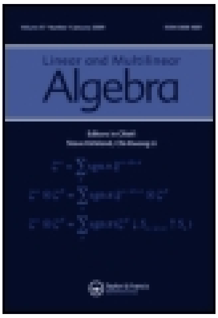

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Reverses and variations of Heinz inequality

Mojtaba Bakherad[a] & Mohammad Sal Moslehian[a]
[a] Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, Mashhad, Iran.
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Reverses and variations of Heinz inequality

Mojtaba Bakherad and Mohammad Sal Moslehian*

[a] *Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, Mashhad, Iran*

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Let A, B be positive definite $n \times n$ matrices. We present several reverse Heinz-type inequalities, in particular

$$\|AX + XB\|_2^2 + 2(\nu - 1)\|AX - XB\|_2^2 \le \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|_2^2,$$

where X is an arbitrary $n \times n$ matrix, $\|\cdot\|_2$ is Hilbert–Schmidt norm and $\nu > 1$. We also establish a Heinz-type inequality involving Hadamard product of the form

$$\begin{aligned} 2|||A^{\frac{1}{2}} \circ B^{\frac{1}{2}}||| &\le |||A^s \circ B^{1-t} + A^{1-s} \circ B^t||| \\ &\le \max\{|||(A+B) \circ I|||, |||(A \circ B) + I|||\}, \end{aligned}$$

in which $s, t \in [0, 1]$ and $|||\cdot|||$ is a unitarily invariant norm.

Keywords: Heinz inequality; Hilbert–Schmidt norm; operator mean; Hadamard product

AMS Subject Classifications: Primary: 47A63; Secondary: 47A60; 15A60; 15A42

1. Introduction and preliminaries

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. In the case when $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive (positive semidefinite for matrices) if $\langle Ax, x\rangle \ge 0$ for all $x \in \mathscr{H}$. The set of all positive invertible operators (respectively, positive definite matrices) is denoted by $\mathbb{B}(\mathscr{H})_{++}$ (respectively, $\mathcal{P}_n$).

The Gelfand map $f(t) \mapsto f(A)$ is an isometrically $*$-isomorphism between the C^*-algebra $C(\sigma(A))$ of all continuous functions on the spectrum $\sigma(A)$ of a self-adjoint operator A and the C^*-algebra generated by A and the identity operator I such that if $f, g \in C(\sigma(A))$, then $f(t) \ge g(t)$ $(t \in \sigma(A))$ implies that $f(A) \ge g(A)$.

If $\{e_j\}$ is an orthonormal basis of $\mathscr{H}$, $V : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is the isometry defined by $Ve_j = e_j \otimes e_j$ and $A \otimes B$ is the tensor product of operators A, B, then the Hadamard product $A \circ B$ regarding to $\{e_j\}$ is expressed by $A \circ B = V^*(A \otimes B)V$.

*Corresponding author. Email: moslehian@um.ac.ir

A unitarily invariant norm $|||\cdot|||$ is defined on a norm ideal $\mathfrak{L}_{|||\cdot|||}$ of $\mathbb{B}(\mathscr{H})$ associated with it and has the property $|||UXV||| = |||X|||$, where U and V are arbitrary unitaries in $\mathbb{B}(\mathscr{H})$ and $X \in \mathfrak{L}_{|||\cdot|||}$. A compact operator $A \in \mathbb{B}(\mathscr{H})$ is called Hilbert–Schmidt if $\|A\|_2 = \left(\sum_{j=1}^{\infty} s_j^2(A)\right)^{1/2} < \infty$, where $s_1(A), s_2(A), \cdots$ are the singular values of A, i.e. the eigenvalues of the positive operator $|A| = (A^*A)^{\frac{1}{2}}$ enumerated as $s_1(A) \geq s_2(A) \geq \cdots$ with their multiplicities counted. The Hilbert–Schmidt norm is a unitarily invariant norm. For $A = [a_{ij}] \in \mathbb{M}_n$, it holds that $\|A\|_2 = \left(\sum_{i,j=1}^{n} |a_{i,j}|^2\right)^{1/2}$. For two operators $A, B \in \mathbb{B}(\mathscr{H})_{++}$, let $A\sharp_\mu B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\mu}A^{\frac{1}{2}}$ $(\mu \in \mathbb{R})$. The operators $A\sharp_{\frac{1}{2}}B$ and $A\nabla B = \frac{A+B}{2}$ are called the operator geometric mean and the operator arithmetic mean, respectively.

The Heinz mean is defined by

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu}b^\nu}{2} \qquad (0 \leq \nu \leq 1,\ a, b > 0).$$

The function H_ν is symmetric about the point $\nu = \frac{1}{2}$. Note that $H_0(a,b) = H_1(a,b) = \frac{a+b}{2}$, $H_{1/2}(a,b) = \sqrt{ab}$ and $H_{1/2}(a,b) \leq H_\nu(a,b) \leq H_0(a,b)$ for all $\nu \in [0,1]$.

The Heinz norm (double) inequality, which is one of the essential inequalities in operator theory, states that for any positive operators $A, B \in \mathbb{B}(\mathscr{H})$, any operator $X \in \mathbb{B}(\mathscr{H})$ and any $\nu \in [0,1]$, the double inequality

$$2\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leq \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\| \leq \|AX + XB\| \tag{1.1}$$

holds; see [1]. Bhatia and Davis [2] proved that (1.1) is valid for any unitarily invariant norm. Fujii et al. [3] proved that the right-hand side inequality at (1.1) is equivalent to several other norm inequalities such as

(i) the McIntosh inequality [4] asserting that $\|A^*AX + XB^*B\| \geq 2\|AXB^*\|$ for all $A, B, X \in \mathbb{B}(\mathscr{H})$;
(ii) the Corach–Porta–Recht inequality $\|AXA^{-1} + A^{-1}XA\| \geq 2\|X\|$, where $A \in \mathbb{B}(\mathscr{H})$ is selfadjoint and invertible and $X \in \mathbb{B}(\mathscr{H})$ (see also [5]), and
(iii) the inequality $\|A^{2m+n}XB^{-n} + A^{-n}XB^{2m+n}\| \geq \|A^{2m}X + XB^{2m}\|$ in which A, B are invertible self-adjoint operators, X is an arbitrary operator in $\mathbb{B}(\mathscr{H})$ and both m and n are nonnegative integers; see also Section 3.9 of the monograph.[6]

Audenaert [7] gave a singular value inequality for the Heinz means of matrices as follows: If $A, B \in \mathbb{M}_n$ are positive semidefinite and $\nu \in [0,1]$, then

$$s_j(A^\nu B^{1-\nu} + A^{1-\nu}B^\nu) \leq s_j(A+B).$$

Kittaneh and Manasrah [8] showed a refinement of the right-hand side of inequality (1.1) for the Hilbert–Schmidt norm as follows:

$$\|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|_2^2 + 2r_0\|AX - XB\|_2^2 \leq \|AX + XB\|_2^2, \tag{1.2}$$

in which $A, B, X \in \mathbb{M}_n$ such that A, B are positive semidefinite, $\nu \in [0, 1]$ and $r_0 = \min\{\nu, 1-\nu\}$. Kaur et al. [9], using the convexity of the function $f(\nu) = |||A^{1-\nu}XB^{\nu} + A^{\nu}XB^{1-\nu}|||$ $(\nu \in [0, 1])$ presented more refinements of the Heinz inequality. More precisely, for $A, B, X \in \mathbb{M}_n$ such that A, B are positive semidefinite and $\nu \in [0, 1]$, they showed the inequality

$$|||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}||| \leq |||4r_1 A^{\frac{1}{2}}XB^{\frac{1}{2}} + (1-2r_1)(AX + XB)|||,$$

where $r_1 = \left\{\nu, |\frac{1}{2} - \nu|, 1-\nu\right\}$. It is shown in [10] a reverse of inequality (1.2) as

$$\|AX + XB\|_2^2 \leq \|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\|_2^2 + 2r_0\|AX - XB\|_2^2, \tag{1.3}$$

where $A, B, X \in \mathbb{M}_n$ such that A, B are positive semidefinite, $\nu \in [0, 1]$ and $r_0 = \max\{\nu, 1-\nu\}$. Singh and Aujla [11] showed that

$$2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \leq |||A^{s}XB^{1-t} + A^{1-s}XB^{t}|||,$$

where $A, B, X \in \mathbb{M}_n$ such that A and B are positive semidefinite, $s, t \in [0, 1]$. It is remarkable that, by using the fact that the function $g(s, t) = |||A^{s}XB^{1-t} + A^{1-s}XB^{t}|||$ attains its maximum at the vertices of the square $[0, 1] \times [0, 1]$, one can see that under the same conditions as above

$$|||A^{s}XB^{1-t} + A^{1-s}XB^{t}||| \leq \max\{|||AX + XB|||, |||AXB + X|||\},$$

Recently, Krnić et al. used the Jensen functional to improve several Heinz-type inequalities. [12]

In this paper, we obtain a reverse of (1.2) and some other operator inequalities. We also show some results on the Hadamard product. In particular, we get the following Heinz-type inequality

$$2|||A^{\frac{1}{2}} \circ B^{\frac{1}{2}}||| \leq |||A^{s} \circ B^{1-t} + A^{1-s} \circ B^{t}||| \leq \max\{|||(A + B) \circ I|||, |||(A \circ B) + I|||\},$$

where $A, B \in \mathcal{P}_n$, $X \in \mathbb{M}_n$ and $s, t \in [0, 1]$.

2. A reverse of the Heinz inequality for matrices

In this section, we present a converse of the Heinz inequality and give several refinements for matrices.

Lemma 2.1 *Let $a, b > 0$ and $\nu \notin [0, 1]$. Then,*

$$a + b \leq a^{\nu}b^{1-\nu} + b^{\nu}a^{1-\nu}. \tag{2.1}$$

Proof Let $\nu \notin [0, 1]$. Assume that $f(t) = t^{1-\nu} - \nu + (\nu - 1)t$ $(t \in (0, \infty))$. It is easy to see that $f(t)$ has a minimum at $t = 1$ in the interval $(0, \infty)$. Hence, $f(t) \geq f(1) = 0$ for all $t > 0$. Assume that $a, b > 0$. Letting $t = \frac{b}{a}$, we get

$$\nu a + (1-\nu)b \leq a^{\nu}b^{1-\nu}. \tag{2.2}$$

Applying (2.2), we obtain

$$\nu a+(1-\nu)b \le a^{\nu}b^{1-\nu} \quad \text{and} \quad \nu b+(1-\nu)a \le b^{\nu}a^{1-\nu},$$

whence

$$a+b \le a^{\nu}b^{1-\nu}+b^{\nu}a^{1-\nu}.$$

□

For $\nu \notin [0,1]$, if we replace ν by $\nu/(2\nu-1)$ and A, B, X by $A^{2\nu-1}$, $B^{2\nu-1}$, $A^{1-\nu}XB^{1-\nu}$ in (1.1), respectively, then we reach the following Theorem, complementary to the right inequality in (1.1).

THEOREM 2.2 *Let $A, B \in \mathcal{P}_n$, $X \in \mathbb{M}_n$ and $\nu \notin [0,1]$. Then,*

$$|||AX+XB||| \le \left|\left|\left|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\right|\right|\right|.$$

In the next theorem, we show a reverse of (1.2). First, we need the following lemma.

LEMMA 2.3 *Let $a, b > 0$ and $\nu \notin [\frac{1}{2}, 1]$. Then,*

(i) $\nu a+(1-\nu)b+(\nu-1)(\sqrt{a}-\sqrt{b})^2 \le a^{\nu}b^{1-\nu}$
(ii) $(a+b)+2(\nu-1)(\sqrt{a}-\sqrt{b})^2 \le a^{\nu}b^{1-\nu}+b^{\nu}a^{1-\nu}$
(iii) $(a+b)^2+2(\nu-1)(a-b)^2 \le (a^{\nu}b^{1-\nu}+b^{\nu}a^{1-\nu})^2$.

Proof Let $a, b > 0$ and $\nu \notin \left[\frac{1}{2}, 1\right]$.

(i) By inequality (2.2),

$$\begin{aligned}\nu a+(1-\nu)b+(\nu-1)(\sqrt{a}-\sqrt{b})^2 &= (2-2\nu)\sqrt{ab}+(2\nu-1)a \\ &\le (\sqrt{ab})^{2-2\nu}a^{2\nu-1} = a^{\nu}b^{1-\nu}.\end{aligned}$$

(ii) It can be proved in a similar fashion as (ii).
(iii) It follows from (ii) by replacing a by a^2 and b by b^2. □

THEOREM 2.4 *Suppose that $A, B \in \mathcal{P}_n$, $X \in \mathbb{M}_n$ and $\nu > 1$. Then,*

$$\|AX+XB\|_2^2+2(\nu-1)\|AX-XB\|_2^2 \le \|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\|_2^2.$$

Proof By the spectral decomposition [13, Theorem 3.4], there are unitary matrices U, $V \in \mathbb{M}_n$ such that $A = U\Lambda U^*$ and $B = V\Gamma V^*$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_n)$, and λ_j, γ_j $(j = 1, \cdots, n)$ are eigenvalues of A and B, respectively. These numbers are positive. If $Z = U^*XV = \left[z_{ij}\right]$, then

$$AX+XB = U\Big(\Lambda Z+Z\Gamma\Big)V^* = U\Big[\Big(\lambda_i+\gamma_j\Big)z_{ij}\Big]V^*, \tag{2.3}$$

$$AX-XB = U\Lambda U^*X-XV\Gamma V^* = U\Big[\Lambda Z-Z\Gamma\Big]V^* = U\Big[\Big(\lambda_i-\gamma_j\Big)z_{ij}\Big]V^* \tag{2.4}$$

and

$$\begin{aligned} A^{\nu} X B^{1-\nu} + A^{1-\nu} X B^{\nu} &= U \Lambda^{\nu} U^* X V \Gamma^{1-\nu} V^* + U \Lambda^{1-\nu} U^* X V \Gamma^{\nu} V^* \\ &= U \Lambda^{\nu} Z \Gamma^{1-\nu} V^* + U \Lambda^{1-\nu} Z \Gamma^{\nu} V^* \\ &= U \Big[\Lambda^{\nu} Z \Gamma^{1-\nu} + \Lambda^{1-\nu} Z \Gamma^{\nu} \Big] V^* \\ &= U \Big[\Big(\lambda_i^{\nu} \gamma_j^{1-\nu} + \lambda_i^{1-\nu} \gamma_j^{\nu} \Big) z_{ij} \Big] V^*. \end{aligned} \tag{2.5}$$

It follows from (2.3), (2.3) and (2.5) that

$$\begin{aligned} &\|AX + XB\|_2^2 + 2(\nu - 1)\|AX - XB\|_2^2 \\ &\quad = \sum_{i,j=1}^{n} \Big(\lambda_i + \gamma_j\Big)^2 |z_{ij}|^2 + 2(\nu - 1) \sum_{i,j=1}^{n} \Big(\lambda_i - \mu_j\Big)^2 |z_{ij}|^2 \\ &\quad \le \sum_{i,j=1}^{n} \Big(\lambda_i^{\nu} \gamma_j^{1-\nu} + \lambda_i^{1-\nu} \gamma_j^{\nu}\Big)^2 |z_{ij}|^2 \quad \text{(by Lemma 2.3 (iii))} \\ &\quad = \|A^{\nu} X B^{1-\nu} + A^{1-\nu} X B^{\nu}\|_2^2. \end{aligned}$$

□

Remark 2.5 Utilizing Lemma 2.3, one can easily see that Theorem 2.4 holds for $\nu < \frac{1}{2}$. The case $\nu < \frac{1}{2}$ is not interesting since the left-hand side is less precise than the left-hand side of Theorem 2.2, but the case of $0 \le \nu \le \frac{1}{2}$ coincides with inequality (1.3).

Theorem 2.4 yields the next two corollaries.

Corollary 2.6 *Suppose that* $A, B \in \mathcal{P}_n$, $X \in \mathbb{M}_n$ *and* $\nu > 1$. *Then,*

$$\|AX + XB\|_2 = \|A^{\nu} X B^{1-\nu} + A^{1-\nu} X B^{\nu}\|_2$$

if and only if $AX = XB$.

Proof If $AX = XB$, then $A^{\nu} X = X B^{\nu}$ and $A^{1-\nu} X = X B^{1-\nu}$. Hence,

$$\|A^{\nu} X B^{1-\nu} + A^{1-\nu} X B^{\nu}\|_2 = \|A^{\nu} A^{1-\nu} X + X B^{1-\nu} B^{\nu}\|_2 = \|AX + XB\|_2.$$

Conversely, assume that $\|AX + XB\|_2 = \|A^{\nu} X B^{1-\nu} + A^{1-\nu} X B^{\nu}\|_2$. It follows from Theorem 2.4 that $\|AX - XB\|_2 = 0$. Thus $AX = XB$. □

Corollary 2.7 *Let* $A, B \in \mathcal{P}_n$ *and* $\nu > 1$. *Then,*

$$s_j(A + B) = s_j(A^{\nu} B^{1-\nu} + A^{1-\nu} B^{\nu}) \qquad (j = 1, 2, \cdots, n)$$

if and only if $A = B$.

Proof If $A = B$, then $A + B = A^{\nu} B^{1-\nu} + A^{1-\nu} B^{\nu}$. Conversely, assume that $s_j(A + B) = s_j(A^{\nu} B^{1-\nu} + A^{1-\nu} B^{\nu})$ $(j = 1, 2, \cdots, n)$. Then, $\|AX + XB\|_2 = \|A^{\nu} X B^{1-\nu} + A^{1-\nu} X B^{\nu}\|_2$. It follows from Corollary 2.6 that $A = B$. □

3. A reverse of the Heinz inequality for operators

In this section, we obtain a reverse of the Heinz inequality for two positive invertible operators as well as some other operator inequalities.

In [14], the authors investigated an operator version of the classical Heinz mean, i.e. the operator

$$H_\nu(A,B)=\frac{A\sharp_\nu B+A\sharp_{1-\nu}B}{2}, \tag{3.1}$$

where $A,B\in\mathbb{B}(\mathscr{H})_{++}$, and $\nu\in[0,1]$. As in the real case, this mean interpolates between arithmetic and geometric mean, that is,

$$A\sharp B\le H_\nu(A,B)\le A\nabla B.$$

On the other hand, since $A,B\in\mathbb{B}(\mathscr{H})_{++}$, the expression (3.1) is also well defined for $\nu\notin[0,1]$. Using inequality (2.2) and the functional calculus for $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ we get the following result.

$$H_{1-\nu}(A,B)=\frac{A\sharp_{1-\nu}B+A\sharp_\nu B}{2}\ge\frac{A\nabla_{1-\nu}B+A\nabla_\nu B}{2}=A\nabla B, \tag{3.2}$$

where $A,B\in\mathbb{B}(\mathscr{H})_{++}$ and $\nu\notin[0,1]$. Applying Lemma 2.3 (ii), we have a refinement of inequality (3.2).

Theorem 3.1 *Let $A,B\in\mathbb{B}(\mathscr{H})_{++}$ and $\nu>1$. Then,*

$$A\nabla B+2(\nu-1)(A\nabla B-A\sharp_{1/2}B)\le H_{1-\nu}(A,B)\,.$$

Proof By Lemma 2.3 (ii), we have $\frac{1+t}{2}+2(\nu-1)(t-2\sqrt{t}+1)\le\frac{t^{1-\nu}+t^\nu}{2}$ $(t>0)$. Hence,

$$\frac{(1+A^{-\frac{1}{2}}BA^{-\frac{1}{2}})}{2}+2(\nu-1)(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}+1)$$
$$\le\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\nu}+(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\nu}}{2}. \tag{3.3}$$

Multiplying $A^{\frac{1}{2}}$ by the both sides of (3.3) we get

$$A\nabla B+2(\nu-1)(A\nabla B-A\sharp_{1/2}B)\le\frac{A\sharp_{1-\nu}B+A\sharp_\nu B}{2}=H_{1-\nu}(A,B)\,.$$ □

Remark 3.2 Theorem 3.1 also holds for $\nu<\frac{1}{2}$. The case when $\nu<\frac{1}{2}$ is not interesting, since it is less precise than inequality (3.2), but the case of $0\le\nu\le\frac{1}{2}$ coincides with the inequality at [14, Corollary 2].

Applying Theorem 3.1, we get immediately the following result.

Corollary 3.3 *Let $A,B\in\mathbb{B}(\mathscr{H})_{++}$ and $\nu>1$. Then,*

$$H_{1-\nu}(A,B)=A\nabla B$$

if and only if $A=B$.

Applying Lemma 2.1 we get

$$a + a^{-1} \leq a^{\nu} + a^{-\nu} \qquad (a > 0,\ \nu > 1).$$

Utilizing this inequality, the functional calculus for $A \otimes B^{-1}$ and the definition of the Hadamard product we get the following result.

PROPOSITION 3.4 *Let $A, B \in \mathbb{B}(\mathscr{H})_{++}$ and $\nu > 1$. Then,*

(i) $A \otimes B^{-1} + A^{-1} \otimes B \leq A^{\nu} \otimes B^{-\nu} + A^{-\nu} \otimes B^{\nu}$
(ii) $A \circ B^{-1} + A^{-1} \circ B \leq A^{\nu} \circ B^{-\nu} + A^{-\nu} \circ B^{\nu}$.

4. Some Heinz-type inequality related to Hadamard product

In this section, using some ideas of [15] and [11], we show some Heinz-type inequalities.

LEMMA 4.1 [16, Theorem 1.1.3] *Let $A, B \in \mathcal{P}_n$ and $X \in \mathbb{M}_n$. Then, the block matrix $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is positive semidefinite if and only if $A \geq XB^{-1}X^*$.*

THEOREM 4.2 *The two variables function*

$$H(s,t) = A^{1+s} \otimes B^{1-t} + A^{1-s} \otimes B^{1+t}$$

is convex on $[-1,1] \times [-1,1]$ and attains its minimum at $(0,0)$ for all $A, B \in \mathcal{P}_n$.

Proof Since H is continuous, it is enough to prove

$$H(s_1,t_1) \leq \frac{1}{2}(H(s_1+s_2, t_1+t_2) + H(s_1-s_2, t_1-t_2))$$

for all $s_1 \pm s_2, t_1 \pm t_2 \in [0,1]$; see [11]. For $A, B \in \mathcal{P}_n$ and $s_1 \pm s_2, t_1 \pm t_2 \in [0,1]$ it follows from Lemma 4.1 that the matrices $\begin{pmatrix} A^{1+s_1+s_2} & A^{1+s_1} \\ A^{1+s_1} & A^{1+(s_1-s_2)} \end{pmatrix}$, $\begin{pmatrix} A^{1-(s_1+s_2)} & A^{1-s_1} \\ A^{1-s_1} & A^{1-(s_1-s_2)} \end{pmatrix}$, $\begin{pmatrix} B^{1+t_1+t_2} & B^{1+t_1} \\ B^{1-t_1} & B^{1+(t_1-t_2)} \end{pmatrix}$ and $\begin{pmatrix} B^{1-(t_1+t_2)} & B^{1-t_1} \\ B^{1-t_1} & B^{1-(t_1-t_2)} \end{pmatrix}$ are positive semidefinite. Hence, the matrices

$$X = \begin{pmatrix} A^{1+s_1+s_2} \otimes B^{1-(t_1+t_2)} + A^{1-(s_1+s_2)} \otimes B^{1+t_1+t_2} & A^{1+s_1} \otimes B^{1-t_1} + A^{1-s_1} \otimes B^{1+t_1} \\ A^{1+s_1} \otimes B^{1-t_1} + A^{1-s_1} \otimes B^{1+t_1} & A^{1+(s_1-s_2)} \otimes B^{1+(t_1-t_2)} + A^{1-(s_1-s_2)} \otimes B^{1-(t_1-t_2)} \end{pmatrix}$$

is positive semidefinite. Similarly,

$$Y = \begin{pmatrix} A^{1+(s_1-s_2)} \otimes B^{1+(t_1-t_2)} + A^{1-(s_1-s_2)} \otimes B^{1-(t_1-t_2)} & A^{1+s_1} \otimes B^{1-t_1} + A^{1-s_1} \otimes B^{1+t_1} \\ A^{1+s_1} \otimes B^{1-t_1} + A^{1-s_1} \otimes B^{1+t_1} & A^{1+s_1+s_2} \otimes B^{1-(t_1+t_2)} + A^{1-(s_1+s_2)} \otimes B^{1+t_1+t_2} \end{pmatrix}$$

is positive semidefinite. Thus,

$$X + Y = \begin{pmatrix} H(s_1+s_2, t_1+t_2) + H(s_1-s_2, t_1-t_2) & 2H(s_1,t_1) \\ 2H(s_1,t_2) & H(s_1+s_2, t_1+t_2) + H(s_1-s_2, t_1-t_2) \end{pmatrix}$$

is positive semidefinite and therefore

$$\begin{pmatrix} I_n & -I_n \\ 0 & 0 \end{pmatrix} (X+Y) \begin{pmatrix} I_n & 0 \\ -I_n & 0 \end{pmatrix}$$

is positive semidefinite. Hence, $H(s_1+s_2, t_1+t_2) + H(s_1-s_2, t_1-t_2) - 2H(s_1, t_1) \geq 0$, which proves the convexity of H. Further note that $H(s,t) = H(-s,-t)$, $s, t \in [0,1]$. This together with the convexity of H imply that H attains its minimum at $(0,0)$. □

If in Theorem 4.2 we replace s, t, A, B by $2s-1, 2t-1, A^{\frac{1}{2}}, B^{\frac{1}{2}}$, respectively, we reach the following result.

Corollary 4.3 *The two variable's function*

$$K(s,t) = A^s \circ B^{1-t} + A^{1-s} \circ B^t \quad (A, B \in \mathcal{P}_n)$$

is convex on $[0,1] \times [0,1]$ *and attains its minimum at* $(\frac{1}{2}, \frac{1}{2})$.

Singh and Aujla [15] showed that

$$2|||A^{\frac{1}{2}} \circ B^{\frac{1}{2}}||| \leq |||A^t \circ B^{1-t} + A^{1-t} \circ B^t||| \leq |||A+B|||,$$

where $A, B \in \mathcal{P}_n$ and $t \in [0,1]$. Now, we are ready to state our last result.

Corollary 4.4 *Let* $A, B \in \mathcal{P}_n$ *and* $s, t \in [0,1]$. *Then,*

$$2|||A^{\frac{1}{2}} \circ B^{\frac{1}{2}}||| \leq |||A^s \circ B^{1-t} + A^{1-s} \circ B^t||| \leq \max\{|||(A+B) \circ I|||, |||(A \circ B) + I|||\}.$$

Proof Let $K(s,t) = A^s \circ B^{1-t} + A^{1-s} \circ B^t$. If we put $G(s,t) = |||K(s,t)|||$, then by the convexity of K and Fan Dominance Theorem [16, p.58] (see also [17]), the function G is convex on $[0,1] \times [0,1]$, and attains minimum at $(\frac{1}{2}, \frac{1}{2})$. Hence, we have the first inequality. In addition, since the function G is continuous and convex on $[0,1] \times [0,1]$, it follows that G attains its maximum at the vertices of the square. Moreover, due to the symmetry there are only two possibilities for the maximum. □

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