B)v_j, v_j\rangle - \alpha_{1-t}^f - \alpha_B^f\right]$$

$$= \sum_{j=1}^{k} \left\langle [t\big(f(A) - (\alpha_t^f + \alpha_A^f)I\big) + (1-t)\big(f(B) - (\alpha_{1-t}^f + \alpha_B^f)I\big)]v_j, v_j \right\rangle$$

$$\leq \sum_{j=1}^{k} \lambda_j\Big(t\big(f(A) - (\alpha_t^f + \alpha_A^f)I\big) + (1-t)\big(f(B) - (\alpha_{1-t}^f + \alpha_B^f)I\big)\Big),$$

where the last inequality follows from Lemma 1.3. $\square$

The power function $f(x) = x^p$ is superquadratic if $p \geq 2$. Theorem 2.1 yields

$$\lambda((tA + (1-t)B)^p) \tag{11}$$
$$\prec_w \lambda\Big(t(A^p - (\alpha_t + \alpha_A)I) + (1-t)(B^p - (\alpha_{1-t} + \alpha_B)I)\Big),$$

where

$$\alpha_s = (1-s)^p \min\left\{|\langle (A-B)w, w\rangle|^p) : w \in \mathbb{C}^n,\ \|w\| = 1\right\} \quad (s \in [0,1])$$

and

$$\alpha_C = \min\left\{\langle |C - \langle Cw, w\rangle|^p w, w\rangle : w \in \mathbb{C}^n,\ \|w\| = 1\right\}, \quad (C \in \mathbb{M}_n^+).$$

Moreover, for $p \in [1,2]$, the function $f(x) = -x^p$ is superquadratic. This fact gives

$$\lambda(tA^p + (1-t)B^p) \tag{12}$$
$$\prec_w \lambda\Big((tA + (1-t)B)^p + (t(\beta_t + \beta_A) + (1-t)(\beta_{1-t} + \beta_B))I\Big),$$

where

$$\beta_s = (1-s)^p \max\left\{|\langle (A-B)w, w\rangle|^p) : w \in \mathbb{C}^n,\ \|w\| = 1\right\}, \quad (s \in [0,1])$$

and

$$\beta_C = \max\left\{\langle |C - \langle Cw, w\rangle|^p w, w\rangle : w \in \mathbb{C}^n,\ \|w\| = 1\right\}, \quad (C \in \mathbb{M}_n^+).$$

EXAMPLE 2.2. Consider the superquadratic function $f\colon [0,\infty) \to \mathbb{R}$ defined by $f(x) = x^4$. Put

$$A = \begin{pmatrix} 3 & 0.3 \\ 0.3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.5 \end{pmatrix},$$

so that A and B are positive matrices.

Using the computer algebra system "Mathematica" we get

$$\alpha_A \approx 0.12, \quad \alpha_B \approx 0.02, \quad \alpha_t \approx 4.23(1-t)^4, \quad \alpha_{1-t} \approx 4.23t^4.$$

Now, for example with $t = 1/3$, we obtain

$$\lambda_1(f(tA + (1-t)B)) \approx 1.98, \quad \lambda_2(f(tA + (1-t)B)) \approx 0.8$$

and

$$\lambda(tf(A) + (1-t)f(B)) \approx (30.12, 4.53).$$

It follows that

$$(0.8, 1.98) \prec_w (0.8 + 1.02, 1.98 + 1.02) \prec_w (4.53, 30.12).$$

Remark 2.3. Let A, B be positive definite matrices. If $p \geq 2$ and $t = 1/2$, it follows from (11) that

$$\lambda(2^{1-p}(A+B)^p) \prec_w \lambda\big(A^p + B^p - (2\alpha_{1/2} + \alpha_A + \alpha_B)I\big).$$

This gives a better majorization than those in [3]. Moreover, if $p \in [1, 2]$ and $t = 1/2$, then (12) yields

$$\lambda(2^{p-1}(A^p + B^p)) \prec_w \lambda\big((A+B)^p + 2^{p-1}(2\beta_{1/2} + \beta_A + \beta_B)I\big). \tag{13}$$

This provides a converse to the results of [3].

Example 2.4. Consider the superquadratic function $f(x) = -x^2$. Assume that A and B are positive definite matrices as in Example 2.2. Then

$$\beta_A \approx 0.34, \quad \beta_B \approx 0.02, \quad \beta_t \approx 8.21(1-t)^2, \quad \beta_{1-t} \approx 8.21t^2.$$

With $t = 1/2$ we obtain

$$\lambda(2(A^2 + B^2)) \approx (7.81, 19.12), \quad \lambda((A+B)^2) \approx (5.7, 10.96)$$

and

$$\lambda((A+B)^2 + 2(2\beta_{1/2} + \beta_A + \beta_B)I) \approx (14.63, 19.89).$$

It follows from (13) that

$$\lambda((A+B)^2) \prec_w \lambda(2(A^2 + B^2)) \prec_w \lambda\big((A+B)^2 + 2(2\beta_{1/2} + \beta_A + \beta_B)I\big).$$

It gives

$$(5.7, 10.96) \prec_w (7.81, 19.12) \prec_w (14.63, 19.89).$$

COROLLARY 2.5. *If $f\colon [0,\infty) \to \mathbb{R}$ is a continuous superquadratic function, then*

$$\begin{aligned}&\left|\left|\left|f(tA+(1-t)B))\right|\right|\right|\\ &\le \left|\left|\left|t(f(A)-(\alpha_t^f+\alpha_A^f)I)+(1-t)(f(B)-(\alpha_{1-t}^f+\alpha_B^f)I)\right|\right|\right|\end{aligned}$$

for all $A, B \ge 0$, $t \in [0,1]$ and every unitarily invariant norm.

REMARK 2.6. In [3], it was shown that if f is an increasing (or decreasing) convex function on an interval J, then

$$\lambda(f(tA+(1-t)B)) \le \lambda\big(tf(A)+(1-t)f(B)\big) \tag{14}$$

for all Hermitian matrices A, B with eigenvalues in J and for all $t \in [0,1]$. Using a similar argument as in [3, Theorem 2.9], it can be shown that if a continuous superquadratic function $f\colon [0,\infty) \to \mathbb{R}$ is increasing (or decreasing), then

$$\begin{aligned}&\lambda(f(tA+(1-t)B))\\ &\le \lambda\big(t\big(f(A)-(\alpha_t^f+\alpha_A^f)I\big)+(1-t)\big(f(B)-(\alpha_{1-t}^f+\alpha_B^f)I\big)\big).\end{aligned}$$

This provides a better bound than (14). For example, if $p \ge 2$, then the superquadratic function $f(x) = x^p$ is increasing. Therefore

$$\lambda((tA+(1-t)B)^p) \le \lambda\big(t\big(A^p-(\alpha_t+\alpha_A)I\big)+(1-t)\big(B^p-(\alpha_{1-t}+\alpha_B)I\big)\big).$$

Furthermore, if $1 \le p \le 2$, then $f(x) = -x^p$ is decreasing and superquadratic. We obtain

$$\lambda\big(t\big(A^p-(\beta_t+\beta_A)I\big)+(1-t)\big(B^p-(\beta_{1-t}+\beta_B)I\big)\big) \le \lambda((tA+(1-t)B)^p),$$

which gives a converse to (14). With $t = 1/2$ we have

$$\lambda((A+B)^p) \le \lambda\big(2^{p-1}\big(A^p+B^p-(2\alpha_{1/2}+\alpha_A+\alpha_B)I\big)\big) \quad (p \ge 2)$$

and

$$\lambda\big(2^{p-1}\big(A^p+B^p-(2\beta_{1/2}+\beta_A+\beta_B)I\big)\big) \le \lambda((A+B)^p) \quad (1 \le p \le 2).$$

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References

[1] S. Abramovich, G. Jameson and G. Sinnamon, Refining Jensen's inequality, *Bull. Math. Soc. Sci. Math. Roumanie*, **47** (2004), 3–14.

[2] T. Ando and X. Zhan, Norm inequalities related to operator monotone functions, *Math. Ann.*, **315** (1999), 771–780.

[3] J. S. Aujla and F. C. Silva, Weak majorization inequalities and convex functions, *Linear Algebra Appl.*, **369** (2003), 217–233.

[4] R. Bhatia, *Matrix Analysis*, Springer Verlag (New York, 1997).

[5] I. Garg and J. S. Aujla, Some singular value inequalities, *Linear Multilinear Algebra*, **66** (2018), 776–784.

[6] I. Halil Gumus, O. Hirzallah and F. Kittaneh, Estimates for the real and imaginary parts of the eigenvalues of matrices and applications, *Linear Multilinear Algebra*, **64** (2016), 2431–2445.

[7] M. Kian, Operator Jensen inequality for superquadratic functions, *Linear Algebra Appl.*, **456** (2014), 82–87.

[8] M. Kian and S. S. Dragomir, Inequalities involving superquadratic functions and operators, *Mediterr. J. Math.*, **11** (2014), 1205–1214.

[9] F. Kittaneh, M. S. Moslehian, and M. Sababheh, Quadratic interpolation of the Heinz means, *Math. Inequal. Appl.*, **21** (2018), 739–757.

[10] J. S. Matharu, M. S. Moslehian, and J. S. Aujla, Eigenvalue extensions of Bohr's inequality, *Linear Algebra Appl.*, **435** (2011), 270–276.