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Some operator Bellman type inequalities

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Abstract

In this paper, we employ the Mond–Pečarić method to establish some reverses of the operator Bellman inequality under certain conditions. In particular, we show

$$\delta I_{\mathscr{H}} + \sum_{j=1}^{n} \omega_j \Phi_j \left((I_{\mathscr{H}} - A_j)^p \right) \geq \left(\sum_{j=1}^{n} \omega_j \Phi_j (I_{\mathscr{H}} - A_j) \right)^p ,$$

where A_j $(1 \leq j \leq n)$ are self-adjoint contraction operators with $0 \leq mI_{\mathscr{H}} \leq A_j \leq MI_{\mathscr{H}}$, Φ_j are unital positive linear maps on $\mathbb{B}(\mathscr{H})$, $\omega_j \in \mathbb{R}_+$ $(1 \leq j \leq n)$ such that $\sum_{j=1}^{n} \omega_j = 1$, $\delta = (1-p)\left(\frac{1}{p}\frac{(1-m)^p-(1-M)^p}{M-m}\right)^{\frac{p}{p-1}} + \frac{(1-M)(1-m)^p-(1-m)(1-M)^p}{M-m}$ and $0 < p < 1$. We also present some refinements of the operator Bellman inequality.

Keywords: Bellman inequality; Operator mean; The Mond–Pečarić method; Positive linear map

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with the identity $I_{\mathscr{H}}$. In the case when $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix

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algebra $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$ and in this case we write $A \geq 0$. We write $A > 0$ if A is a positive invertible operator. The set of all positive invertible operators is denoted by $\mathbb{B}(\mathscr{H})_+$. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $A \leq B$ if $B - A \geq 0$. Also, an operator $A \in \mathbb{B}(\mathscr{H})$ is said to be contraction, if $A^*A \leq I_{\mathscr{H}}$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a self-adjoint operator A and the C^*-algebra generated by A and $I_{\mathscr{H}}$. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \geq g(A)$.

Let f be a continuous real valued function defined on an interval J. It is called operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for all self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ with spectra in J; see [4] and references therein for some recent results. It is said to be operator concave if $\lambda f(A)+(1-\lambda)f(B) \leq f(\lambda A+(1-\lambda)B)$ for all self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ with spectra in J and all $\lambda \in [0, 1]$. Every nonnegative continuous function f is operator monotone on $[0, +\infty)$ if and only if f is operator concave on $[0, +\infty)$; see [5, Theorem 8.1]. A map $\Phi : \mathbb{B}(\mathscr{H}) \longrightarrow \mathbb{B}(\mathscr{K})$ is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$, where $\mathscr{K}$ is a complex Hilbert space and is said to be unital if $\Phi(I_{\mathscr{H}}) = I_{\mathscr{K}}$. We denote by $\mathbf{P}_N[\mathbb{B}(\mathscr{H}), \mathbb{B}(\mathscr{K})]$ the set of all unital positive linear maps $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$.

The axiomatic theory for operator means of positive invertible operators have been developed by Kubo and Ando [7]. A binary operation σ on $\mathbb{B}(\mathscr{H})_+$ is called a connection, if the following conditions are satisfied:

(i) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$;
(ii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$, where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \cdots$ and $A_n \to A$ as $n \to \infty$ in the strong operator topology;
(iii) $T^*(A\sigma B)T \leq (T^*AT)\sigma(T^*BT)$ $(T \in \mathbb{B}(\mathscr{H}))$.

There exists an affine order isomorphism between the class of connections and the class of positive operator monotone functions f defined on $(0, \infty)$ via $f(t)I_{\mathscr{H}} = I_{\mathscr{H}}\sigma_f(tI_{\mathscr{H}})$ $(t > 0)$. In addition, $A\sigma_f B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$ for all $A, B \in \mathbb{B}(\mathscr{H})_+$. The operator monotone function f is called the representing function of σ_f. A connection σ_f is a mean if it is normalized, i.e. $I_{\mathscr{H}}\sigma_f I_{\mathscr{H}} = I_{\mathscr{H}}$. The function $f_{\nabla_\mu}(t) = (1-\mu)+\mu t$ and $f_{\sharp_\mu}(t) = t^\mu$ on $(0, \infty)$ for $\mu \in (0, 1)$ give the operator weighted arithmetic mean $A\nabla_\mu B = (1-\mu)A+\mu B$ and the operator weighted geometric mean $A\sharp_\mu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\mu} A^{\frac{1}{2}}$, respectively. The case $\mu = 1/2$, the operator weighted geometric mean gives rise to the so-called geometric mean $A\sharp B$.

Bellman [2] proved that if p is a positive integer and a, b, a_j, b_j $(1 \leq j \leq n)$ are positive real numbers such that $\sum_{j=1}^n a_j^p \leq a^p$ and $\sum_{j=1}^n b_j^p \leq b^p$, then

$$\left(a^p - \sum_{j=1}^n a_j^p\right)^{1/p} + \left(b^p - \sum_{j=1}^n b_j^p\right)^{1/p} \leq \left((a+b)^p - \sum_{j=1}^n (a_j+b_j)^p\right)^{1/p}.$$

A multiplicative analogue of this inequality is due to J. Aczél; see [1] and its operator version in [10]. In 1956, Aczél [1] proved that if a_j, b_j $(1 \leq j \leq n)$ are positive real numbers such that $a_1^2 - \sum_{j=2}^n a_j^2 > 0$ or $b_1^2 - \sum_{j=2}^n b_j^2 > 0$, then

$$\left(a_1^2 - \sum_{j=2}^n a_j^2\right)\left(b_1^2 - \sum_{j=2}^n b_j^2\right) \leq \left(a_1b_1 - \sum_{j=2}^n a_jb_j\right)^2.$$

Popoviciu [11] extended the following Aczél's inequality

$$\left(a_1^p-\sum_{j=2}^n a_j^p\right)\left(b_1^p-\sum_{j=2}^n b_j^p\right)\le\left(a_1b_1-\sum_{j=2}^n a_jb_j\right)^p,$$

where $p\ge 1$ and $a_1^p-\sum_{j=2}^n a_j^p>0$ or $b_1^p-\sum_{j=2}^n b_j^p>0$.

During the last decades several generalizations, refinements and applications of the Bellman inequality in various settings have been given and some results related to integral inequalities are presented; see [3,8,9,12] and references therein.

In [9] the authors showed an operator Bellman inequality as follows:

$$\Phi\left((I_{\mathscr{H}}-A)^p\nabla_\lambda(I_{\mathscr{H}}-B)^p\right)\le(\Phi(I_{\mathscr{H}}-A\nabla_\lambda B))^p,$$

whenever A, B are positive contraction operators, Φ is a unital positive linear map on $\mathbb{B}(\mathscr{H})$ and $0<p<1$. They also [8] showed the following generalization of the Bellman operator inequality

$$\left(I_{\mathscr{H}}-\sum_{j=1}^n A_j\right)\sigma_{f^p}\left(I_{\mathscr{H}}-\sum_{j=1}^n B_j\right)\le\left(I_{\mathscr{H}}-\left(\sum_{j=1}^n A_j\sigma_f B_j\right)\right)^p, \tag{1.1}$$

where A_j, B_j $(1\le j\le n)$ are positive operators such that $\sum_{j=1}^n A_j\le I_{\mathscr{H}}, \sum_{j=1}^n B_j\le I_{\mathscr{H}}$, σ_f is a mean with the representing function f and $0<p<1$.

In this paper, we use the Mond–Pečarić method to present some reverses of the operator Bellman inequality under some mild conditions. We also show some refinements of (1.1).

2. Some reverses of the Bellman type operator inequality

The operator Choi–Davis–Jensen inequality says that if f is an operator concave function on an interval J and $\Phi\in\mathbf{P}_N[\mathbb{B}(\mathscr{H}),\mathbb{B}(\mathscr{K})]$, then $f(\Phi(A))\ge\Phi(f(A))$ for all self-adjoint operators A with spectrum in J. The Mond–Pečarić method [5, Chapter 2] present that if f is a strictly concave differentiable function on an interval $[m,M]$ with $m<M$ and $\Phi\in\mathbf{P}_N[\mathbb{B}(\mathscr{H}),\mathbb{B}(\mathscr{K})]$,

$$\begin{aligned}&\mu_f=\frac{f(M)-f(m)}{M-m},\qquad \nu_f=\frac{Mf(m)-mf(M)}{M-m}\quad\text{and}\\&\gamma_f=\max\left\{\frac{f(t)}{\mu_f t+\nu_f}:m\le t\le M\right\},\end{aligned} \tag{2.1}$$

then

$$\gamma_f\Phi(f(A))\ge f(\Phi(A)). \tag{2.2}$$

In inequality (2.2), if we put $\Phi(X):=\Psi(A)^{-1/2}\Psi(A^{1/2}XA^{1/2})\Psi(A)^{-1/2}$, where Ψ is an arbitrary unital positive linear map and take f to be the representing function of an operator mean σ_f, then we reach the inequality

$$\left(\max_{m\le t\le M}\frac{f(t)}{\mu_f t+\nu_f}\right)\Psi(A\sigma_f B)\ge\Psi(A)\sigma_f\Psi(B) \tag{2.3}$$

whenever $0<mA\le B\le MA$.

Finally, if we take Ψ in (2.3) to be the positive unital linear map defined on the diagonal blocks of operators by $\Psi(\operatorname{diag}(A_1, \ldots, A_n)) = \frac{1}{n}\sum_{j=1}^n A_j$, then

$$\gamma_f \sum_{j=1}^n A_j \sigma_f B_j \ge \left(\sum_{j=1}^n A_j\right) \sigma_f \left(\sum_{j=1}^n B_j\right), \tag{2.4}$$

where γ_f is given by (2.1) and $0 < mA_j \le B_j \le MA_j$ $(1 \le j \le n)$.

In the following theorem we show a reversed operator Bellman type inequality.

Theorem 2.1. *Suppose that* $0 < mA_j \le B_j \le MA_j$ $(1 \le j \le n)$ *and* $0 < m\left(I_{\mathscr{H}} - \gamma_f \sum_{j=1}^n A_j\right) \le I_{\mathscr{H}} - \gamma_f \sum_{j=1}^n B_j \le M\left(I_{\mathscr{H}} - \gamma_f \sum_{j=1}^n A_j\right)$ *for some positive real numbers* m, M *such that* $m < 1 < M$, γ_f *is given by* (2.1), σ_f *is an operator mean with the representing function* f *and* $p \in [0, 1]$. *Then*

$$\gamma_f^p \left(\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \sigma_f \left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right)^p \ge \left(I_{\mathscr{H}} - \gamma_f \left(\sum_{j=1}^n A_j \sigma_f B_j\right)\right)^p. \tag{2.5}$$

Proof. By using (2.4) we have

$$\gamma_f \sum_{j=1}^{n+1} X_j \sigma_f Y_j \ge \left(\sum_{j=1}^{n+1} X_j\right) \sigma_f \left(\sum_{j=1}^{n+1} Y_j\right),$$

where $0 < mX_j \le Y_j \le MX_j$ $(1 \le j \le n+1)$. If we take $X_j = A_j, Y_j = B_j$ $(1 \le j \le n)$, $X_{n+1} = I_{\mathscr{H}} - \sum_{j=1}^n A_j \ge 0$ and $Y_{n+1} = I_{\mathscr{H}} - \sum_{j=1}^n B_j \ge 0$, then

$$\gamma_f \left[\left(\sum_{j=1}^n A_j\right) \sigma_f \left(\sum_{j=1}^n B_j\right) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \sigma_f \left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right]$$
$$\ge I_{\mathscr{H}} \sigma_f I_{\mathscr{H}} = I_{\mathscr{H}},$$

whence

$$\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \sigma_f \left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right) \ge \frac{1}{\gamma_f} I_{\mathscr{H}} - \left(\sum_{j=1}^n A_j\right) \sigma_f \left(\sum_{j=1}^n B_j\right). \tag{2.6}$$

It follows from inequality (2.6) and the Löwner–Heinz inequality [5, Theorem 1.8] that

$$\left[\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \sigma_f \left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right]^p \ge \left[\frac{1}{\gamma_f} I_{\mathscr{H}} - \left(\sum_{j=1}^n A_j\right) \sigma_f \left(\sum_{j=1}^n B_j\right)\right]^p. \quad \square$$

Lemma 2.2. *Suppose that* $C, X \in \mathbb{B}(\mathscr{H})$ *such that* C *is a contraction operator,* $0 < mI_{\mathscr{H}} \le X \le MI_{\mathscr{H}}$, f *is a concave and operator monotone function on* $[m, M]$ *and* γ_f *is given by* (2.1). *Then*

$$\gamma_f \left[C^* f(X) C + f(m)(I_{\mathscr{H}} - C^*C)\right] \ge f(C^*XC). \tag{2.7}$$

Proof. Let $D = (I_{\mathscr{H}} - C^*C)^{\frac{1}{2}}$. Consider the positive unital linear map $\Phi\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = C^*XC + D^*YD$ $(X, Y \in \mathbb{B}(\mathscr{H}))$. Using inequality (2.2) and the operator monotonicity of f we have

$$\begin{aligned} f(C^*XC) &\leq f(C^*XC + D^*mD) \\ &= f\left(\Phi\left(\begin{bmatrix} X & 0 \\ 0 & m \end{bmatrix}\right)\right) \\ &\leq \gamma_f\left(\Phi\left(\begin{bmatrix} f(X) & 0 \\ 0 & f(m) \end{bmatrix}\right)\right) \quad \text{(by (2.2))} \\ &= \gamma_f\left[C^* f(X)C + D^* f(m)D\right], \end{aligned}$$

whenever $0 < mI_{\mathscr{H}} \leq X \leq MI_{\mathscr{H}}$. □

Lemma 2.3. *Let* $0 < mA \leq B \leq MA$ *with* A *contraction. Let* σ_f *be an operator mean with the representing function* f *and* h *be an operator monotone function on* $[0, +\infty)$. *Then*

$$\gamma_h\big[h(f(m))\big(I_{\mathscr{H}} - A\big) + \big(A\sigma_{hof}B\big)\big] \geq h\left(A\sigma_f B\right), \tag{2.8}$$

where $\mu_h = \frac{h(f(M))-h(f(m))}{f(M)-f(m)}$, $\nu_h = \frac{f(M)h(f(m))-f(m)h(f(M))}{f(M)-f(m)}$ *and* $\gamma_h = \max_{f(m)\leq t\leq f(M)} \frac{h(t)}{\mu_h t+\nu_h}$.

Proof. It follows from $f(m) \leq f\left(A^{-1/2}BA^{-1/2}\right) \leq f(M)$ and inequality (2.7) that

$$\begin{aligned} h\left(A\sigma_f B\right) &= h\left(A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2}\right) \\ &\leq \gamma_h\left(A^{1/2} h\left(f\left(A^{-1/2}BA^{-1/2}\right)\right) A^{1/2} + (I_{\mathscr{H}} - A)^{1/2}\, h(f(m))\, (I_{\mathscr{H}} - A)^{1/2}\right) \\ &\quad \text{(by (2.7))} \\ &= \gamma_h\left[\left(A\sigma_{hof}B\right) + h(f(m))\,(I_{\mathscr{H}} - A)\right] \end{aligned}$$

□

Applying the operator monotone function $f(t) = (1 - \lambda) + \lambda t$ $(\lambda \in [0, 1])$ in Theorem 2.1, due to

$$\gamma_f = \max_{m\leq t\leq M} \frac{f(t)}{\mu_f t + \nu_f} = \max_{m\leq t\leq M} \frac{(1-\lambda) + \lambda t}{(1-\lambda) + \lambda t} = 1$$

and using Lemma 2.3 for the special case $h(t) = t^p$ $(p \in [0, 1])$, due to

$$\gamma_h = \max_{f(m)\leq t\leq f(M)} \frac{t^p}{\mu_h t + \nu_h} = \frac{p^p(f(M) - f(m))(f(M)f(m)^p - f(m)f(M)^p)^{p-1}}{(1-p)^{p-1}(f(M)^p - f(m)^p)^p}$$

we have the following result; see [5, p. 77].

Corollary 2.4 (*A Reverse Operator Bellman Inequality*)**.** *Let* $0 < mA_j \leq B_j \leq MA_j$ $(1 \leq j \leq n)$ *and* $0 < m\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \leq I_{\mathscr{H}} - \sum_{j=1}^n B_j \leq M\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right)$ *for some positive real numbers* m, M *such that* $m < 1 < M$ *and* $p \in [0, 1]$. *Then*

$$\begin{aligned} &\delta\left(f(m)^p\left(\sum_{j=1}^n A_j\right) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right)\sigma_{((1-\lambda)+\lambda t)^p}\left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right) \\ &\geq \left(I_{\mathscr{H}} - \left(\sum_{j=1}^n A_j\sigma_f B_j\right)\right)^p, \end{aligned}$$

where $\delta = \frac{p^p(f(M)-f(m))(f(M)f(m)^p-f(m)f(M)^p)^{p-1}}{(1-p)^{p-1}(f(M)^p-f(m)^p)^p}$ *and* $f(t) = (1 - \lambda) + \lambda t$ $(\lambda \in [0, 1])$.

Proof. We have

$$\begin{aligned}
&\delta\left(f(m)^p\left(\sum_{j=1}^n A_j\right)+\left(I_{\mathscr{H}}-\sum_{j=1}^n A_j\right)\sigma_{((1-\lambda)+\lambda t)^p}\left(I_{\mathscr{H}}-\sum_{j=1}^n B_j\right)\right)\\
&\geq\left(\left(I_{\mathscr{H}}-\sum_{j=1}^n A_j\right)\nabla_\lambda\left(I_{\mathscr{H}}-\sum_{j=1}^n B_j\right)\right)^p \quad \text{(by (2.8))}\\
&\geq\left(I_{\mathscr{H}}-\left(\sum_{j=1}^n A_j\nabla_\lambda B_j\right)\right)^p \quad \text{(by (2.5))}. \quad \square
\end{aligned}$$

In [6], the authors showed another way to find a reverse Choi–Davis–Jensen inequality. If f is a strictly concave differentiable function on an interval $[m, M]$ with $m < M$ and Φ is a unital positive linear map, then

$$\beta_f I_{\mathscr{H}} + \Phi(f(A)) \geq f(\Phi(A)), \tag{2.9}$$

where $A \in \mathbb{B}(\mathscr{H})$ is a self-adjoint operator with spectrum in $[m, M]$ and $\beta_f = \max_{m\leq t\leq M}\left\{f(t)-\mu_f t-\nu_f\right\}$.

In inequality (2.9), if we put $\Psi(X) := \Psi(A)^{-1/2}\Psi(A^{1/2}XA^{1/2})\Psi(A)^{-1/2}$, where Ψ is an arbitrary unital positive linear map and take f to be the representing function of an operator mean σ_f, then we reach the inequality

$$\beta_f\Psi(X) + \Psi(X\sigma_f Y) \geq \Psi(X)\sigma_f\Psi(Y), \tag{2.10}$$

where $0 < mX \leq Y \leq MX, \sigma_f$ is an operator mean with representing function f and $\beta_f = \max_{m\leq t\leq M}\left\{f(t)-\mu_f t-\nu_f\right\}$ that is the unique solution of the equation $f'(t) = \mu_f$, whenever $\mu_f = \frac{f(M)-f(m)}{M-m}$ and $\nu_f = \frac{Mf(m)-mf(M)}{M-m}$.

Applying (2.10) to the positive unital linear map defined on the diagonal blocks of operators by $\Psi(\mathrm{diag}(X_1,\ldots,X_{n+1})) = \frac{1}{n}\sum_{j=1}^{n+1}X_j$, we get

$$\beta_f\sum_{j=1}^{n+1}X_j+\sum_{j=1}^{n+1}Y_j\sigma_f X_j \geq \left(\sum_{j=1}^{n+1}X_j\right)\sigma_f\left(\sum_{j=1}^{n+1}Y_j\right), \tag{2.11}$$

where $0 < mX_j \leq Y_j \leq MX_j$ $(1 \leq j \leq n+1), \sigma_f$ is an operator mean with representing function f and $\beta_f = \max_{m\leq t\leq M}\left\{f(t)-\mu_f t-\nu_f\right\}$.

Now, we have the next result.

Proposition 2.5. *Suppose that* $0 < mA_j \leq B_j \leq MA_j$ $(1 \leq j \leq n)$ *and* $0 < m\Big(I_{\mathscr{H}} - \sum_{j=1}^n A_j\Big) \leq I_{\mathscr{H}} - \sum_{j=1}^n B_j \leq M\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right)$ *for some positive real numbers* m, M *such that* $m < 1 < M$ *and* σ_f *is an operator mean with representing function* f*. Then*

$$\left(\beta_f I_{\mathscr{H}}+\left(I_{\mathscr{H}}-\sum_{j=1}^n A_j\right)\sigma_f\left(I_{\mathscr{H}}-\sum_{j=1}^n B_j\right)\right)^p \geq \left(I_{\mathscr{H}}-\left(\sum_{j=1}^n A_j\sigma_f B_j\right)\right)^p,$$

whenever $p \in [0, 1]$ *and* $\beta_f = \max_{m\leq t\leq M}\left\{f(t)-\mu_f t-\nu_f\right\}$.

Proof. If we take $X_j = A_j, Y_j = B_j$ $(1 \le j \le n), X_{n+1} = I_{\mathscr{H}} - \sum_{j=1}^n A_j$ and $Y_{n+1} = I_{\mathscr{H}} - \sum_{j=1}^n B_j$ in inequality (2.11), then we get

$$\beta_f I_{\mathscr{H}} + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \sigma_f \left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right) \ge \left(I_{\mathscr{H}} - \left(\sum_{j=1}^n A_j \sigma_f B_j\right)\right). \tag{2.12}$$

By the operator monotonicity of $h(t) = t^p$ and (2.12) we reach the desired inequality. □

Corollary 2.6 (*A Reverse Aczél Type Inequality*). *Let* $0 < mA_j \le B_j \le MA_j$ $(1 \le j \le n), 0 < m\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \le I_{\mathscr{H}} - \sum_{j=1}^n B_j \le M\left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right)$ *for some positive real numbers* m, M *such that* $m < 1 < M$. *Then*

$$\left(\zeta I_{\mathscr{H}} + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \sharp_\lambda \left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right)^p \ge \left(I_{\mathscr{H}} - \left(\sum_{j=1}^n A_j \sharp_\lambda B_j\right)\right)^p,$$

where $p, \lambda \in [0, 1]$ *and* $\zeta = (1-p)\left(\frac{M^p - m^p}{p(M-m)}\right)^{\frac{p}{p-1}} - \frac{Mm^p - mM^p}{M-m}$.

We can generalize the operator Bellman inequality in Corollary 2.2 of [9] as follows

$$\left(\Phi\left(I_{\mathscr{H}} - \sum_{j=1}^n \omega_j A_j\right)\right)^p \ge \Phi\left(\sum_{j=1}^n \omega_j (I_{\mathscr{H}} - A_j)^p\right), \tag{2.13}$$

where $\Phi \in \mathbf{P}_N[(\mathscr{H}), (\mathscr{K})]$, A_j are contractions, $0 < p < 1$ and ω_j are real positive numbers such that $\sum_{j=1}^n \omega_j = 1$.

In [9], the authors presented an equivalent form of Bellman inequality. With a similar argument in the proof of [9, Theorem 2.5], the following theorem holds.

Theorem 2.7. *The following equivalent statements hold:*

(i) *If* m, n *are positive integers,* $0 < p < 1, \omega_1, \ldots, \omega_n$ *are any finite number of positive real numbers such that* $\sum_{j=1}^n \omega_j = 1$ *and* a_{ij} $(j = 1, \ldots, n,\ i = 1, \ldots, m)$ *are positive real numbers such that* $\sum_{i=1}^m a_{ij}^{1/p} \le 1$ *for all* $j = 1, \ldots, n$, *then*

$$\sum_{j=1}^n \omega_j \left(1 - \sum_{i=1}^m a_{ij}^{\frac{1}{p}}\right)^p \le \left(1 - \sum_{i=1}^m \Big(\sum_{j=1}^n \omega_j a_{ij}\Big)^{\frac{1}{p}}\right)^p. \tag{2.14}$$

(ii) *(Generalization of Classical Bellman Inequality) If* n *is a positive integer,* $0 < p < 1$ *and* M_i, a_{ij} $(j = 1, \ldots, n,\ i = 1, \ldots, m)$ *are nonnegative real numbers such that* $\sum_{i=1}^m a_{ij}^{1/p} \le M_j^{1/p}$ *for all* $j = 1, \ldots, n$, *then*

$$\sum_{j=1}^n \left(M_j^{\frac{1}{p}} - \sum_{i=1}^m a_{ij}^{\frac{1}{p}}\right)^p \le \left(\left(\sum_{j=1}^n M_j\right)^{\frac{1}{p}} - \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij}\right)^{\frac{1}{p}}\right)^p. \tag{2.15}$$

Proof. Let m, n be positive integers, $0 < p < 1, \omega_1, \ldots, \omega_n \in \mathbb{R}_+$ are any finite number of positive real numbers such that $\sum_{j=1}^{n} \omega_j = 1$ and a_{ij} $(j = 1, \ldots, n,\ i = 1, \ldots, m)$ are positive real numbers such that $\sum_{i=1}^{m} a_{ij}^{1/p} \le 1$ for all $j = 1, \ldots, n$. Set $A_j = \begin{bmatrix} \sum_{i=1}^{m} a_{ij}^{\frac{1}{p}} & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$ $(j = 1, \ldots, n)$. Then

$$\left(I_2 - \sum_{j=1}^{n} \omega_j A_j\right)^p = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \sum_{j=1}^{n}\sum_{i=1}^{m} \omega_j a_{ij}^{\frac{1}{p}} & 0 \\ 0 & 1 \end{bmatrix}\right)^p$$
$$= \begin{bmatrix} \left(1 - \sum_{j=1}^{n}\sum_{i=1}^{m} \omega_j a_{ij}^{\frac{1}{p}}\right)^p & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \left(\sum_{j=1}^{n} \omega_j \left(1 - \sum_{i=1}^{m} a_{ij}^{\frac{1}{p}}\right)\right)^p & 0 \\ 0 & 0 \end{bmatrix}.$$

and

$$\sum_{j=1}^{n} \omega_j (I_2 - A_j)^p = \sum_{j=1}^{n} \omega_j \begin{bmatrix} 1 - \sum_{i=1}^{m} a_{ij}^{\frac{1}{p}} & 0 \\ 0 & 0 \end{bmatrix}^p$$
$$= \begin{bmatrix} \sum_{j=1}^{n} \omega_j \left(1 - \sum_{i=1}^{m} a_{ij}^{\frac{1}{p}}\right)^p & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows from (2.13) with the identity map Φ that

$$\sum_{j=1}^{n} \omega_j \left(1 - \sum_{i=1}^{m} a_{ij}^{\frac{1}{p}}\right)^p \le \left(\sum_{j=1}^{n} \omega_j \Big(1 - \sum_{i=1}^{m} a_{ij}^{\frac{1}{p}}\Big)\right)^p$$
$$= \left(1 - \sum_{j=1}^{n} \omega_j \Big(\sum_{i=1}^{m} a_{ij}^{\frac{1}{p}}\Big)\right)^p$$
$$= \left(1 - \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} \omega_j a_{ij}^{\frac{1}{p}}\Big)\right)^p$$
$$\le \left(1 - \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} \omega_j a_{ij}\Big)^{\frac{1}{p}}\right)^p$$

(by the convexity of $t^{1/p}$ for $0 < p < 1$),

which gives (2.14).

(i)⇒(ii) Set $\omega_j = \frac{M_j}{\sum_{j=1}^n M_j}$ and replace a_{ij} by a_{ij}/M_j, respectively, in (2.14) to get

$$\begin{aligned}\frac{1}{\sum\limits_{j=1}^n M_j}\sum_{j=1}^n\left(M_j^{\frac{1}{p}}-\sum_{i=1}^m a_{ij}^{\frac{1}{p}}\right)^p &= \sum_{j=1}^n \frac{M_j}{\sum\limits_{j=1}^n M_j}\left(1-\sum_{i=1}^m\Big(\frac{a_{ij}}{M_j}\Big)^{\frac{1}{p}}\right)^p\\ &= \sum_{j=1}^n \omega_j\left(1-\sum_{i=1}^m\Big(\frac{a_{ij}}{M_j}\Big)^{\frac{1}{p}}\right)^p\\ &\le \left(1-\sum_{i=1}^m\Big(\sum_{j=1}^n\omega_j\frac{a_{ij}}{M_j}\Big)^{\frac{1}{p}}\right)^p\\ &= \left(1-\sum_{i=1}^m\Big(\sum_{j=1}^n\frac{M_j}{\sum\limits_{j=1}^n M_j}\frac{a_{ij}}{M_j}\Big)^{\frac{1}{p}}\right)^p\\ &= \left(1-\frac{1}{(\sum\limits_{j=1}^n M_j)^{\frac{1}{p}}}\sum_{i=1}^m\Big(\sum_{j=1}^n a_{ij}\Big)^{\frac{1}{p}}\right)^p\\ &= \frac{1}{\sum\limits_{j=1}^n M_j}\left(\Big(\sum_{j=1}^n M_j\Big)^{\frac{1}{p}}-\sum_{i=1}^m\Big(\sum_{j=1}^n a_{ij}\Big)^{\frac{1}{p}}\right)^p.\end{aligned}$$

We therefore deduce the desired inequality (2.15).

(ii)⇒(i) Set $M_j = \omega_j$, and replace a_{ij} by $\omega_j a_{ij}$ in (2.15) to get (2.14). □

Let $A_j \in \mathbb{B}(\mathscr{H})$ $(1 \le j \le n)$ be self-adjoint operators with $\mathrm{sp}(A_j) \subseteq [m, M]$ for some scalars $m < M$, Φ_j be unital positive linear maps on $\mathbb{B}(\mathscr{H})$, $\omega_1, \ldots, \omega_n \in \mathbb{R}_+$ be any finite number of positive real numbers such that $\sum_{j=1}^n \omega_j = 1$ and f be a strictly concave differentiable function. If we take the positive unital linear map $\Phi(\mathrm{diag}(A_1, \ldots, A_n)) = \sum_{j=1}^n \omega_j \Phi_j(A_j)$, in inequality (2.9), then

$$\beta_f I_{\mathscr{H}} + \sum_{j=1}^n \omega_j \Phi_j\big(f(A_j)\big) \ge f\left(\sum_{j=1}^n \omega_j \Phi_j(A_j)\right), \tag{2.16}$$

where $\beta_f = \max_{m\le t\le M}\left\{f(t) - \mu_f t - \nu_f\right\}$; see also [5, Corollary 2.16].

Now, we state reverse of (2.13) by following result.

Corollary 2.8 (*A Second Type Reverse Operator Bellman Inequality*). *Let A_j, Φ_j, ω_j, $j = 1, \ldots, n$ be as above, A_j be contractions such that $0 \le mI_{\mathscr{H}} \le A_j \le MI_{\mathscr{H}}$ and $0 < p < 1$. Then*

$$\delta I_{\mathscr{H}} + \sum_{j=1}^n \omega_j \Phi_j\left((I_{\mathscr{H}} - A_j)^p\right) \ge \left(\sum_{j=1}^n \omega_j \Phi_j(I_{\mathscr{H}} - A_j)\right)^p, \tag{2.17}$$

where $\delta = (1-p)\left(\frac{1}{p}\frac{(1-m)^p-(1-M)^p}{M-m}\right)^{\frac{p}{p-1}} + \frac{(1-M)(1-m)^p-(1-m)(1-M)^p}{M-m}$.

Proof. Note that the function $g(t) = t^r$ is operator concave on $(0, \infty)$ when $0 \le r \le 1$ and so is the function $f(t) = (1-t)^p$ on $(0, 1)$ when $0 \le p \le 1$. It follows from the linearity and the normality of Φ_j that

$$\begin{aligned}\left(\sum_{j=1}^n \omega_j \Phi_j(I_{\mathscr{H}} - A_j)\right)^p &= \left(\sum_{j=1}^n \omega_j \big(I_{\mathscr{K}} - \Phi_j(A_j)\big)\right)^p \\ &= \left(I_{\mathscr{H}} - \sum_{j=1}^n \omega_j \Phi_j(A_j)\right)^p \\ &= f\left(\sum_{j=1}^n \omega_j \Phi_j(A_j)\right) \\ &\le \sum_{j=1}^n \omega_j \Phi_j(f(A_j)) + \beta_f I_{\mathscr{H}} \quad \text{(by (2.16))} \\ &= \sum_{j=1}^n \omega_j \Phi_j\left((I_{\mathscr{H}} - A_j)^p\right) + \beta_f I_{\mathscr{K}}.\end{aligned}$$

Since $f(t) = (1-t)^p$ is a differentiable function on $(0, 1)$, the function $h(t) := (1-t)^p - \frac{(1-M)^p-(1-m)^p}{M-m} t - \frac{M(1-m)^p - m(1-M)^p}{M-m}$ $(m \le t \le M)$ attained its maximum value in $t_0 = 1 - \left(\frac{1}{p}\frac{(1-m)^p-(1-M)^p}{M-m}\right)^{\frac{1}{p-1}}$ which is equal to

$$\begin{aligned}\delta = \max_{m \le t \le M} h(t) &= (1-p)\left(\frac{1}{p}\frac{(1-m)^p - (1-M)^p}{M-m}\right)^{\frac{p}{p-1}} \\ &\quad + \frac{(1-M)(1-m)^p - (1-m)(1-M)^p}{M-m}. \quad \square\end{aligned}$$

Corollary 2.9. *If m, n are positive integers, $0 < p < 1$, $\omega_1, \ldots, \omega_n \in \mathbb{R}_+$ are any finite number of positive real numbers such that $\sum_{j=1}^n \omega_j = 1$ and a_{ij} $(j = 1, \ldots, n,\ i = 1, \ldots, m)$ are positive real numbers such that $1 \ge \sum_{i=1}^m a_{ij}^{1/p}$ $(j = 1, \ldots, n)$, then*

$$(1-p)p^{\frac{p}{1-p}} + \sum_{j=1}^n \omega_j \left(1 - \sum_{i=1}^m a_{ij}^{\frac{1}{p}}\right)^p \ge \left(1 - \sum_{i=1}^m \sum_{j=1}^n \omega_j a_{ij}^{\frac{1}{p}}\right)^p. \tag{2.18}$$

Proof. Let m, n be positive integers, $0 < p < 1$, $0 \le \omega_j \le 1$ and a_{ij} $(1 \le j \le n,\ 1 \le i \le m)$ are positive real numbers such that $1 \ge \sum_{i=1}^m a_{ij}^{1/p}$ $(j = 1, \ldots, n)$. Set $A_j = \begin{bmatrix} \sum_{i=1}^m a_{ij}^{1/p} & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$ $(j = 1, \ldots, n)$. Then

$$\begin{aligned}\sum_{j=1}^n \omega_j (I_2 - A_j)^p &= \sum_{j=1}^n \omega_j \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \sum_{i=1}^m a_{ij}^{\frac{1}{p}} & 0 \\ 0 & 1 \end{bmatrix}\right)^p \\ &= \sum_{j=1}^n \omega_j \begin{bmatrix} \left(1 - \sum_{i=1}^m a_{ij}^{\frac{1}{p}}\right)^p & 0 \\ 0 & 0 \end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} \sum_{j=1}^{n} \omega_j \left(1 - \sum_{i=1}^{m} a_{ij}^{\frac{1}{p}}\right)^p & 0 \\ 0 & 0 \end{bmatrix}.$$

and

$$\begin{aligned} \left(\sum_{j=1}^{n} \omega_j (I_2 - A_j)\right)^p &= \left(\sum_{j=1}^{n} \omega_j \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \sum_{i=1}^{m} a_{ij}^{\frac{1}{p}} & 0 \\ 0 & 1 \end{bmatrix}\right)\right)^p \\ &= \begin{bmatrix} \sum_{j=1}^{n} \omega_j \left(1 - \sum_{i=1}^{m} a_{ij}^{\frac{1}{p}}\right) & 0 \\ 0 & 0 \end{bmatrix}^p \\ &= \begin{bmatrix} \left(1 - \sum_{j=1}^{n} \sum_{i=1}^{m} \omega_j a_{ij}^{\frac{1}{p}}\right)^p & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

It follows from (2.17) with the identity map Φ that

$$\delta I_2 + \begin{bmatrix} \sum_{j=1}^{n} \omega_j \left(1 - \sum_{i=1}^{m} a_{ij}^{\frac{1}{p}}\right)^p & 0 \\ 0 & 0 \end{bmatrix} \geq \begin{bmatrix} \left(1 - \sum_{j=1}^{n} \sum_{i=1}^{m} \omega_j a_{ij}^{\frac{1}{p}}\right)^p & 0 \\ 0 & 0 \end{bmatrix},$$

where $\delta = (1-p)p^{\frac{p}{1-p}}$ by taking $m = 0$ and $M = 1$ in (2.17), which gives (2.18). □

Corollary 2.10. *Let $\Phi \in \mathbf{P}_N[\mathcal{B}(\mathscr{H}), \mathcal{B}(\mathscr{K})]$, $0 < mI_{\mathscr{H}} \leq A_j \leq MI_{\mathscr{H}}$ be positive operators and $0 \leq \omega_j \leq 1$ $(j = 1, \ldots, n)$ such that $\sum_{j=1}^{n} \omega_j = 1$. Then*

$$\log \left[\frac{1}{e} \left(\frac{M^m}{m^M}\right)^{\frac{1}{M-m}} L(m, M)\right] + \Phi\left(\sum_{j=1}^{n} \omega_j \log A_j\right) \geq \log \left(\sum_{j=1}^{n} \omega_j \Phi(A_j)\right)$$

where $L(a,b) = \begin{cases} \frac{b-a}{\log b - \log a} & ; a \neq b \\ a & ; a = b \end{cases}$ is the logarithmic mean of positive real numbers a and b.

Proof. Put $f(t) = \log t$ and $\Phi_j = \Phi$ in (2.16). □

3. Some refinements of the Bellman operator inequality

In this section, we present some refinements of the operator Bellman inequality by using some ideas of [3]. First we need the following lemmas.

Lemma 3.1. *Let A, B, A_j, B_j, $(1 \leq j \leq n)$ be positive operators such that $\sum_{j=1}^{n} A_j \leq A$, $\sum_{j=1}^{n} B_j \leq B$ and let σ_f be an operator mean with the representing function f. Then*

$$\left(A - \sum_{j=1}^{n} A_j\right) \sigma_f \left(B - \sum_{j=1}^{n} B_j\right) \leq (A \sigma_f B) - \sum_{j=1}^{n} \left(A_j \sigma_f B_j\right). \tag{3.1}$$

Proof. The subadditivity of operator mean says that [5, Theorem 5.7]

$$\sum_{j=1}^{n+1}\left(X_j\sigma_f Y_j\right) \le \left(\sum_{j=1}^{n+1} X_j\right)\sigma_f\left(\sum_{j=1}^{n+1} Y_j\right), \tag{3.2}$$

where X_j, Y_j, $(1 \le j \le n+1)$ are positive operators. If we put $X_j = A_j, Y_j = B_j$ $(1 \le j \le n)$, $X_{n+1} = A - \sum_{j=1}^{n} A_j$ and $Y_{n+1} = A - \sum_{j=1}^{n} B_j$ in inequality (3.2), then we reach

$$\sum_{j=1}^{n}\left(A_j\sigma_f B_j\right) + \left(A - \sum_{j=1}^{n} A_j\right)\sigma_f\left(B - \sum_{j=1}^{n} B_j\right) \le A\sigma_f B.$$

Therefore

$$\left(A - \sum_{j=1}^{n} A_j\right)\sigma_f\left(B - \sum_{j=1}^{n} B_j\right) \le \left(A\sigma_f B\right) - \sum_{j=1}^{n}\left(A_j\sigma_f B_j\right). \quad \square$$

Lemma 3.2 (*[8, Lemma 2.1]*). *Let $A, B \in \mathbb{B}(\mathscr{H})$ be positive operators such that A is contraction, h is a nonnegative operator monotone function on $[0, +\infty)$ and σ_f be an operator mean with the representing function f. Then*

$$A\sigma_{hof} B \le h(A\sigma_f B).$$

In the next theorem, we show a refinement of (1.1).

Theorem 3.3. *Let A_j, B_j, $(1 \le j \le n)$ be positive operators such that $\sum_{j=1}^{n} A_j \le I_{\mathscr{H}}$, $\sum_{j=1}^{n} B_j \le I_{\mathscr{H}}$, σ_f be an operator mean with the representing function f and $p \in [0, 1]$. Then*

$$\begin{aligned}\left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right)\sigma_{f^p}\left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right) &\le \left(\left(I_{\mathscr{H}} - \sum_{j=1}^{k} A_j\right)\sigma_f\left(I_{\mathscr{H}} - \sum_{j=1}^{k} B_j\right)\right.\\ &\qquad \left. - \sum_{j=k+1}^{n}\left(A_j\sigma_f B_j\right)\right)^p\\ &\le \left(I_{\mathscr{H}} - \sum_{j=1}^{n}\left(A_j\sigma_f B_j\right)\right)^p,\end{aligned}$$

in which $k = 1, 2, \ldots, n-1$.

Proof. For $k = 1, 2, \ldots, n-1$ we have

$$\begin{aligned}&\left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right)\sigma_f\left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)\\ &= \left(\left(I_{\mathscr{H}} - \sum_{j=1}^{k} A_j\right) - \sum_{j=k+1}^{n} A_j\right)\sigma_f\left(\left(I_{\mathscr{H}} - \sum_{j=1}^{k} B_j\right) - \sum_{j=k+1}^{n} B_j\right)\\ &\le \left(I_{\mathscr{H}} - \sum_{j=1}^{k} A_j\right)\sigma_f\left(I_{\mathscr{H}} - \sum_{j=1}^{k} B_j\right) - \sum_{j=k+1}^{n}\left(A_j\sigma_f B_j\right) \quad \text{(by (3.1))}\end{aligned}$$

$$\leq \left(\left(I_{\mathscr{H}} \sigma_f I_{\mathscr{H}}\right) - \sum_{j=1}^{k} (A_j \sigma_f B_j) \right) - \sum_{j=k+1}^{n} \left(A_j \sigma_f B_j\right) \quad \text{(by (3.1))}$$
$$= I_{\mathscr{H}} - \sum_{j=1}^{n} \left(A_j \sigma_f B_j\right),$$

whence for the spacial case $g(t) = t^p$ ($p \in [0, 1]$) we have

$$\left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right) \sigma_{f^p} \left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)$$
$$\leq \left(\left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right) \sigma_f \left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right) \right)^p \quad \text{(by Lemma 3.2)}$$
$$\leq \left(\left(I_{\mathscr{H}} - \sum_{j=1}^{k} A_j\right) \sigma_f \left(I_{\mathscr{H}} - \sum_{j=1}^{k} B_j\right) - \sum_{j=k+1}^{n} (A_j \sigma_f B_j) \right)^p$$
$$= \left(I_{\mathscr{H}} - \sum_{j=1}^{n} (A_j \sigma_f B_j)\right)^p. \quad \square$$

Using the same idea as in the proof of Theorem 3.3 we improve inequality (1.1) in the next theorem.

Theorem 3.4. *Let A_j, B_j, $(1 \leq j \leq n)$ be positive operators such that $\sum_{j=1}^{n} A_j \leq I_{\mathscr{H}}$, $\sum_{j=1}^{n} B_j \leq I_{\mathscr{H}}$, σ_f be an operator mean with the representing function f and $p \in [0, 1]$. Then*

$$\left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right) \sigma_{f^p} \left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)$$
$$\leq \left(\left(I_{\mathscr{H}} - \sum_{j=1}^{n} t_j A_j\right) \sigma_f \left(I_{\mathscr{H}} - \sum_{j=1}^{n} t_j B_j\right) - \sum_{j=1}^{n} (1 - t_j) \left(A_j \sigma_f B_j\right) \right)^p$$
$$\leq \left(I_{\mathscr{H}} - \sum_{j=1}^{n} (A_j \sigma_f B_j)\right)^p,$$

where $t_j \in [0, 1]$ ($j = 1, \ldots, n$).

Proof. Let $t_j \in [0, 1]$ ($j = 1, \ldots, n$). It follows from $I_{\mathscr{H}} - \sum_{j=1}^{n} t_j A_j \geq \sum_{j=1}^{n} (1 - t_j) A_j$, $I_{\mathscr{H}} - \sum_{j=1}^{n} t_j B_j \geq \sum_{j=1}^{n} (1 - t_j) B_j$ and inequality (3.1) that

$$\left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right) \sigma_f \left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)$$
$$= \left(\left(I_{\mathscr{H}} - \sum_{j=1}^{n} t_j A_j\right) - \sum_{j=1}^{n} (1 - t_j) A_j \right) \sigma_f \left(\left(I_{\mathscr{H}} - \sum_{j=1}^{n} t_j B_j\right) - \sum_{j=1}^{n} (1 - t_j) B_j \right)$$
$$\leq \left(\left(I_{\mathscr{H}} - \sum_{j=1}^{n} t_j A_j\right) \sigma_f \left(I_{\mathscr{H}} - \sum_{j=1}^{n} t_j B_j\right) - \sum_{j=1}^{n} \left((1 - t_j) A_j \sigma_f (1 - t_j) B_j\right) \right)$$

(by (3.1))

$$= \left(\left(I_{\mathscr{H}} - \sum_{j=1}^{n} t_j A_j \right) \sigma_f \left(I_{\mathscr{H}} - \sum_{j=1}^{n} t_j B_j \right) - \sum_{j=1}^{n} (1 - t_j) \left(A_j \sigma_f B_j \right) \right)$$

(by property (iii) of operator means)

$$\leq \left((I_{\mathscr{H}} \sigma_f I_{\mathscr{H}}) - \sum_{j=1}^{n} (t_j A_j \sigma_f t_j B_j) - \sum_{j=1}^{n} (1 - t_j) \left(A_j \sigma_f B_j \right) \right) \quad \text{(by (3.1))}$$

$$= \left(I_{\mathscr{H}} - \sum_{j=1}^{n} t_j (A_j \sigma_f B_j) - \sum_{j=1}^{n} (1 - t_j) \left(A_j \sigma_f B_j \right) \right)$$

(by property (iii) of operator means)

$$= \left(I_{\mathscr{H}} - \sum_{j=1}^{n} \left(A_j \sigma_f B_j \right) \right). \tag{3.3}$$

Using Lemma 3.2, the operator monotone function $g(t) = t^p$ $(p \in [0, 1])$ and inequality (3.3) we get the desired result. □

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