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Refined Berezin number inequalities via superquadratic and convex functions

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Abstract. In this paper, we generalize and refine some Berezin number inequalities for Hilbert space operators. Namely, we refine the Hermite-Hadamard inequality and some other recent results by using the concept of superquadraticity and convexity. Then we extend these inequalities for the Berezin number. Among other inequalities, it is shown that if $S, T \in \mathcal{L}(\mathcal{H}(\Omega))$ such that $\mathbf{ber}(\mathbf{T}) \leq \mathbf{ber}(|\mathbf{S}|)$ and f is a non-negative superquadratic function, then

$$f\left(\mathbf{ber}\left(T\right)\right) \leq \mathbf{ber}(f\left(|S|\right)) - \ell_{\mathbf{ber}}\left(f\left(\left||S| - \mathbf{ber}\left(T\right)\right|\right)\right).$$

1. Introduction and preliminaries

Let $\mathcal{H}$ be a complex Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. Let $\mathcal{L}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators from $\mathcal{H}$ into itself, with the identity $I_{\mathcal{H}}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called positive if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and we then write $T \geq 0$. A linear map $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ positive linear map, is positive if $\Phi(T) \geq 0$ whenever $T \geq 0$. It is said to be normalized if $\Phi(I_{\mathcal{H}}) = I_{\mathcal{H}}$. For $S, T \in \mathcal{L}(\mathcal{H})$ assume that $S \oplus T$ is the operator matrix $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$ on $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$.

A functional Hilbert space is a Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on a (non-empty) set $\Omega \subset \mathbb{C}$, which has the property that point evaluations are continuous, i.e., for each $\lambda \in \Omega$ the map $f \longmapsto f(\lambda)$ is a continuous (linear) functional on $\mathcal{H}$. Then the Riesz representation theorem ensures that for each $\lambda \in \Omega$ there is a unique element k_λ of $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda\rangle$ for all $f \in \mathcal{H}$. The collection $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of $\mathcal{H}$. If $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by $k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z)$. For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathcal{H}$. For a bounded linear operator T on $\mathcal{H}$, the function $\widetilde{T}$ defined on Ω

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by $\widetilde{T}(\lambda) = \langle T\hat{k}_\lambda, \hat{k}_\lambda \rangle$, is the Berezin transform of T, which was introduced by Berezin [7, 8]. The Berezin set and the Berezin number of the operator T are defined by

$$\mathbf{Ber}(T) = \{\tilde{T}(\lambda) : \lambda \in \Omega\} \quad \text{and} \quad \mathbf{ber}(T) = \sup\{|\tilde{T}(\lambda)| : \lambda \in \Omega\},$$

respectively. For an operator $T \in \mathcal{L}(\mathcal{H}(\Omega))$ we define

$$\|T\|_{\mathbf{ber}} = \sup\{|\langle T\hat{k}_{\lambda_1}, \hat{k}_{\lambda_2}\rangle| : \lambda_1, \lambda_2 \in \Omega\} \quad \text{and} \quad \ell_{\mathbf{ber}}(T) = \inf\{|\langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle| : \lambda \in \Omega\}.$$

Clearly, the Berezin symbol $\tilde{T}$ is a bounded function on Ω whose values lie in the numerical range of the operator T, and hence

$$\mathbf{Ber}(T) \subseteq W(T) \text{ and } \mathbf{ber}(T) \leq w(T).$$

The Berezin number of operators S and T satisfies the following properties(see [2–5, 10, 13, 27] and the references therein):
(i) $\mathbf{ber}(I_{\mathcal{H}}) = 1$;
(ii)If $T \in \mathcal{L}(\mathcal{H}(\Omega))$ is selfadjoint, then $\widetilde{T}(\lambda) \in \mathbb{R}$;
(iii)If $T \in \mathcal{L}(\mathcal{H}(\Omega))$ is positive, then $\widetilde{T}(\lambda) \geq 0$;
(iv)$\mathbf{ber}(\beta T) = |\beta|\mathbf{ber}(T)$ for all $\beta \in \mathbb{C}$;
(v) $\mathbf{ber}(T + S) \leq \mathbf{ber}(T) + \mathbf{ber}(S)$.

The Berezin symbol has been studied in detail for Toeplitz and Hankel operators on Hardy and Bergman spaces. A nice property of the Berezin symbol is mentioned next. If $\tilde{S}(\lambda) = \tilde{T}(\lambda)$ for all $\lambda \in \Omega$, then $S = T$. Therefore, the Berezin symbol uniquely determines the operator. Some excellent results about the Berezin number were found in [11, 12, 16–18, 25, 26] very recently.

Moreover, if $T \in \mathcal{L}(\mathcal{H}(\Omega))$, then

$$|\langle T\hat{k}_\lambda, \hat{k}_\mu\rangle|^2 \leq \langle |T|^{2\nu}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |T^*|^{2(1-\nu)}\hat{k}_\mu, \hat{k}_\mu\rangle \tag{1}$$

for all $\hat{k}_\lambda, \hat{k}_\mu \in \mathcal{H}(\Omega)$ and $\nu \in [0, 1]$; see [19].

A function $f : [0, \infty) \to \mathcal{R}$ is called superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathcal{R}$ such that

$$f(t) \geq f(x) + C_x(t - x) + f(|t - x|) \tag{2}$$

for all $t \geq 0$. We say that f is subquadratic if $-f$ is superquadratic. Thus, for a superquadratic function we require that f lie above its tangent line plus a translation of f itself [1]. Moreover, the following results hold for superquadratic function.

Lemma 1.1. *[1] Let f be a differentiable superquadratic function and $f(0) = f'(0) = 0$, then $C_x = f'(x)$ for all $x \geq 0$. Also, if f is differentiable superquadratic such that $f(x) \geq 0$ $(x \geq 0)$, then f is convex, $f(0) = f'(0) = 0$ and if C_s in* (2) *is non-negative.*

Remark 1.2. *A superquadratic function looks to be stronger than convex function itself but if f takes negative values then it may be considered as a weaker function. If f is superquadratic and non-negative, then f is convex and increasing. Also, if f' is convex and $f(0) = f'(0) = 0$, then f is superquadratic. The converse is not true. For more information about see [1] and references therein.*

A real valued continuous function $f(t)$ on $[0, \infty)$ is superquadratic if and only if

$$f(\alpha a + (1 - \alpha) b) \leq \alpha [f(a) - f((1 - \alpha)|a - b|)] + (1 - \alpha)[f(b) - f(\alpha|a - b|)] \tag{3}$$

holds for all $\alpha \in [0, 1]$, and for all $a, b \in [0, \infty)$, see [1].

The Hermite-Hadamard inequality involving convex function asserts that

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f((1-t)a+tb)\,dt \leq \frac{f(a)+f(b)}{2}, \tag{4}$$

where $f : J \longrightarrow \mathbb{R}$ is a convex function on the interval J and $a, b \in J$. We have the following version of the Hermite-Hadamard inequality for superquadratic functions.

Theorem 1.3. *[6] Let $f : [0, \infty) \to \mathcal{R}$ be an integrable superquadratic function and $0 \leq a \leq b$. Then*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f\left(\left|t - \frac{a+b}{2}\right|\right)dt \\ & \leq \int_0^1 f\left(ta+(1-t)b\right)dt \\ & \leq \frac{f(a)+f(b)}{2} - \frac{1}{(b-a)^2}\int_a^b \left[(b-a) f(t-a) + (t-a) f(b-t)\right]dt. \end{aligned} \tag{5}$$

We note that, in the left hand side of (5) we have

$$\begin{aligned} \frac{1}{b-a}\int_a^b f\left(\left|t-\frac{a+b}{2}\right|\right)dt &= \int_0^1 f\left(\left|ta+(1-t)b-\frac{a+b}{2}\right|\right)dt \\ &= \int_0^1 f\left(\left|t-\frac{1}{2}\right||a-b|\right)dt. \end{aligned} \tag{6}$$

For recent results concerning supequadraticity the reader may refer to the comprehensive book [23].

Recently, the authors [21] and [22] proved the following operator version of Jensen's inequality for superquadratic functions.

Theorem 1.4. *Let $T \in \mathcal{L}(\mathcal{H}(\Omega))$ be a positive operator and $\hat{k}_\lambda \in \mathcal{H}(\Omega)$. If $f : [0, \infty) \to \mathcal{R}$ is superquadratic and $\Phi : \mathcal{L}(\mathcal{H}(\Omega)) \to \mathcal{L}(\mathcal{H}(\Omega))$ is a normalized positive linear map, then*

$$\left\langle f(T)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \geq f\left(\left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) + \left\langle f\left(\left|T - \left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle I_{\mathcal{H}}\right|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle. \tag{7}$$

Moreover, we have

$$\left\langle \Phi(f(T))\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \geq f\left(\left\langle \Phi(T)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) + \left\langle \Phi\left(f\left(\left|T - \left\langle \Phi(T)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle I_{\mathcal{H}}\right|\right)\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle. \tag{8}$$

In this paper, we get some upper bounds for the Berezin number of the $f(\mathbf{ber}(T))$ and $f(\mathbf{ber}(S^*T))$ on reproducing kernel Hilbert spaces (RKHS), where $S, T \in \mathcal{L}(\mathcal{H}(\Omega))$ is arbitrary and f is a superquadratic increasing, by using some ideas from [22, 24]. We also present some inequalities involving positive linear maps. In the last section, we show a refinement of the Hermite-Hadamard inequality for convex function, and then assert it for the Berezin number.

2. Berezin number inequalities via superquadratic functions

In the first section, we show some Berezin number inequalities involving superquadratic functions. We demonstrate one of our results.

Theorem 2.1. *Let $T \in \mathcal{L}(\mathcal{H}(\Omega))$ and $f : [0,\infty) \to \mathcal{R}$ be increasing superquadratic on $[0,\infty)$. Then*

$$f(\mathbf{ber}(T)) \leq \mathbf{ber}\left(\int_0^1 f(t|T| + (1-t)|T^*|)\,dt\right) - \inf_{\lambda\in\Omega}\left\langle \int_0^1 f\left(\left|[t|T| + (1-t)|T^*|] - \left\langle [t|T| + (1-t)|T^*|]\hat{k}_\lambda, \hat{k}_\lambda\right\rangle 1_{\mathcal{H}}\right|\right) dt \hat{k}_\lambda, \hat{k}_\lambda \right\rangle - \inf_{\lambda\in\Omega}\int_0^1 f\left(\left|\left\langle (|T| - |T^*|)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\left|t - \frac{1}{2}\right|\right) dt. \tag{9}$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}(\Omega)$. It follows from f is increasing on $[0,\infty)$, inequality (1) for $\nu = \frac{1}{2}$ and the arithmetic-geometric inequality(AM-GM) that

$$\begin{aligned} f\left(\left|\left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right) &\leq f\left(\sqrt{\left\langle |T|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \left\langle |T^*|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle}\right) \\ &\leq f\left(\frac{\left\langle |T|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left\langle |T^*|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle}{2}\right) \qquad \text{(by AM-GM)} \\ &\leq \int_0^1 f\left(t\left\langle |T|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + (1-t)\left\langle |T^*|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) dt \\ &\quad - \int_0^1 f\left(\left|\left\langle (|T| - |T^*|)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\left|t - \frac{1}{2}\right|\right) dt. \qquad \text{(by (5))} \end{aligned} \tag{10}$$

Now, since f is superquadratic we have

$$\begin{aligned} & f\left(t\left\langle |T|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + (1-t)\left\langle |T^*|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) \\ &= f\left(\left\langle [t|T| + (1-t)|T^*|]\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) \\ &\leq \left\langle f(t|T| + (1-t)|T^*|)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\ &\quad - \left\langle f\left(\left|[t|T| + (1-t)|T^*|] - \left\langle [t|T| + (1-t)|T^*|]\hat{k}_\lambda, \hat{k}_\lambda\right\rangle I_{\mathcal{H}}\right|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \qquad \text{(by (7))}. \end{aligned}$$

Integrating the previous inequalities over t on $[0,1]$, we get

$$\begin{aligned} & \int_0^1 f\left(t\left\langle |T|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + (1-t)\left\langle |T^*|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) dt \\ &\leq \left\langle \int_0^1 f(t|T| + (1-t)|T^*|)\,dt\,\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\ &\quad - \left\langle \int_0^1 f\left(\left|[t|T| + (1-t)|T^*|] - \left\langle [t|T| + (1-t)|T^*|]\hat{k}_\lambda, \hat{k}_\lambda\right\rangle I_{\mathcal{H}}\right|\right) dt\,\hat{k}_\lambda, \hat{k}_\lambda\right\rangle. \end{aligned} \tag{11}$$

Combining (10) and (11), we get

$$\begin{aligned}
f\left(\left|\left\langle T\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|\right) &\leq \int_0^1 f\left(t\left\langle |T|\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle + (1-t)\left\langle |T^*|\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right)dt \\
&\quad - \int_0^1 f\left(\left|\left\langle (|T|-|T^*|)\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt \\
&\leq \left\langle \int_0^1 f\left(t\,|T| + (1-t)\,|T^*|\right)dt\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle \\
&\quad - \left\langle \int_0^1 f\left(\left|[t\,|T|+(1-t)\,|T^*|] - \left\langle [t\,|T|+(1-t)\,|T^*|]\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle I_{\mathcal{H}}\right|\right)dt\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle \\
&\quad - \int_0^1 f\left(\left|\left\langle (|T|-|T^*|)\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt.
\end{aligned}$$

If we get supremum over $\hat{k}_\lambda \in \mathcal{H}(\Omega)$ we reach our result. □

Remark 2.2. *Note, if we assume that f is convex in Theorem 2.1, then by using inequality* (10) *and the Hermite-Hadamard inequality we can deduce that*

$$f\left(\mathbf{ber}(T)\right) \leq \mathbf{ber}\left(\int_0^1 f\left(t\,|T| + (1-t)\,|T^*|\right)dt\right). \tag{12}$$

Now, if we suppose that superquadratic functions are non-negative, then our result is a better upper bound for the Berezin inequality, because in this case in the right side of inequality (9) *the values*

$$\inf_{\lambda\in\Omega}\left\langle \int_0^1 f\left(\left|[t\,|T|+(1-t)\,|T^*|] - \left\langle [t\,|T|+(1-t)\,|T^*|]\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle 1_{\mathcal{H}}\right|\right)dt\hat{k}_\lambda,\hat{k}_\lambda\right\rangle$$

and

$$\inf_{\lambda\in\Omega}\int_0^1 f\left(\left|\left\langle (|T|-|T^*|)\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt$$

are non-negative.

The function $f(x) = x^r$ $(r \geq 2)$ is an increasing(non-negative) superquadratic function, so we have the next result; see [1].

Corollary 2.3. *Let $T \in \mathcal{L}(\mathcal{H}(\Omega))$ and $r \geq 2$. Then*

$$\begin{aligned}
&\mathbf{ber}^r(T) \\
&\leq \mathbf{ber}\left(\int_0^1 \left(t\,|T| + (1-t)\,|T^*|\right)^r dt\right) \\
&\quad - \inf_{\lambda\in\Omega}\left\langle \int_0^1 \left(\left|[t\,|T|+(1-t)\,|T^*|] - \left\langle [t\,|T|+(1-t)\,|T^*|]\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle I_{\mathcal{H}}\right|\right)^r dt\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle \\
&\quad\quad - \inf_{\lambda\in\Omega}\int_0^1 \left(\left|\left\langle (|T|-|T^*|)\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|\left|t-\frac{1}{2}\right|\right)^r dt
\end{aligned} \tag{13}$$

Theorem 2.4. *Let $S, T \in \mathcal{L}(\mathcal{H}(\Omega))$. If $f : [0, \infty) \to \mathcal{R}$ is increasing superquadratic, then*

$$
\begin{aligned}
& f\left(\mathbf{ber}\left(S^*T\right)\right) \\
& \leq \mathbf{ber}\left(\int_0^1 f\left(t\,|T|^2 + (1-t)\,|S|^2\right) dt\right) \qquad (14)\\
& \quad - \inf_{\lambda\in\Omega} \int_0^1 f\left(\left|\left[t\,|T|^2 + (1-t)\,|S|^2\right] - \left\langle \left[t\,|T|^2 + (1-t)\,|S|^2\right] \hat{k}_\lambda, \hat{k}_\lambda \right\rangle I_{\mathcal{H}}\right|\right) dt \\
& \qquad - \inf_{\lambda\in\Omega} \int_0^1 f\left(\left|\left\langle \left(|T|^2 - |S|^2\right) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right| \left|t - \frac{1}{2}\right|\right) dt.
\end{aligned}
$$

Proof. For an increasing superquadratic function $f : [0, \infty) \to \mathcal{R}$ we have

$$
\begin{aligned}
f\left(\left|\left\langle S^*T\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|\right) &= f\left(\left|\left\langle T\hat{k}_\lambda, S\hat{k}_\lambda \right\rangle\right|\right) \\
&\leq f\left(\left\|T\hat{k}_\lambda\right\| \left\|S\hat{k}_\lambda\right\|\right) \qquad \text{(by the Cauchy-Schwarz inequality)} \\
&= f\left(\sqrt{\left\langle |T|^2 \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \left\langle |S|^2 \hat{k}_\lambda, \hat{k}_\lambda \right\rangle}\right) \\
&\leq f\left(\frac{\left\langle |T|^2 \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \left\langle |S|^2 \hat{k}_\lambda, \hat{k}_\lambda \right\rangle}{2}\right) \qquad \text{(by AM-GM)} \qquad (15)\\
&\leq \int_0^1 f\left(t\left\langle |T|^2 \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + (1-t)\left\langle |S|^2 \hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right) dt \\
&\quad - \int_0^1 f\left(\left|\left\langle \left(|T|^2 - |S|^2\right) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right| \left|t - \frac{1}{2}\right|\right) dt \qquad \text{(by (5))},
\end{aligned}
$$

where $S, T \in \mathcal{L}(\mathcal{H}(\Omega))$ and $\lambda \in \Omega$. Now, by a similar argument as used in the proof of Theorem 2.1 we have the result. □

If we put $S = T$, we have the next result.

Corollary 2.5. *Let $T \in \mathcal{L}(\mathcal{H}(\Omega))$. If $f : [0, \infty) \to \mathcal{R}$ is increasing superquadratic, then*

$$
\begin{aligned}
& f\left(\mathbf{ber}\left(|T|^2\right)\right) \\
& \leq \mathbf{ber}\left(\int_0^1 f\left(t\,|T|^2 + (1-t)\,|T^*|^2\right) dt\right) \qquad (16)\\
& \quad - \inf_{\lambda\in\Omega} \int_0^1 f\left(\left|\left[t\,|T|^2 + (1-t)\,|T^*|^2\right] - \left\langle \left[t\,|T|^2 + (1-t)\,|T^*|^2\right] \hat{k}_\lambda, \hat{k}_\lambda \right\rangle I_{\mathcal{H}}\right|\right) dt \\
& \qquad - \inf_{\lambda\in\Omega} \int_0^1 f\left(\left|\left\langle \left(|T|^2 - |T^*|^2\right) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right| \left|t - \frac{1}{2}\right|\right) dt
\end{aligned}
$$

In the next theorem we investigate an upper bound for the Berezin number involving a normalized positive linear map.

Theorem 2.6. *Let $\Phi : \mathcal{L}(\mathcal{H}(\Omega)) \to \mathcal{L}(\mathcal{H}(\Omega))$ be a normalized positive linear map and $T \in \mathcal{L}(\mathcal{H}(\Omega))$. If f :*

$[0,\infty) \to \mathcal{R}$ is increasing superquadratic, then

$$
\begin{aligned}
& f(\mathbf{ber}^2(\Phi(T))) \\
& \leq \sup_{\lambda\in\Omega}\left(\int_0^1 f\left(\left\|\left(\Phi(t|T|^2+(1-t)|T^*|^2)\right)^{\frac{1}{2}}\hat{k}_\lambda\right\|^2\right)dt\right) \\
& \qquad - \inf_{\lambda\in\Omega}\left(\int_0^1 f\left(\left|\langle(\Phi(T^*T)-\Phi(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt\right) \\
& \leq \mathbf{ber}\left(\Phi\left(\int_0^1 f(t|T|^2+(1-t)|T^*|^2)dt\right)\right) \\
& \qquad - \inf_{\lambda\in\Omega}\left(\left\langle\Phi\left(\int_0^1 f\left(\left|t|T|^2+(1-t)|T^*|^2)-\langle\Phi(t|T|^2+(1-t)|T^*|^2)\hat{k}_\lambda,\hat{k}_\lambda\rangle I_{\mathcal{H}}\right|\right)\right)dt\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right) \\
& \qquad\quad - \inf_{\lambda\in\Omega}\left(\int_0^1 f\left(\left|\langle(\Phi(T^*T)-\Phi(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt\right).
\end{aligned}
$$

Proof. If $\hat{k}_\lambda \in \mathcal{H}(\Omega)$, then we have

$$
\begin{aligned}
f\left(|\langle\Phi(T)\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2\right) &\leq f\left(\frac{\langle\Phi(T^*T)\hat{k}_\lambda,\hat{k}_\lambda\rangle+\langle\Phi(TT^*)\hat{k}_\lambda,\hat{k}_\lambda\rangle}{2}\right) \quad \text{(by [24, Theorem 2.2])} \\
&\leq \int_0^1 f(t\langle\Phi(T^*T)\hat{k}_\lambda,\hat{k}_\lambda\rangle+(1-t)\langle\Phi(TT^*)\hat{k}_\lambda,\hat{k}_\lambda\rangle)dt \\
&\qquad - \int_0^1 f\left(\left|\langle(\Phi(T^*T)-\Phi(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt \\
&\qquad\qquad \text{(by (3))} \\
&= \int_0^1 f(\langle(t\Phi(T^*T)+(1-t)\Phi(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle)dt \\
&\qquad - \int_0^1 f\left(\left|\langle(\Phi(T^*T)-\Phi(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt \\
&= \int_0^1 f(\langle\Phi(t(T^*T)+(1-t)(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle)dt \\
&\qquad - \int_0^1 f\left(\left|\langle(\Phi(T^*T)-\Phi(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt \qquad (17)
\end{aligned}
$$

Hence

$$\begin{aligned} f\left(|\langle\Phi(T)\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2\right) &\leq \int_0^1 f(\langle\Phi(t(T^*T)+(1-t)(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle)dt \\ &\qquad - \int_0^1 f\left(\left|\langle(\Phi(T^*T)-\Phi(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt \\ &= \int_0^1 f\left(\left\langle\left(\Phi(t|T|^2+(1-t)|T^*|^2)\right)^{\frac{1}{2}}\hat{k}_\lambda,\left(\Phi(t|T|^2+(1-t)|T^*|^2)\right)^{\frac{1}{2}}\hat{k}_\lambda\right\rangle\right)dt \\ &\qquad - \int_0^1 f\left(\left|\langle(\Phi(T^*T)-\Phi(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt \\ &= \int_0^1 f\left(\left\|\left(\Phi(t|T|^2+(1-t)|T^*|^2)\right)^{\frac{1}{2}}\hat{k}_\lambda\right\|^2\right)dt \\ &\qquad - \int_0^1 f\left(\left|\langle(\Phi(T^*T)-\Phi(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt. \end{aligned} \tag{18}$$

Moreover, since f is superquadratic we get,

$$\begin{aligned} & f\left(\left\|\left(\Phi(t|T|^2+(1-t)|T^*|^2)\right)^{\frac{1}{2}}\hat{k}_\lambda\right\|^2\right) \\ &= f(\langle\Phi(tT^*T+(1-t)TT^*)\hat{k}_\lambda,\hat{k}_\lambda\rangle) \\ &\leq \langle\Phi(f(t|T|^2+(1-t)|T^*|^2))\hat{k}_\lambda,\hat{k}_\lambda\rangle \\ &\qquad - \left\langle\Phi\left(f\left(\left(\left|t|T|^2+(1-t)|T^*|^2\right)-\langle\Phi(t|T|^2+(1-t)|T^*|^2)\hat{k}_\lambda,\hat{k}_\lambda\rangle I_{\mathcal{H}}\right|\right)\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle \quad \text{(by (8))}. \end{aligned}$$

Integrating on the previous inequality over t on $[0,1]$, we have

$$\begin{aligned} & \int_0^1 f\left(\left\|\left(\Phi(t|T|^2+(1-t)|T^*|^2)\right)^{\frac{1}{2}}\hat{k}_\lambda\right\|^2\right)dt \\ &= \int_0^1 f(\langle\Phi(t|T|^2+(1-t)|T^*|^2)\hat{k}_\lambda,\hat{k}_\lambda\rangle)dt \qquad (19) \\ &\leq \int_0^1 \langle\Phi(f(t|T|^2+(1-t)|T^*|^2))\hat{k}_\lambda,\hat{k}_\lambda\rangle \\ &\qquad - \int_0^1 \left\langle\Phi\left(f\left(\left(\left|t|T|^2+(1-t)|T^*|^2\right)-\langle\Phi(t|T|^2+(1-t)|T^*|^2)\hat{k}_\lambda,\hat{k}_\lambda\rangle I_{\mathcal{H}}\right|\right)\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle \\ &= \left\langle\Phi\left(\int_0^1 f(t|T|^2+(1-t)|T^*|^2)dt\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle \\ &\qquad - \left\langle\Phi\left(\int_0^1 f\left(\left(\left|t|T|^2+(1-t)|T^*|^2\right)-\langle\Phi(t|T|^2+(1-t)|T^*|^2)\hat{k}_\lambda,\hat{k}_\lambda\rangle I_{\mathcal{H}}\right|\right)\right)dt\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle \\ &\qquad\qquad \text{(by the linearity of the inner product and } \Phi\text{)}, \end{aligned}$$

whence by (18) and (19) we get

$$
\begin{aligned}
& f\left(|\langle\Phi(T)\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2\right)\\
&\le \int_0^1 f\left(\left\langle\left(\Phi(t|T|^2+(1-t)|T^*|^2)\right)^{\frac{1}{2}}\hat{k}_\lambda,\left(\Phi(t|T|^2+(1-t)|T^*|^2)\right)^{\frac{1}{2}}\hat{k}_\lambda\right\rangle\right)dt\\
&\qquad\qquad -\int_0^1 f\left(\left|\langle(\Phi(T^*T)-\Phi(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt\\
&= \int_0^1 f\left(\left\|\left(\Phi(t|T|^2+(1-t)|T^*|^2)\right)^{\frac{1}{2}}\hat{k}_\lambda\right\|^2\right)dt\\
&\qquad\qquad -\int_0^1 f\left(\left|\langle(\Phi(T^*T)-\Phi(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt\\
&\le \left\langle\Phi\left(\int_0^1 f(t|T|^2+(1-t)|T^*|^2)dt\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\\
&\qquad\qquad -\left\langle\Phi\left(\int_0^1 f\left(\left|(t|T|^2+(1-t)|T^*|^2)-\langle\Phi(t|T|^2+(1-t)|T^*|^2)\hat{k}_\lambda,\hat{k}_\lambda\rangle I_{\mathcal{H}}\right|\right)dt\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\\
&\qquad\qquad\qquad -\int_0^1 f\left(\left|\langle(\Phi(T^*T)-\Phi(TT^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt.
\end{aligned}
$$

Now, we get supremum over $\hat{k}_\lambda\in\mathcal{H}(\Omega)$ we reach our results. □

Corollary 2.7. *Let $T_j\in\mathcal{L}(\mathcal{H}(\Omega))$ $(1\le j\le n)$ and $\Phi_j:\mathcal{L}(\mathcal{H}(\Omega))\longrightarrow\mathcal{L}(\mathcal{H}(\Omega))$ $(1\le j\le n)$ be normalized positive linear maps. If $f:[0,\infty)\to\mathcal{R}$ is increasing superquadratic, then*

$$
\begin{aligned}
& f\left(\mathbf{ber}^2\left(\sum_{j=1}^n\Phi_j(T_j)\right)\right)\\
&\le \sup_{\lambda\in\Omega}\left(\int_0^1 f\left(\left\|\left(\sum_{j=1}^n\Phi_j(t|T_j|^2+(1-t)|T_j^*|^2)\right)^{\frac{1}{2}}\hat{k}_\lambda\right\|^2\right)dt\right)\\
&\qquad\qquad -\inf_{\lambda\in\Omega}\left(\int_0^1 f\left(\left|\langle(\sum_{j=1}^n\Phi_j(T_j^*T_j-T_jT_j^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt\right)\\
&\le \mathbf{ber}\left(\sum_{j=1}^n\Phi_j\left(\int_0^1 f(t|T_j|^2+(1-t)|T_j^*|^2)dt\right)\right)\\
&\qquad\qquad -\inf_{\lambda\in\Omega}\left(\left\langle\sum_{j=1}^n\Phi_j\left(\int_0^1 f\left(\left|(t|T_j|^2+(1-t)|T_j^*|^2)-\langle\Phi(t|T|^2+(1-t)|T^*|^2)\hat{k}_\lambda,\hat{k}_\lambda\rangle I_{\mathcal{H}}\right|\right)dt\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right)\\
&\qquad\qquad\qquad -\inf_{\lambda\in\Omega}\left(\int_0^1 f\left(\left|\langle(\sum_{j=1}^n\Phi_j(T_j^*T_j-T_jT_j^*))\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|\left|t-\frac{1}{2}\right|\right)dt\right).
\end{aligned}
$$

Proof. Assume that the normalized positive linear map Φ on $\mathcal{L}(\mathcal{H}(\Omega)\oplus\cdots\oplus\mathcal{H}(\Omega))$ by $\Phi(A)=\Phi(A_1)\oplus\cdots\oplus\Phi(A_n)$, where $A=A_1\oplus\cdots\oplus A_n$ is an operator in $\mathcal{L}(\mathcal{H}(\Omega)\oplus\cdots\oplus\mathcal{H}(\Omega))$. Then by applying Theorem 2.6 we get the desired inequalities. □

The next result shows another upper bound for $f(\mathbf{ber}(S^*T))$.

Proposition 2.8. *Let $S,T \in \mathcal{L}(\mathcal{H}(\Omega))$. If $f:[0,\infty)\to\mathcal{R}$ is superquadratic increasing, then*

$$f(\mathbf{ber}(S^*T)) \le \sup_{\lambda\in\Omega}\left(\int_0^1 f\left(\left\|\left(t|T|^2+(1-t)|S|^2\right)^{\frac{1}{2}}\hat{k}_\lambda\right\|^2\right)dt\right) - \inf_{\lambda\in\Omega}\int_0^1 f\left(\left|\left\langle\left(|T|^2-|S|^2\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt. \tag{20}$$

Proof. Let $\hat{k}_\lambda \in \mathcal{H}(\Omega)$. Since f is increasing on $[0,\infty)$, by (15) we have

$$\begin{aligned}
f\left(\left|\left\langle S^*T\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|\right) &\le f\left(\frac{\left\langle|T|^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle+\left\langle|S|^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle}{2}\right) \qquad \text{(by (15))} \qquad (21)\\
&\le \int_0^1 f\left(t\left\langle|T|^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle+(1-t)\left\langle|S|^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right)dt\\
&\quad - \int_0^1 f\left(\left|\left\langle\left(|T|^2-|S|^2\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt \qquad \text{(by (5))}\\
&= \int_0^1 f\left(\left\langle\left(t|T|^2+(1-t)|S|^2\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right)dt\\
&\quad - \int_0^1 f\left(\left|\left\langle\left(|T|^2-|S|^2\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt\\
&= \int_0^1 f\left(\left\langle\left(t|T|^2+(1-t)|S|^2\right)^{\frac{1}{2}}\hat{k}_\lambda,\left(t|T|^2+(1-t)|S|^2\right)^{\frac{1}{2}}\hat{k}_\lambda\right\rangle\right)dt\\
&\quad - \int_0^1 f\left(\left|\left\langle\left(|T|^2-|S|^2\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt\\
&= \int_0^1 f\left(\left\|\left(t|T|^2+(1-t)|S|^2\right)^{\frac{1}{2}}\hat{k}_\lambda\right\|^2\right)dt\\
&\quad - \int_0^1 f\left(\left|\left\langle\left(|T|^2-|S|^2\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|\left|t-\frac{1}{2}\right|\right)dt.
\end{aligned}$$

Now, by taking supremum over all $\hat{k}_\lambda \in \mathcal{H}(\Omega)$, we get the desired result. □

In the next theorem we compare the Berezin number two operators under some mild conditions.

Theorem 2.9. *If $S,T\in\mathcal{L}(\mathcal{H}(\Omega))$ such that $\mathbf{ber}(\mathbf{T})\le\mathbf{ber}(|\mathbf{S}|)$ and f is a non-negative superquadratic function, then*

$$f(\mathbf{ber}(T)) \le \mathbf{ber}(f(|S|)) - \ell_{\mathbf{ber}}\left(f\left(\left||S|-\mathbf{ber}(T)\right|\right)\right).$$

In particular,

$$f(\mathbf{ber}(T)) \le \mathbf{ber}(f(|T|)) - \ell_{\mathbf{ber}}\left(f\left(\left||T|-\mathbf{ber}(T)\right|\right)\right).$$

Proof. If we put $x=\mathbf{ber}(T)$ in (2), we have

$$f(t) \ge f(\mathbf{ber}(T)) + C_{\mathbf{ber}(T)}(t-\mathbf{ber}(T)) + f(|t-\mathbf{ber}(T)|)$$

By using the functional calculus for the operator $|S|$ we get

$$f(|S|) \ge f(\mathbf{ber}(T)) + C_{\mathbf{ber}(T)}(|S|-\mathbf{ber}(T)) + f(||S|-\mathbf{ber}(T)|),$$

whence

$$\langle f(|S|)\hat{k}_\lambda, \hat{k}_\lambda\rangle \geq f(\mathbf{ber}(T)) + C_{\mathbf{ber}(T)}\Big(\langle |S|\hat{k}_\lambda, \hat{k}_\lambda\rangle - \mathbf{ber}(T)\Big) + \langle f(||S| - \mathbf{ber}(T)|)\hat{k}_\lambda, \hat{k}_\lambda\rangle,$$

for $\hat{k}_\lambda \in \mathcal{H}(\Omega)$. So

$$\langle f(|S|)\hat{k}_\lambda, \hat{k}_\lambda\rangle - \langle f(||S| - \mathbf{ber}(T)|)\hat{k}_\lambda, \hat{k}_\lambda\rangle \geq f(\mathbf{ber}(T)) + C_{\mathbf{ber}(T)}\Big(\langle |S|\hat{k}_\lambda, \hat{k}_\lambda\rangle - \mathbf{ber}(T)\Big)$$

for $\hat{k}_\lambda \in \mathcal{H}(\Omega)$. We take supremum over all $\hat{k}_\lambda \in \mathcal{H}(\Omega)$, and we get

$$\begin{aligned}\mathbf{ber}(f(|S|)) - \ell_{\mathbf{ber}} f(||S| - \mathbf{ber}(T)|) &= \sup_{\lambda\in\Omega}\langle f(|S|)\hat{k}_\lambda, \hat{k}_\lambda\rangle - \inf_{\lambda\in\Omega}\langle f(||S| - \mathbf{ber}(T)|)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\ &\geq f(\mathbf{ber}(T)) + C_{\mathbf{ber}(T)}\left(\sup_{\lambda\in\Omega}(\langle |S|\hat{k}_\lambda, \hat{k}_\lambda\rangle) - \mathbf{ber}(T)\right) \\ &= f(\mathbf{ber}(T)) + C_{\mathbf{ber}(T)}(\mathbf{ber}(|S|) - \mathbf{ber}(T))\end{aligned}$$

Note that $C_{\mathbf{ber}(T)}(\mathbf{ber}(|S|) - \mathbf{ber}(T)) \geq 0$, then

$$\mathbf{ber}(f(|S|)) - \ell_{\mathbf{ber}}(f(||S| - \mathbf{ber}(T)|)) \geq f(\mathbf{ber}(T)).$$

Thus we have the first inequality. If we take $S = T$ in the first inequality, we get the second inequality. □

Corollary 2.10. *If $T \in \mathcal{L}(\mathcal{H}(\Omega))$, then*

$$\mathbf{ber}^r(T) \leq \mathbf{ber}(|T|^r) - \ell_{\mathbf{ber}}((||T| - \mathbf{ber}(T)|)^r),$$

where $r \geq 2$.

3. Berezin number inequalities via convex functions

In this section, we establish some Berezin number inequalities involving convex functions. For this propose we need the following lemma, which deduce from [9, Theorem 4].

Lemma 3.1. *Let $f : J \to \mathcal{R}$ be a convex function on the interval J and $a, b \in J$. Then*

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leq (1-\alpha) f\left(\frac{(1-\alpha)a + (1+\alpha)b}{2}\right) + \alpha f\left(\frac{(2-\alpha)a + \alpha b}{2}\right) \\ &\leq \int_0^1 f((1-t)a + tb)\,dt \qquad (22)\\ &\leq \frac{1}{2}\left(f((1-\alpha)a + \alpha b) + (1-\alpha)f(a) + \alpha f(b)\right) \qquad (23)\\ &\leq \frac{f(a) + f(b)}{2}, \qquad (24)\end{aligned}$$

where $\alpha \in [0, 1]$.

Proof. By the Hermite-Hadamard inequality (4) we have

$$\begin{aligned}f\left(\frac{(1-\alpha)a + (1+\alpha)b}{2}\right) &\leq \int_0^1 f\Big((1-t)((1-\alpha)a + \alpha b) + tb\Big)\,dt \\ &\leq \frac{f\Big((1-\alpha)a + \alpha b\Big) + f\Big(b\Big)}{2}\end{aligned}$$

and

$$f\left(\frac{(2-\alpha)a+\alpha b}{2}\right) \le \int_0^1 f\Big((1-t)a+t((1-\alpha)a+\alpha b)\Big)\,dt \\ \le \frac{f\big(a\big)+f\Big((1-\alpha)a+\alpha b\Big)}{2},$$

where $\alpha \in [0,1]$. Now we multiply $1-\alpha$ in the first inequalities and α in the second inequalities and use this fact (see [9, Lemma 1]):

$$(1-\alpha)\int_0^1 f\Big((1-t)((1-\alpha)a+\alpha b)+tb\Big)\,dt+\alpha\int_0^1 f\Big((1-t)a+t((1-\alpha)a+\alpha b)\Big)\,dt=\int_0^1 f((1-t)a+tb)\,dt,$$

we get

$$(1-\alpha)f\left(\frac{(1-\alpha)a+(1+\alpha)b}{2}\right)+\alpha f\left(\frac{(2-\alpha)a+\alpha b}{2}\right) \le \int_0^1 f((1-t)a+tb)dt \\ \le \frac{1}{2}[f((1-\alpha)a+\alpha b)+(1-\alpha)f(a)+\alpha f(b)],$$

whence we have inequalities (22) and (23). Moreover

$$f\Big(\frac{a+b}{2}\Big)=f\left((1-\alpha)\frac{(1-\alpha)a+(1+\alpha)b}{2}+\alpha\frac{(2-\alpha)a+\alpha b}{2}\right) \\ \le (1-\alpha)f\left(\frac{(1-\alpha)a+(1+\alpha)b}{2}\right)+\alpha f\left(\frac{(2-\alpha)a+\alpha b}{2}\right) \qquad \text{(by the convexity)}$$

and

$$\frac{1}{2}\left(f((1-\alpha)a+\alpha b)+(1-\alpha)f(a)+\alpha f(b)\right) \le \frac{1}{2}\left((1-\alpha)f(a)+\alpha f(b)+(1-\alpha)f(a)+\alpha f(b)\right) \\ \text{(by the convexity)} \\ =\frac{f(a)+f(b)}{2},$$

and hence we get the desired inequalities. □

The next result shows a refinement of inequality (12).

Theorem 3.2. *Let $f: J \subseteq \mathcal{R}^+ \to \mathcal{R}$ be an increasing convex function and $T \in \mathcal{L}(\mathcal{H}(\Omega))$. Then*

$$f(\mathbf{ber}\,(T)) \le \mathbf{ber}\left(\int_0^1 f\Big(t|T|+(1-t)|T^*|\Big)dt\right) \\ \le \mathbf{ber}\left(\frac{f((1-\alpha)|T|+\alpha|T^*|)+(1-\alpha)f(|T^*|)+\alpha f(|T|)}{2}\right) \\ \le \mathbf{ber}\left(\frac{f(|T|)+f(|T^*|)}{2}\right),$$

where $\alpha \in [0,1]$.

Proof. Let $\hat{k}_\lambda \in \mathcal{H}(\Omega)$. We have

$$
\begin{aligned}
f\left(\left|\left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right) &\leq f\left(\frac{\left\langle |T|\,\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left\langle |T^*|\,\hat{k}_\lambda, \hat{k}_\lambda\right\rangle}{2}\right) && \text{(by (10))}\\
&\leq \left\langle \int_0^1 f((1-t)|T| + t|T^*|)dt\,\hat{k}_\lambda, \hat{k}_\lambda\right\rangle && \text{(by (22))}\\
&\leq \frac{1}{2}\left\langle [f((1-\alpha)|T| + \alpha|T^*|) + (1-\alpha)f(|T^*|) + \alpha f(|T|)]\hat{k}_\lambda, \hat{k}_\lambda\right\rangle && \text{(by (23))}\\
&\leq \left\langle \left(\frac{f(|T|) + f(|T^*|)}{2}\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle && \text{(by (24)).}
\end{aligned}
$$

□

Because $f(x) = x^r$ $(r \geq 1)$ is increasing convex that we have the next corollary.

Corollary 3.3. *Let $T \in \mathcal{L}(\mathcal{H}(\Omega))$. Then*

$$
\begin{aligned}
\mathbf{ber}^r(T) &\leq \mathbf{ber}\left(\int_0^1 \left(t|T| + (1-t)|T^*|\right)^r dt\right)\\
&\leq \mathbf{ber}\left(\frac{((1-\alpha)|T| + \alpha|T^*|)^r + (1-\alpha)|T^*|^r + \alpha|T|^r}{2}\right)\\
&\leq \mathbf{ber}\left(\frac{|T|^r + |T^*|^r}{2}\right),
\end{aligned}
$$

in which $r \geq 1$ and $\alpha \in [0,1]$.

With a similar argument in the proof of Theorem 3.2, we reach the following result.

Proposition 3.4. *Let $f : J \subseteq \mathcal{R}^+ \to \mathcal{R}$ be an increasing convex function and $S, T \in \mathcal{L}(\mathcal{H}(\Omega))$. Then*

$$
\begin{aligned}
f(\mathbf{ber}(S^*T)) &\leq \mathbf{ber}\left(\int_0^1 f\left(t|T|^2 + (1-t)|S^*|^2\right)dt\right)\\
&\leq \mathbf{ber}\left(\frac{f((1-\alpha)|T|^2 + \alpha|S^*|^2) + (1-\alpha)f(|S^*|) + \alpha f(|T|)}{2}\right)\\
&\leq \mathbf{ber}\left(\frac{f(|T|^2) + f(|S^*|^2)}{2}\right),
\end{aligned}
$$

where $\alpha \in [0,1]$.

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